OPTIMALITY CONDITIONS FOR A CLASS OF HYBRID SYSTEMS

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Abstract

A class of hybrid systems models introduced by the author (Valuev, 1996; Valuev, 2005) gives the possibility to represent production processes in complex industrial systems such as opencast mines which elements change their qualitative states subsequently or cyclically. The problem of resource planning for project scheduling as well as some problems of inventory control may be treated in the same way. The way of changing the succession of events and optimality conditions are presented that show the principal method of finding the optimum control. In the case of linear models optimality conditions are both necessary and sufficient and the optimum may be found with a finite method based on decomposition techniques.

Key words

Hybrid systems, control, optimality conditions, decomposition

1 Introduction. Origin of the model class

The study of production systems show many examples of processes in which after any qualitative state change, or an event (such as changes of equipment units work modes, origination and termination of partial production processes, switches of materials flows destination and so on) the set of relationships between production system variables alter. Events subdivide the entire process period into stages; the succession of events, or a process *scenario*, within a given period is fixed neither in order nor in the number and depends on the process control.

The general formulation of the new models class is given in the paper. These models give the possibility to apply exact optimization techniques for determination of values of some parameters that earlier might be appointed only by experts and to embrace in the sole problem statement a lot of plan problem variants that traditionally may be regarded only separately. Such kind of models may be treated either as deterministic or stochastic that lead to broad possibilities of controlled processes modelling in the context of planning as well as regulation due to various disturbances, but now we concentrate the study on deterministic models only and present some general results pertaining to them.

2 General formulation of the problem

An event-switched process is an N-staged process in which instants of stages ends are moments of the advent of one or more events (for an arbitrary k-th stage the set of these events is $S(k) \subseteq \{1,\ldots, L\}$ where L denotes the number of events types). For an arbitrary k-th stage, i.e. for the flowing time interval $[T(k), T(k+1))$ vectors of qualitative state $d(k) \in A_D (A_D$ is a finite set) and control $u(k) \in R^m$ are constant and the relationship between the final ($x^1(k) \in R^n$) and initial ($x^0(k) \in R^n$) state vectors and the stage duration $t(k)$ has a form of difference equations

$$x^1(k) = Y(d(k),\ x^0(k),\ u(k),\ t(k)), \qquad (1)$$

where $Y(d(k), x^0(k), u(k), t)$ denotes the solution of the Cauchy problem for the ODE system

$$dx(t,\ k)/dt = f(d(k), x(t,\ k), u(k)) \qquad (2)$$

with the initial conditions t=0, $x(0,\ k) = x^0(k)$. For the s-th event type there are the sets of components of I_{Xs}, I_{Ds} of $x(t,\ k)$, $d(k)$ (the latter forming vectors $x^{(s)}(t, k)$, $d^{(s)}(k)$, respectively), so that $I_{Xs'} \cap I_{Xs} = I_{Ds'} \cap I_{Ds} = \emptyset$ for $s' \neq s$ and $i(s) \in I_{Xs}$ exists for which

$$f_{i(s)}(d(k), x(t,\ k), u(k)) \geq f_{min} > 0. \qquad (3)$$

The conditions for the stage termination are

$$\begin{aligned} &r^Y_{i(s)}(d^{(s)}(k), x^{1(s)}(k)) \equiv x^1_{i(s)}(k) - x_{s0} \\ &(d^{(s)}(k)) = 0, s \in S(k), \end{aligned} \qquad (4)$$

$$r^{Y}_{i(s)}(d^{(s)}(k), x^{1(s)}(k)) < 0, s \notin S(k), \quad (5)$$

resulting in no events within the stage. The values of some components of both state vectors change as a result of the above events, so that:

$$\left.\begin{array}{l} d_i(k+1) = D_{is}(d^{(s)}(k)), i \in I_{Ds}, s \in S(k), \\ d_i(k+1) = d_i(k), i \notin I_{Ds}, s \in S(k), \end{array}\right\} \quad (6)$$

$$\left.\begin{array}{l} x^0_i(k+1) = X_{is}(d^{(s)}(k),\ x^{1\ (s)}(k)), \\ i \in I_{Xs}, s \in S(k), \\ \\ x^0_i(k+1) = x^1_i(k), i \notin I_{Xs}, s \in S(k). \end{array}\right\} \quad (7)$$

