MEIJER G-FUNCTIONS SERIES AS SOLUTIONS FOR SOME EULER-LAGRANGE EQUATIONS OF FRACTIONAL MECHANICS

Malgorzata Klimek
Institute of Mathematics and Computer Science
Czestochowa University of Technology
ul. Dabrowskiego 73, 42-200 Czestochowa, Poland
klimek@imi.pcz.czest.pl
fax 0048 34 3250 324

Abstract

Fractional oscillator problem on a finite time interval is studied. This equation, obtained by the minimum action principle, contains the left- and the right-sided fractional derivatives. The Mellin transform is applied and the general continuous solution in the form of Meijer G-functions series is derived. Solutions of the fractional oscillator equation derived using Mellin transform are compared with that obtained via Banach fixed point theorem. As a result analytical relations for certain iterated fractional integrals including the left- and the right-sided operators appear. These integrals will be applied in solving a general class of variational fractional equations.

Key words

Fractional derivatives, Euler-Lagrange equations, Meijer G-functions.

1 Introduction

In fractional calculus the notions of integer order derivatives and integrals were extended to the non integer order operators (see results and references enclosed in monograph [Samko, Kilbas and Marichev, 1993]). Fractional operators are now being applied in physics, mechanics, engineering, bioengineering and finances [Agrawal, Tenreiro Machado and Sabatier, 2004; Hilfer, 2000; Magin, 2006; Metzler and Klafter, 2004; Sabatier, Agrawal and Tenreiro Machado, 2007; West, Bologna and Grigolini, 2003; Herrmann, 2007]. In mathematical modelling of many systems fractional differential equations, both ordinary and partial ones, were obtained. Hence, the derivation of solutions for such equations is an important problem in fractional calculus.

Methods for solving fractional problems include fixed point theorems, integral transforms, compositional and operational methods [Kilbas, Srivastava and Trujillo, 2006] (see also earlier results in [Miller and Ross, 1993; Podlubny, 1999]). However, the ordinary fractional equations were solved exactly under the assumption that they contain only one type of derivative, namely the left- or the right-sided one.

Let us point out that due to mixing of the left- and the right-sided fractional derivatives in the integration by parts formula [Agrawal, 2006; Samko, Kilbas and Marichev, 1993], the fractional variational principle always leads to the Euler-Lagrange equations with both types of derivatives. This feature is characteristic for models of fractional mechanics as well as for field theory equations. Fractional mechanics was studied first by Riewe in [Riewe, 1996; Riewe, 1997] where both in Lagrange and Hamilton formulations the mixing of derivatives appears. Subsequent results by Agrawal [Agrawal, 2002; Agrawal, 2006], Klimek [Klimek, 2001; Klimek, 2002] as well as obtained in papers by Baleanu and collaborators [Baleanu and Avkar, 2004; Baleanu and Muslish 2005; Baleanu 2006] and recently by Cresson [Cresson 2007], yield similar equations.

The problem of derivation the exact solutions for fractional differential equations with mixed derivatives is therefore an important and emerging area in fractional calculus.

Let us notice that interesting results were obtained in this field by Agrawal in [Agrawal, 2007], where he applied composition rules of fractional calculus. The present paper is devoted to derivation of exact continuous solution for one of the simplest variational equations of fractional mechanics, namely we discuss application of the Mellin transform to the fractional oscillator equation. As a solution Meijer G-functions series appears.

The paper is organized as follows. In section 2 we present all relevant formulas from fractional calculus and integral transforms theory. Then in section 3 we solve fractional oscillator equation on a finite time interval using the Mellin transform. The solutions for the irrational order α are described in section 3.1 and for the rational order in section 3.2. Sections 3.3 and 3.4 contain examples for the irrational and rational order α. In section 4 some analytical results concerning the integration in fractional calculus are discussed.