Equations (6)–(7) may be denoted as

$$\begin{array}{l} d(k+1) = D(S(k),\ d(k)), \\ x^0(k+1) = X(S(k),\ d(k),\ x^1(k)). \end{array}$$

The number of the process stages N is determined from the process termination condition

$$T(N+1) = T(0) + T_1. \quad (8)$$

Constraints on the process have two types: the constraints for any stage

$$\left.\begin{array}{l} r^U_j(d(k), u(k)) \le 0, j \in J_1(d(k)), \\ r^U_j(d(k), u(k)) = 0, j \in J_2(d(k)), \end{array}\right\} \quad (9)$$

and the constraints for a definite event (including terminal constraints)

$$\left.\begin{array}{l} r^Y_j(x^1(k)) \le 0, j \in K_0(d(k)), \\ r^Y_j(x^1(k)) \le 0, j \in K_1(d(k), S(k)). \end{array}\right\} \quad (10)$$

It is supposed that for any $d(k) \in A_D$ the set $U_0(d(k))$ of $u(k)$ satisfying (9) is non-empty and bounded. The problem consists in the determination of the process scenario $S = (S(1), \ldots, S(N))$ and control (i.e., the succession $v = (v(1), \ldots, v(N))$ of vectors $v(k) = (u(k),\ \ t(k))$ with trajectories in continuous- and discrete-valued state variables $d = (d(1), \ldots, d(N))$, $x = (x^0(1), x^1(1), \ldots, x^0(N), x^1(N))$ corresponding to S, v due to (1),(4)–(7) so that restrictions (8)–(10) are satisfied and the target functional

$$F_0(x^1(N)). \quad (11)$$

has the minimum value. We assume that for every $d' \in A_D$, $x' \in R^n$, $u' \in U^0_\Delta(d')$ (where the constant $\Delta > 0$) all the functions $f_i(d', x', u'$, $r^U_j(d', u')$, $r^Y_j(d'^{(s)}, x'^{(s)})$ are determined and continuously differentiated with respect to x', u' and for all their first partial derivatives the generalized Lipschitz condition $|g(y') - g(y)| \le K||y' - y||^\beta$ is valid (here $y = x', u'$) and the constants $K > 0$, $\beta \in$(0, 1] do not depend on a function $g(y)$).

3 Process scenario and other representations of the problem

We assume further that the model (1)–(11) satisfies some general properties (for the base model they are obviously satisfied and additional form of models relationships listed below are likely not to violate them).

Condition 1. For any $d(k) \in A_D$ the set $U_0(d(k))$ of $u(k)$ satisfying (9) is non-empty and bounded.

Condition 2. For all the $d' \in A_D$, $x' \in R^n$, $t' \geq 0$, $u' \in U_{0\Delta}(d') = \{u'' \in R^m \mid r^U_j(d', u'') \le \Delta, j \in J_1(d')\}$ where $\Delta > 0$ is a constant, the functions $Y_i(d', x', u', t')$, $r^U_j(d', u')$, $r^Y_j(x')$ are defined, continuously differentiated with respect to x', u', t' and all their 1^{st} order partial derivatives satisfy Lipschitz condition having the form $|g(y') - g(y)| \le K||y' - y||$ where $y = (x, u, t)$, $y' = (x', u', t')$ and $K > 0$.

Condition 3. For all the $s = 1,\ \ldots,\ L, d' \in A_D$,

1)

$$r^Y_{i(s)}(x^0(1)) < 0; \quad (12)$$

2) for all $x' \in R^n$, $u' \in U_{0\,\Delta}(d')$ the function $r^Y_{i(s)}(Y(d',\, x',\, u',\ t))$ rises monotonously with respect to t;

3) for all $S' \subseteq \{1,\ \ldots,\ L\}$ for which $s \in S'$ and all $x' \in R^n$ satisfying $r^Y_{i(s)}(x') = 0$ the inequality is valid

$$r^Y_{i(s)}(X(S', d', x')) = r_{s0} < 0; \quad (13)$$

4) for all $S', s \notin S'$ and all $x' \in R^n$

$$r^Y_{i(s)}(X(S', d', x')) = r^Y_{i(s)}(x'). \quad (14)$$

Each possible process of the project fulfillment is characterized with the control v, or a succession of vectors $v(k)$=$(u_1(k), \ldots, u_m(k), t(k))$, the scenario, or a succession of sets $S = (S(1),\ldots,\ S(N))$, the trajectory $x = (x_0(1), x_1(1), \ldots,\ x_0(N), x_1(N))$ and the discrete trajectory $d = (d(1), \ldots,\ d(N))$. According to (6) the discrete trajectory is the function of the scenario and according to (1) and (7) the trajectory is the function of the scenario and the control. Subdividing the whole set of possible processes into the sets of processes with the definite scenario we determine $V_0(S)$ as the set of all possible v where $u(k) \in U_0(d(k))$ for any k that generates the trajectory satisfying restrictions (4), (5), (8)–(10). Conditions (3), (12)–(14) guarantee that for all the $s = 1, ..., L$, $k = 1, \ldots, N$ $r^Y_{i(s)}(x^0(k)) < 0$, so from $r^Y_{i(s)}(Y(d(k), x^0(k), u(k), t(k))) = 0$, $s \in S(k)$, we conclude that the obligatory relationship $t(k) > 0$ takes place.

But $V_0(S)$ is not a closed set and for $v^* = \lim\limits_{r\to\infty} v^{(r)}$, $v^{(r)} \in V_0(S)$, we can say that for the corresponding x^* $r^Y_{i(s)}(x^{*1}(k)) \le 0$, $s \notin S(k)$. So we determine another model M_1 with the set of relationships (1)–(3), (6)–(10) and

$$r^Y_{i(s)}(x^1(k)) \le 0, s \notin S(k). \tag{15}$$

Analogously for the model M_1 we conclude formally that for any k $t(k) \ge 0$. The values of $u(k)$ for stages with $t(k)$=0 do not affect the sequence of $x^0(k)$, $x^1(k)$ for stages with $t(k) > 0$. So for any control corresponding to the scenario having $\dim S(k) > 1$ for a certain k we can use other scenario representations. Both properties are used in the iterative search of the optimum scenario.

Other representation of the model M_1 is the model M_2 determined with the set of relationships (1)–(4), (6)–(10) and

$$t(k) \ge 0 \tag{16}$$

for $k = 1, \ldots, N$ and (22) for $k = N$. For M_1 and M_2 $V_1(S)$ and $V_2(S)$ are determined analogously to $V_0(S)$. The equivalence of both representation is asserted by the following lemma.

Lemma 1. $V_1(S) = V_2(S)$.

For a given scenario the set of the model M_2 relationships defines the optimization problem for a discrete-time process with known optimality conditions (Ashchepkov, 1985; Boltyanski, 1973; Propoy, 1973) and efficient numerical methods including (Valuev, 1990; Valuev, 1987). However, we are interested in the project optimization regardless of events succession.

4 Change of the process scenario and necessary optimality conditions

So we consider two aims related to the scenario change for a given $v \in V_2(S)$: first, to separate two simultaneous events sets $S_1 = S(k'-1)$ and $S_2 = S(k')$ for which $t(k') = 0$ with a short stage and second, to make simultaneous two events sets $S_1 = S(k'-1)$ and $S_2 = S(k')$ initially separated with a short stage. To reach both aims we seek for $v_A \in V_2(S)$ for which $v_A(k)$ $=v(k) + \varepsilon\delta v(k) + O(\varepsilon^2)$, $k \neq k'$ and $u_A(k') = u' \in U_0(D(S(k'-1), d(k'-1)))$, $t_A(k') = \varepsilon$ for the 1^{st} aim and $u_A(k') = u(k')$, $t_A(k')$=0 for the second aim.