2 Fractional integrals and derivatives

We recall some definitions of fractional operators and describe their properties. The fractional integrals of real, positive order $\alpha \in R_+$ are defined as follows [Kilbas, Srivastava and Trujillo, 2006; Samko, Kilbas and Marichev, 1993]:

$$I^{\alpha}_{0+} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)ds}{(t-s)^{1-\alpha}} \;\; t > 0 \quad (1)$$

$$I^{\alpha}_{b-} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(s)ds}{(s-t)^{1-\alpha}} \;\; t < b \quad (2)$$

$$I^{\alpha}_{-} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^{\infty} \frac{f(s)ds}{(s-t)^{1-\alpha}} \;\; t < \infty. \quad (3)$$

The first two integrals I^{α}_{0+} and I^{α}_{b-} are known as the left-sided and respectively the right-sided Riemann-Liouville fractional integral and the last one is called right-sided Liouville integral.

Using the above fractional integrals two types of derivatives are constructed. The first family are the left- and the right-sided Riemann-Liouville derivatives which for real order $\alpha \in (n-1, n)$ look as follows (we have denoted the classical derivative as $D := \frac{d}{dt}$) [Kilbas, Srivastava and Trujillo, 2006; Samko, Kilbas and Marichev, 1993]:

$$D^{\alpha}_{0+} f(t) := D^n I^{n-\alpha}_{0+} f(t) \quad (4)$$

$$D^{\alpha}_{b-} f(t) := (-D)^n I^{n-\alpha}_{b-} f(t). \quad (5)$$

In the limit $\alpha \longrightarrow n^+$ for $\alpha \in (n, n+1)$ we recover the classical integer order derivatives.

When we change the order of integral and differential operators in (4,5) we obtain the Caputo fractional derivatives which for the order $\alpha \in (n-1, n)$ are given by the following formulas for absolutely continuous function $f \in AC^n[0, b]$ [Kilbas, Srivastava and Trujillo, 2006; Samko, Kilbas and Marichev, 1993]:

$${}^cD^{\alpha}_{0+} f(t) := I^{n-\alpha}_{0+} D^n f(t) \quad (6)$$

$${}^cD^{\alpha}_{b-} f(t) := I^{n-\alpha}_{b-} (-D)^n f(t). \quad (7)$$

We propose to apply in solution of some variational fractional problems one of the integral transforms, namely we shall use the Mellin transform. It looks as follows for sufficiently good functions [Glaeske, Prudnikov and Skórnik, 2006; Kilbas and Saigo, 2004]:

$$\mathcal{M}[f](s) := \int_0^{\infty} t^{s-1} f(t) dt. \quad (8)$$

Let us recall the Mellin transforms for power functions. These transforms look as follows when the functions are multiplied with the Heaviside's function (we assume $Re(s) > 0, \;\; Re(\gamma) > 0$ for the first equality):

$$\mathcal{M}[H(1-t)(1-t)^{\gamma-1}](s) = \frac{\Gamma(s)\Gamma(\gamma)}{\Gamma(s+\gamma)}$$

$$\mathcal{M}[t^{\gamma} \Delta H](s) = \frac{b^{\gamma+s}}{\gamma+s}, \quad (9)$$

where we have denoted as ΔH the following difference of Heaviside's functions: $\Delta H(t) = H(t) - H(t-b)$. Similarly to the Laplace transform, the Mellin transform also has its convolution defined by the formula:

$$f * g(t) := \int_0^{\infty} f(u) g\left(\frac{t}{u}\right) \frac{du}{u}. \quad (10)$$

When the Mellin transform acts on the Mellin convolution of two functions the result is the multiplication of corresponding transforms of both functions:

$$\mathcal{M}[f * g](s) = \mathcal{M}[f](s) \cdot \mathcal{M}[g](s). \quad (11)$$

The Mellin transform obeys also the translation property:

$$\mathcal{M}[t^{\gamma} f](s) = \mathcal{M}[f](s+\gamma) \quad (12)$$

Finally, we quote Lemma describing the Mellin transform for fractional integrals after monograph by Kilbas *et al.* [Kilbas, Srivastava and Trujillo, 2006].