The set of the model restrictions for a given scenario may be represented in the following general form:

$$\begin{aligned} &F_j(v, S) \le 0,\ j \in I_1(S),\\ &F_j(v,\ S) = 0,\ j \in I_2(S). \end{aligned} \tag{17}$$

The target functional is treated as $F_0(v,S)$ as well. Let us denote (for a feasible control v and $\varepsilon \ge 0$) the set of ε–active restrictions for any $J_1 \subseteq I_1(S)$ as $J_{1\varepsilon}(v,\ S)$=$\{j \in J_1 \mid F_j(v,\ S) \ge -\varepsilon\}$. We define $I_\varepsilon(v, S)$ as $I_{1\varepsilon}(v, S) \cup I_2(S)$ and introduce obvious notation $I^Y(k, S)$ and $I^U(k, S) = J_1(d(k))$. We denote for $J \subseteq I_1(S) \cup I_2(S)$, $v' \in V_2(S)$ $\ F(v', S, J)$ as the vector of $F_j(v', S)$, $j \in J$, and $b_j(k; v', S) = \nabla_{v(k)} F_j(v', S)$, $B_j(v', S)$ as the vector resulting from concatenation of all the $b_j(k; v', S)$, $k \neq k'$, and $B(v', S, J)$ the matrix which rows are $B_j(v', S)$, $j \in J$. We suppose that $B_j(v', S)$, $j \in J$, are linearly independent that is guaranteed with the following

Condition 4(regularity condition).
1) for an arbitrary $v \in V_0(S)$ vectors $F_{jv}(v,\ S)$, $j \in I_0(v, S)$, are linearly independent;
2) for an arbitrary $u(k)$ satisfying (9) vectors $F_{ju}(d(k), u(k))$, $j \in J_{10}(d(k))$, are linearly independent.

If the Condition 4 is valid then ε_0 exists, such that for any $0 \le \varepsilon \le \varepsilon_0$ it is valid not only for 0-active restrictions but for ε-active ones as well.

Let $C(v', S, J)$ be a $\dim(J) \times \dim(J)$ submatrix of $B(v', S, J)$ with the minimum inverse matrix norm. The Condition 4 yields $c_{inv} > 0$ for which $||(C(v', S, J))^{-1}|| \le c_{inv}$ for all $v' \in V_2(S)$, $J \subseteq I_\varepsilon(v, S)$, $0 \le \varepsilon \le \varepsilon_0$.

For both aims $v_A(k')$ satisfy the respective restrictions (9). All other restrictions (17) will be satisfied provided that $||\delta v|| \le n_V$ if for a $\varepsilon \le \varepsilon_0/n_V$ a control v_A satisfies the equations set for the given v:

$$\begin{aligned} &G_j(v_A, S) \equiv F_j(v_A, S) - F_j(v, S) = 0,\\ &j \in I'_\varepsilon = I_\varepsilon(v, S) \setminus J_{1\varepsilon}(d(k')). \end{aligned}$$

We propose a Newton-like method of its solution with initial $v^{(0)}$ where $v^{(0)}(k) = v(k)$, $k \neq k'$, $v^{(0)}(k') = v_A(k')$ and recursive relationships

$$B(v^{(r)}, S, I'_\varepsilon)(v^{(r+1)} - v^{(r)}) = -G(v^{(r)}, S, I'_\varepsilon) \tag{18}$$

from which the vector $v^{C(r+1)}$ of $v^{(r+1)}$ components corresponding to columns of C may be determined as $v^{C(r)} - (C(v^{(r)}, S, I'_\varepsilon))^{-1} G(v^{(r)}, S, I'_\varepsilon)$, the rest components being zeros that yields the unique solution $v^{(r+1)}$. Complying $(C(v^{(r)}, S, I'_\varepsilon))^{-1}$ with zero columns to the dim$(J) \times M$ matrix $Q(v^{(r)}, S, I'_\varepsilon)$ we represent (18) as