Lemma 2.1

(1) Let $s \in C$ and $\int_0^{\infty} \mid t^{s+\alpha-1} f(t) \mid dt < \infty$. The following formula holds for $Re(s) < 1 - \alpha$:

$$\mathcal{M}[I^{\alpha}_{0+} f](s) = \frac{\Gamma(1-\alpha-s)}{\Gamma(1-s)} \mathcal{M}[f](s+\alpha). \quad (13)$$

(2) Let $s \in C$ and $\int_0^{\infty} \mid t^{s+\alpha-1} f(t) \mid dt < \infty$. The following formula holds for $Re(s) > 0$:

$$\mathcal{M}[I^{\alpha}_{-} f](s) = \frac{\Gamma(s)}{\Gamma(s+\alpha)} \mathcal{M}[f](s+\alpha). \quad (14)$$

3 Mellin transform applied to fractional oscillator equation

We shall study the fractional oscillator equation on a finite time interval $[0, b]$ and apply the Mellin transform method to obtain its general continuous solution.

The fractional oscillator equation is derived using the minimum action principle. Such an equation contains both: the left- and the right-sided derivatives. Let us consider the following action:

$$S = \int_0^b \left[\frac{1}{2}(D^\alpha_{0+} f)^2 - \frac{\lambda}{2} f^2 \right] dt.$$

Applying the minimum action principle and the integration by parts formula [Agrawal, 2006]:

$$\int_0^b f(t) \cdot D^\alpha_{0+} g(t) dt = \int_0^b g(t) \cdot {}^c D^\alpha_{b-} f(t) dt \quad (15)$$

we obtain the fractional oscillator equation of the form:

$$[{}^c D^\alpha_{b-} D^\alpha_{0+} - \lambda] f(t) = 0 \qquad t \in [0, b]. \quad (16)$$

Similarly to the case of ordinary differential equation, we have previously reformulated the above problem to its integral form on the space of continuous functions [Klimek, 2007; Klimek, 2008a]. In this procedure we have derived the explicit form of the continuous stationary function for fractional operator ${}^c D^\alpha_{b-} D^\alpha_{0+}$ when $\alpha \in (n-1, n)$:

$${}^c D^\alpha_{b-} D^\alpha_{0+} \phi_0(t) = 0 \Longleftrightarrow \quad (17)$$

$$\phi_0(t) = \sum_{k=-n+1}^{n-1} A_k t^{\alpha+k} \Delta H(t),$$

where $A_k \in R$ are arbitrary real coefficients and ΔH is the difference of Heaviside's functions.
As we intend to study solutions of (16) on a finite time interval $[0, b]$ thus we can include ΔH function in the formula for stationary function and the general solution can be written as $f_0 = f \Delta H$ for $t \in R_+$ since on the interval $[0, b]$ both functions f and f_0 coincide.
After application of the corresponding composition rules for fractional integrals and derivatives we reformulate the problem (16) to the integral equation, which is an equivalent to (16) when we look for continuous solutions $f_0 \in C[0, b]$:

$$(1 - \lambda I^\alpha_{0+} I^\alpha_{b-}) f_0(t) = \phi_0(t) \quad (18)$$

and the stationary function ϕ_0 is given by (17).
Let us notice that when we study the problem on a finite time interval $[0, b]$ using the form of solution f_0, then the following fractional integrals of this function coincide:

$$I^\alpha_{b-} f_0(t) = I^\alpha_{-} f_0(t). \quad (19)$$

Equation (18) contains now fractional integrals on a half axis:

$$(1 - \lambda I^\alpha_{0+} I^\alpha_{-}) f_0(t) = \phi_0(t). \quad (20)$$

Applying the Mellin transform we rewrite (20) as the following difference equation (valid on a strip on complex plane described as $Re(s + \alpha) \in (0, 1)$):

$$(1 - \lambda g(s) T_{2\alpha}) \mathcal{M}[f_0](s) = \Phi_0(s). \quad (21)$$

We have denoted in the above formula as g the following function of the complex variable $s \in C$:

$$g(s) = \frac{\Gamma(1 - \alpha - s)\Gamma(s + \alpha)}{\Gamma(1 - s)\Gamma(s + 2\alpha)} \quad (22)$$

and as Φ_0 - the Mellin transform of the stationary function ϕ_0 for $\alpha \in (n-1, n)$:

$$\Phi_0(s) = \sum_{k=-n+1}^{n-1} A_k \frac{b^{s+\alpha+k}}{s + \alpha + k}. \quad (23)$$

The operator $T_{2\alpha}$ is a translation operator on complex plane: $T_{2\alpha} h(s) = h(s + 2\alpha)$.
The above difference equation is solved by the following series [Klimek, 2008a]:

$$\mathcal{M}[f_0](s) = \sum_{m=0}^{\infty} \lambda^m [g(s) T_{2\alpha}]^m \Phi_0(s). \quad (24)$$

Absolute convergence of the above series was explicitly proved in [Klimek, 2008a] for $Re(s + \alpha) \in (0, 1)$.

3.1 The case of the order α - an arbitrary irrational number

Let us notice that the component $[g(s) T_{2\alpha}]^m$ can be written as the Mellin transform of the Meijer G-function, namely the following relation is valid [Kilbas and Saigo, 2004; Kilbas, Srivastava and Trujillo, 2006]:

$$\mathcal{G}^{m,m}_{2m,2m} \left[\begin{array}{c} \vec{a}_m \\ \vec{b}_m \end{array} \mid s \right] = \prod_{l=0}^{m-1} g(s + 2l\alpha)$$

$$\vec{a}_m = [\alpha, 3\alpha, ..., (2m-1)\alpha, 2\alpha, 4\alpha, ..., 2m\alpha]$$

$$\vec{b}_m = [\alpha, 3\alpha, ..., (2m-1)\alpha, 0, 2\alpha, ..., 2(m-1)\alpha]$$

provided the condition for the order α is fulfilled [Kilbas and Saigo, 2004; Kilbas, Srivastava and Trujillo, 2006]:

$$\alpha \neq \frac{-1-k-l}{2(j-i)}, \tag{25}$$

where $j, i = 1, ..., m$ and $k, l \in N_0$. We clearly see that for α rational this condition can not be fulfilled starting from a certain number $m \in N$.
Thus we shall describe now the solution f_0 in the case when the order α is an irrational number.
We arrive at the explicit form for the Mellin transform of the solution f_0 expressed in terms of Mellin transforms of the Meijer G-functions:

$$\mathcal{M}[f_0](s) = \tag{26}$$

$$= \Phi_0(s) + \sum_{m=1}^{\infty} \lambda^m \mathcal{G}^{m,m}_{2m,2m} \left[\begin{array}{c} \vec{a}_m \\ \vec{b}_m \end{array} \mid s \right] \Phi_0(s + 2m\alpha).$$

We see that in the above series each factor $\mathcal{G}^{m,m}_{2m,2m}$ is multiplied with $\Phi_0(s + 2m\alpha) = \mathcal{M}[t^{2m\alpha}\phi_0(t)]$.
We know from classical results on Fox and Meijer G-functions (see for example [Kilbas and Saigo, 2004; Kilbas, Srivastava and Trujillo, 2006]) that the inverse Mellin transform of the first factor looks as follows:

$$\mathcal{M}^{-1}\left[\mathcal{G}^{m,m}_{2m,2m} \left[\begin{array}{c} \vec{a}_m \\ \vec{b}_m \end{array} \mid s \right]\right](t) = \tag{27}$$

$$= G^{m,m}_{2m,2m} \left[\begin{array}{c} \vec{a}_m \\ \vec{b}_m \end{array} \mid t \right]$$

provided the corresponding vertical contour $\mathcal{L}_{i\gamma\infty}$ can be placed in the strip $Re(s + \alpha) \in (0, 1)$. Parameters defining the contour look as follows for our transforms (compare for example formulas (1.11.16-18) and (1.12.6) for Fox functions from [Kilbas, Srivastava and Trujillo, 2006]):