$$v^{(r+1)} = v^{(r)} - Q(v^{(r)}, S, I'_\varepsilon) G(v^{(r)}, S, I'_\varepsilon)$$

and the iteration process (18) converges superlinearly if ε is sufficiently small and $||(C(v^{(r)}, S, I'_\varepsilon))^{-1}|| \leq c_{inv}$ for all r.

To determine $F_{jv}(v', S)$ we can use the formula (Propoy, 1973) for $j \in \{0\} \cup I^Y(k_j, S)$

$$\begin{aligned} &\delta F_j = (p_j^1(S, k'), \delta x^1(k')) + \\ &\sum_{k=k'+1}^{k_j} (p_j^0(S, k), Y_v(d(k), x^0(k), v(k)) \delta v(k)) \end{aligned} \tag{19}$$

where $k_0 = N$ and for conjugate variables $p_j^0(S, k'),\ \ p_j^1(S, k')$ we have:

$$\begin{aligned} &p_j^1(S, k_j) = (r_{jx}^Y(x^1(k_j)))^T; \\ &p_j^0(S, k) = Y_x^T(d(k), x^0(k), v(k)) \cdot p_j^1(S, k), \\ &p_j^1(S, k-1) = X_x^T(S(k-1), d(k-1), \\ &x^1(k-1)) \cdot p_j^0(S, k), \quad k = k_j, \ldots, 1, \end{aligned} \tag{20}$$

$$\begin{aligned} &F_0(v^*, S) = F_0(v, S) + \\ &\varepsilon \cdot (q_{00}(v, S),\, Y_t(d(k'), x^0(k'), u_A(k'), 0) \\ &+ O(\varepsilon^2) \end{aligned} \tag{21}$$

where

$$\begin{aligned} q_{00}(v,\ S) = p_0^1(S, k') - F_{ov}(v,\ S) \times \\ \sum_{j \in I_0^Y(v,S)} Q_j(v^{(0)}, S, I'_0) \cdot p_j^1(S, k'). \end{aligned}$$

From (21) we may come to the necessary optimality condition formulated in the (Valuev, 2005), namely

Theorem 1. *If the pair* $(S,\ v \in V_2(S))$ *gives the solution of the problem (1)–(11) and for some* k' $\dim(S(k'-1)) > 1$ *then for any* S_A, $S_A(k) = S(k)$, $k < k'-1$, $S_A(k'-1) \cup S_A(k') = S(k'-1)$ *and* $S_A(k) = S(k-1)$, $k = k'+1, \ldots, N+1$, *there exists a vector* $q_{00}(v,\ S_A)$ *for which for any* $u_A(k') \in U_0(d_A(k'))$

$$(q_{00}(v,\ S_A), Y_t(d(k'), x^0(k'), u_A(k'), 0)) \geq 0.$$

5 Resource planning as a problem of a transforming process optimization

The paper (Valuev, 2007) introduced the representation of the problem in question as an optimization problem for a hybrid system (Branicky *et al.*, 1998); this form of the model enables to perform non-local optimization with the use of special optimality conditions and iteration method of branch-and-bound type. In this paper additional opportunities resulting from linear form of relationships are studied. It was noticed in (Valuev, 2007) that the process may have different *scenarios*, i.e., sequences $D = (d(1), \ldots, d(N))$ of qualitative states of the project. According to this approach the search of the optimum solution is based on three types of calculations: optimization within a given scenario, testing the present scenario optimality and shifting to a better adjacent scenario.