$$\Delta = 0 \qquad \delta = 1 \qquad a^* = 0 \qquad \mu = -2m\alpha. \tag{28}$$

We conclude that for such values of parameters the required contour exists when $\alpha > 1/2$ and in this case $t \in R_+$.
Now we are ready to derive the solution f_0 as the inverse Mellin transform using the formula (11) for the Mellin convolution:

$$f_0(t) = \tag{29}$$

$$= \phi_0(t) + \sum_{m=1}^{\infty} \lambda^m G^{m,m}_{2m,2m} \left[\begin{array}{c} \vec{a}_m \\ \vec{b}_m \end{array} \mid t \right] * t^{2m\alpha}\phi_0(t).$$

In the above version of the general solution of (16) the Mellin convolution appears in each term on the right-hand side. This convolution was calculated for components $t^{2m\alpha+\alpha+k}$ using properties of Fox and Meijer G-functions [Kilbas and Saigo, 2004] providing the following result:

$$G^{m,m}_{2m,2m} \left[\begin{array}{c} \vec{a}_m \\ \vec{b}_m \end{array} \mid t \right] * t^{2m\alpha} t^{\alpha+k} \Delta H(t) = \tag{30}$$

$$= b^{2m\alpha+\alpha+k} G^{m+1,m}_{2m+1,2m+1} \left[\begin{array}{c} \vec{A}_{m,k} \\ \vec{B}_{m,k} \end{array} \mid \frac{t}{b} \right].$$

In this formula vectors $\vec{a}_m$ and $\vec{b}_m$ are replaced with the new vectors $\vec{A}_{m,k}$ and $\vec{B}_{m,k}$ defined as follows:

$$\vec{A}_{m,k} = [\vec{a}_m, (2m+1)\alpha + k + 1] \tag{31}$$

$$\vec{B}_{m,k} = [(2m+1)\alpha + k, \vec{b}_m]. \tag{32}$$

Using the above result on covolution we can present the new version of the solution f_0 for the fractional variational problem (16). It appears that this general solution is an arbitrary linear combination of Meijer G-functions series, each corresponding to one component of the continuous stationary function (17):

$$f_0(t) = \sum_{k=-n+1}^{n-1} c_k f_0^k(t) \tag{33}$$

$$f_0^k(t) = \tag{34}$$

$$t^{\alpha+k} + b^{\alpha+k} \sum_{m=1}^{\infty} (\lambda b^{2\alpha})^m G^{m+1,m}_{2m+1,2m+1} \left[\begin{array}{c} \vec{A}_{m,k} \\ \vec{B}_{m,k} \end{array} \mid \frac{t}{b} \right],$$

where $\alpha > 1/2$ is an arbitrary irrational number.
Let us notice that even for the rational value of α the approximate solutions can be studied and they are fully described by finite sums of the Meijer G-functions provided $\lambda b^{2\alpha}$ is small. We write explicitly the approximate solution including only the first term from the series (34):

$$f_0^k(t) \approx t^{\alpha+k} + \tag{35}$$

$$+ \lambda b^{3\alpha+k} G^{2,1}_{3,3} \left[\begin{array}{c} [\alpha, 2\alpha, 3\alpha + k + 1] \\ [3\alpha + k, \alpha, 0] \end{array} \mid \frac{t}{b} \right].$$

3.2 The case of the order α - an arbitrary rational number

Now let us discuss the case when the order α is a rational number. In this case some of the poles of the Gamma functions in the numerator of the expression $\prod_{l=0}^{m-1} g(s+2l\alpha)$ can coincide for certain values of m. These specific terms cannot be the Mellin transforms of Meijer G-functions but lead to new special functions - Mellin convolutions of the Meijer G-functions. In general to avoid the ambiguity we can split this product into two parts:

$$\prod_{l=0}^{m-1} g(s+2l\alpha) = \tag{36}$$

$$= \prod_{l=0}^{m-1} \frac{\Gamma(1-\alpha-s-l\alpha)}{\Gamma(1-s-l\alpha)} \prod_{l=0}^{m-1} \frac{\Gamma(s+\alpha+l\alpha)}{\Gamma(s+2\alpha+l\alpha)}.$$

Instead of vectors $\vec{a}_m, \vec{b}_m \in R^{2m}$ we shall now consider the following vectors from the space R^m:

$$\vec{a'}_m = [\alpha, 3\alpha, ..., (2m-1)\alpha] \tag{37}$$

$$\vec{b'}_m = [0, 2\alpha, ..., (2m-2)\alpha] \tag{38}$$

$$\vec{a''}_m = [2\alpha, 4\alpha, ..., 2m\alpha] \tag{39}$$

$$\vec{b''}_m = [\alpha, 3\alpha, ..., (2m-1)\alpha]. \tag{40}$$

We clearly see that both products in (36) obey the condition (25). Thus each of them is the Mellin transform of the respective Meijer G-function:

$$\prod_{l=0}^{m-1} \frac{\Gamma(1-\alpha-s-l\alpha)}{\Gamma(1-s-l\alpha)} = \mathcal{G}^{0,m}_{m,m} \left[\begin{array}{c} \vec{a'}_m \\ \vec{b'}_m \end{array} \mid s \right] \tag{41}$$

$$\prod_{l=0}^{m-1} \frac{\Gamma(s+\alpha+l\alpha)}{\Gamma(s+2\alpha+l\alpha)} = \mathcal{G}^{m,0}_{m,m} \left[\begin{array}{c} \vec{a''}_m \\ \vec{b''}_m \end{array} \mid s \right]. \tag{42}$$

We check now whether the inverse Mellin transform for these $\mathcal{G}$ functions can be calculated in the strip $Re(s+\alpha) \in (0,1)$, that means whether the vertical contour $\mathcal{L}_{i\gamma\infty}$ exists [Kilbas, Srivastava and Trujillo, 2006; Kilbas and Saigo, 2004]. We start calculation with parameters of the above $\mathcal{G}$ functions:

$$\Delta' = 0 \quad \delta' = 1 \quad a'^* = 0 \quad \mu' = -m\alpha \tag{43}$$

$$\Delta'' = 0 \quad \delta'' = 1 \quad a''^* = 0 \quad \mu'' = -m\alpha \tag{44}$$

and conclude that the inversion in the vertical strip $Re(s+\alpha) \in (0,1)$ requires the following conditions:

$$\Delta'\gamma + Re(\mu') < -1 \Longleftrightarrow \alpha > 1 \tag{45}$$

$$\Delta''\gamma + Re(\mu'') < -1 \Longleftrightarrow \alpha > 1. \tag{46}$$

The case $\alpha \in (0,1)$ should be studied separately.
Using the inverse Mellin transform we obtain the general continuous solution for the order α rational and $\alpha > 1$ in the form of the following series:

$$f_0(t) = \phi_0(t) + \sum_{m=1}^{\infty} \lambda^m \times \tag{47}$$

$$\times G^{0,m}_{m,m} \left[\begin{array}{c} \vec{a'}_m \\ \vec{b'}_m \end{array} \mid t \right] * G^{m,0}_{m,m} \left[\begin{array}{c} \vec{a''}_m \\ \vec{b''}_m \end{array} \mid t \right] * t^{2m\alpha} \phi_0(t).$$

3.3 Example: solution for $\alpha = \frac{\sqrt{2}}{2}$

Let us write the solutions (29) and (34) for the equation of the order $\alpha = \frac{\sqrt{2}}{2}$. The continuous stationary function has in this case only one component, namely:

$$\phi_0(t) = A_0 t^{\alpha} \Delta H(t).$$

The general solution (29) looks in this case as follows:

$$f_0(t) = A_0 t^{\alpha} + \tag{48}$$

$$+ A_0 \sum_{m=1}^{\infty} \lambda^m G^{m,m}_{2m,2m} \left[\begin{array}{c} \vec{a}_m \\ \vec{b}_m \end{array} \mid t \right] * t^{2m\alpha+\alpha} \Delta H(t)$$

with vectors $\vec{a}_m, \vec{b}_m$ defined by the formulas:

$$\vec{a}_m = [\tfrac{1}{2}, \tfrac{3}{2}, ..., m - \tfrac{1}{2}, 1, 2, ..., m] \cdot \sqrt{2} \tag{49}$$

$$\vec{b}_m = [\tfrac{1}{2}, \tfrac{3}{2}, ..., m - \tfrac{1}{2}, 0, 1, ..., m-1] \cdot \sqrt{2}. \tag{50}$$