For a given scenario the optimum schedule is found from dynamic linear programming problem (DLP):

$$T(N) \to \min; \tag{22}$$

$$\begin{aligned} &T(0) = 0; \quad T(k) = T(k-1) + t(k), \\ &k = 1, \ldots, N; \\ &x(0) = 0; \quad x_i(k) = x_i(k-1) + y_i(k); \end{aligned} \tag{23}$$

$$\begin{aligned} &u_{\min i} t(k) \leq y_i(k) \leq u_{\max i} t(k), \\ &i \in I_1(d(k));\ y_i(k) = 0, i \notin I_1(d(k)); \\ &\sum_{i \in I_{Rj}} y_i(k) \leq u_{Rj} t(k), j = 1, \ldots, m; \\ &t(k) \geq 0; \end{aligned} \tag{24}$$

$$x_i(k) + y_i(k) = x_{Ti},\ i \in I_2(d(k+1)). \tag{25}$$

First of all, DLP problem is a particular case of a linear programming problem, so its exact solution may be found with a finite method. Besides, there are decomposition methods that enhance the efficiency of optimum search, e.g. (Krivonozhko *et al.*, 1987). All of these method guarantee reaching the optimum (within a given scenario). If more than one work terminates at the end of some stage, then other scenario representations of the project schedule exist and it is necessary to test whether the same schedule is optimal within these adjacent scenarios.

6 Treating the problem with the use of a decomposition scheme

The method proposed here originates from author's generalization (Valuev, 1987) of the computational construction proposed by Boltyanski (Boltyanski, 1973) to simplify optimality conditions for discrete-time processes.

Let $v \in V(D)$, $K_0(v, D) = \emptyset$. Let us determine for any $k = 1, \ldots, N$ $P(k) \subseteq I_T(k, D)$ and $N_M(k)$ according to the conditions:

$$\begin{aligned}
&\dim(I_{V10}(v,k,D) \cup I_{V2}(k,D) \cup P_T(k)) \le n+1,\\
&L(k) = I_T(k,D) \setminus P_T(k),\ N_L = \dim(L(1)) + \ldots +\\
&\dim(L(N)),\ N_M(k) \le n+1-\dim(I_{V10}(v,k,D)\\
&\cup I_{V2}(k,D) \cup P_T(k)),\\
&N_{MS}(k) = N_M(1) + \ldots + N_M(k) \ge N_{LS}(k) =\\
&\dim(L(1)) + \ldots + \dim(L(N)), N_{MS}(N) = N_{LS}(N),\\
&M(k) = \{N_{MS}(k-1)+1, \ldots, N_{MS}(k)\}.
\end{aligned}$$

We can determine $(n+1) \times (n+1)$ matrices $C(k)$ and a set of linearly independent vectors $g^m(k) \in R^{n+1}$, $m \in M(k)$, from the following systems of linear equations:

$$\begin{aligned}
&b_{Zi}^T(k,D) + b_{Vi}^T(k,D)C(k) = 0,\\
&b_{Vi}^T(k,D)g^l(k) = 0,\ l \in M(k),\\
&i \in I_{V10}(v,k,D) \cup I_{V2}(k,D) \cup P_T(k).
\end{aligned} \tag{26}$$

It is shown (Valuev, 1987) for a more general model than (22)–(25) that any feasible direction δv may be defined stage-wise in such a way:

$$\delta v(k) = \delta_0 v(k) + C(k)\delta z(k) + \sum_{l \in M(k)} \mu_l g^l(k), \tag{27}$$

where the following condition is valid for every $\delta_0 v(k)$

$$\begin{aligned}
&b_{Vi}^T(k,D)\delta_0 v(k) \le 0,\ i \in I_{V10}(k,D),\\
&b_{Vi}^T(k,D)\delta_0 v(k) = 0,\\
&i \in I_{V2}(k,D) \cup P_T(k)),
\end{aligned} \tag{28}$$

and the below conditions on variables μ_l, $l \in M(k)$, are valid. Note that for any $\delta z \in R^{n+1}$ and any μ_l, $l \in M(k)$, we have for $\delta F_i = F_i(v,D) - F_i(v+\delta v, D)$, $i \in I_{V10}(v,k,D) \cup I_{V2}(k,D) \cup P_T(k)$, the formula

$$\begin{aligned}
&b_{Zi}^T(k,D)\delta z(k) +\\
&b_{Vi}^T(k,D)\delta v(k) = b_{Vi}^T(k,D)\delta_0 v(k).
\end{aligned} \tag{29}$$

Using the following conjugate equations for the target functional and restrictions from $L(k)$:

$$\begin{aligned}
&p^0(N+1) = (0, \ldots, 0, -1);\\
&p^0(k) = (E + C(k))p^0(k+1), k = N, \ldots, 1;
\end{aligned} \tag{30}$$

$$\begin{aligned}
&p^i(k'+1) = 0,\ k' = k+2, \ldots, N,\\
&p^i(k+1) = b_{Zi}^T(k,D),\\
&p^i(k') = (E + C(k'))p^i(k'+1),\\
&k' = k, \ldots, 1,
\end{aligned} \tag{31}$$

and letting $L_S = L(1) ... \cup L(N)$ we get the following formulas for their variations:

$$\begin{aligned}
&\delta F_i(v) = \sum_{k=1}^{N} (p^i(k+1), \delta_0 v(k) +\\
&\sum_{l \in M(k)} \mu_l g^l(k)),\ i \in \{0\} \cup L_S.
\end{aligned} \tag{32}$$

The relationships to determine all values of μ_l, $l \in M(k)$, $k = 1, \ldots, N$, are

$$\delta F_i(v) = 0,\ \ i \in L_S. \tag{33}$$

With the formulas (32) they are reduced to a system of linear equations. Let $G_{il} = (p^i(k+1), g^l(k))$, $l \in L_S$, $l \in M(k)$, $Q = G^{-1}$, then

$$\begin{aligned}
&\mu_l = -\sum_{i \in L_S} Q_{il} \sum_{k=1}^{N} (p^i(k+1), \delta_0 v(k)),\\
&l \in M(k), k = 1, ..., N.
\end{aligned} \tag{34}$$

With the substitution of (34) to (32) we have the final expression for $\delta F_0(v)$

$$\begin{aligned}
&\delta F_0(v,D) = \sum_{k=1}^{N} (q(k+1), \delta_0 v(k)),\\
&Q_i(k) = \left(\sum_{k'=1}^{N} \sum_{l \in M(k')} Q_{il} \bullet \left(p^0(k'+1), g^l(k')\right) \right)\\
&q(k+1) = p^0(k+1) + \sum_{i \in L_S} Q_i(k) p^i(k+1).
\end{aligned} \tag{35}$$

The efficiency of the decomposition scheme depends mainly on its dimension, i.e., $N_{LS}(N)$; in practice, as a rule, $N_{LS}(N)$ is much less than the dimension of v.

If $K_0(v,D) \ne \emptyset$, then the control v' received from v by cancelling stages of zero duration (and hence $v(k) = 0$) and joining $I_T(k)$ to $I_T(k-1)$ corresponds to another scenario D'. For $v' \in V'(D')$ the optimality criterion of the Theorem 2 may be tested. It is possible, however, to test the optimality of v within the original scenario and other adjacent scenarios with the below optimality conditions.

Other scenario representations exist for the process with $v \in V(D)$ for which $K_0(v,D) = \emptyset$ and $K_1(D) = \{k \mid \dim(I_T(k)) > 1\} \ne \emptyset$. We treat the scenario D' as adjacent to D if a set of stages $K' \subseteq K_1(D)$ exists for which every $k \in K'$ (for D) corresponds in D' to the succession of stages that we numerate with compound indices $(k,1)$ or $k, (k,2), \ldots, (k, n(k))$, these stages terminating with sets of finished works $I_T(k,1), \ldots,$ $I_T(k, n(k))$ where $I_T(k) = I_T(k,1) \cup \ldots \cup I_T(k, n(k))$, the rest stages $k \notin K'$ having the same $I_T(k)$. Obviously $v' \in V'(D')$, if $v'(k) = v(k)$, $k = 1, \ldots, N$, $v'(k,i) = 0$, $k \in K'$, $i = 1, \ldots, n(k)$; we denote $v_{TR}(v, D, D')$ such a v'.

Let $v \in V(D)$ be the solution of the problem (22)–(25). To establish whether the adjacent scenario

D' is not better than D, it is necessary to compare $F_0(v_{TR}(v, D, D'),\ D') = F_0(v, D)$ with $F_0(v', D')$ for near controls $v' \in V(D')$ with at least one $v'(k, i) \neq 0$. To construct such a v' we use the following variant of the formula (27):

$$\delta v(k) = C(k)\left(\delta z(k) + \sum_{r=1}^{n(k)} v(k,r)\right) + \sum_{l} \in M(k)\mu_l g^l. \tag{36}$$

We have for $\delta F_i(v', D')$, $i \in L_T(k)$, $k = 1, \ldots, N$, and for $\delta z(k+1)$, $k \in K'$, the same expression as for the scenario D with

$$\delta_0 v(k) = (E + C(k)) \sum_{r=1}^{n(k)} v(k,r). \tag{37}$$

Let us determine μ_l, $l \in M(k)$, k =1,…,N, from (34) as for the scenario D with $\delta_0 v(k)$=0 for $k \notin K'$, using the formula (37) for $k \in K'$. Then we have from (35)

$$\delta F_0(v') = \sum_{k \in K'} \left(q(k+1), (E + C(k)) \sum_{r=1}^{n(k)} v(k,r) \right).$$

Theorem 2. *Let the pair $(D, v \in V(D))$ for which $K_1(D) \neq \emptyset$ and let $K_0(v, D) = \emptyset$ be the solution of the problem (22)–(25). The optimum values of the problem (22)–(25) for all adjacent scenarios satisfy $F^*(D') \geq F^*(D) = F_0(v, D)$ if and only if for any $k \in K_1(D), I_{T1}(k) \subset I_T(k)$, $I_{T1}(k) \neq \emptyset$, for any $\delta v' = (\delta y'_1, \ldots, \delta y'_n, 1)$ satisfying*

$$\begin{aligned} &u_{\min i} \leq \delta y'_i \leq u_{\max i},\ i \in I_1(d(k,2)); \\ &\delta y'_i = 0,\ i \notin I_1(d(k,2)); \\ &\sum_{i \in I_{Rj}} \delta y'_i \leq u_{Rj},\ j = 1, \ldots, m; \end{aligned} \tag{38}$$

where $d(k, 2) = D_+(d(k),\ I_{T1}(k))$, the inequality is valid

$$\delta F_0(v) = (q(k+1), (E + C(k))\delta v') \geq 0. \tag{39}$$

7 Principal construction of the computational method

As it was formulated above, the numerical method based on the above decomposition constructions consists in the interchange of the solution of optimization problems (22)–(25) within a given D and the search of better adjacent scenarios by testing optimality conditions of the theorem 3. Most of the necessary calculations are reduced to direct computation of conjugate trajectories with (30)–(31), solution of algebraic linear equations (26) and (33), some linear transformations and testing optimality conditions by solution of linear programming problems, the latter being 1) minimization of $q^T(k+1)\delta_0 v(k)$ under constraints (28) and $|\delta_0 v_i(k)| \leq$1, $i = 1, \ldots, n$; and 2) minimization of $q^T(k+1)(E + C(k))\delta v'$ under constraints (38). The dimension (n+1 and n variables, respectively) and the structure of both problems are very similar, no singularity being displayed.

The author's hypothesis is that in the set of D there are no local minima. It means that if $F(D)$ is less that $F(D')$ for all adjacent scenarios D' it gives the global minimum. No contradictions with this hypothesis was found, some evidence is found in particular cases, but its formal substantiation is not found as well. If it is always true it is not necessary to build a solution tree, because in that case every minimizing succession of scenarios lead to the globally optimum solution.

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