Using explicit formula for the Mellin convolutions in (48) we obtain the solution of (16) for the order $\alpha = \frac{\sqrt{2}}{2}$ as the Meijer G-functions series:

$$f_0(t) = A_0 t^{\alpha} + \tag{51}$$

$$+ A_0 b^{\alpha} \sum_{m=1}^{\infty} (\lambda b^{2\alpha})^m G^{m+1,m}_{2m+1,2m+1} \left[\begin{array}{c} \vec{A}_m \\ \vec{B}_m \end{array} \mid \frac{t}{b} \right]$$

with vectors $\vec{A}_m, \vec{B}_m$ given as follows:

$$\vec{A}_m = [\vec{a}_m, (m+\tfrac{1}{2})\sqrt{2}+1] \quad (52)$$
$$\vec{B}_m = [(m+\tfrac{1}{2})\sqrt{2}, \vec{b}_m]. \quad (53)$$

Finally we write the approximate solution for the order $\alpha = \frac{\sqrt{2}}{2}$:

$$f_0(t) \approx A_0 t^{\sqrt{2}/2} + \quad (54)$$

$$+A_0 \lambda b^{3\sqrt{2}/2} G^{2,1}_{3,3}\left[\begin{array}{c} [\frac{\sqrt{2}}{2}, \sqrt{2}, \frac{3\sqrt{2}}{2}+1] \\ [\frac{\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}, 0] \end{array} \mid \frac{t}{b}\right].$$

3.4 Example: solution for $\alpha = \frac{3}{2}$

We add an example with the order $\alpha = \frac{3}{2}$ - being the rational number. The continuous stationary function has now three components:

$$\phi_0(t) = [A_{-1} t^{\frac{1}{2}} + A_0 t^{\frac{3}{2}} + A_1 t^{\frac{5}{2}}]\Delta H(t).$$

The general solution (47) looks on the time interval $[0,b]$ as follows:

$$f_0(t) = A_{-1} t^{\frac{1}{2}} + A_0 t^{\frac{3}{2}} + A_1 t^{\frac{5}{2}} + \sum_{m=1}^{\infty} \lambda^m \times \quad (55)$$

$$\times G^{0,m}_{m,m}\left[\begin{array}{c} \vec{a'}_m \\ \vec{b'}_m \end{array} \mid t\right] * G^{m,0}_{m,m}\left[\begin{array}{c} \vec{a''}_m \\ \vec{b''}_m \end{array} \mid t\right] *$$

$$* t^{3m}(A_{-1} t^{\frac{1}{2}} + A_0 t^{\frac{3}{2}} + A_1 t^{\frac{5}{2}})\Delta H(t)$$

with vectors given by the formulas:

$$\vec{a'}_m = \tfrac{3}{2} \cdot [1, 3, ..., (2m-1)] \quad (56)$$
$$\vec{b'}_m = \tfrac{3}{2} \cdot [0, 2, ..., (2m-2)] \quad (57)$$
$$\vec{a''}_m = \tfrac{3}{2} \cdot [2, 4, ..., 2m] \quad (58)$$
$$\vec{b''}_m = \tfrac{3}{2} \cdot [1, 3, ..., (2m-1)]. \quad (59)$$

Let us assume now that the value of λ is small enough to consider an approximate solution, where only the first term in the series is included. Such an approximation looks as follows:

$$f_0(t) \approx A_{-1} t^{\frac{1}{2}} + A_0 t^{\frac{3}{2}} + A_1 t^{\frac{5}{2}} + \quad (60)$$

$$+\lambda(t-1)^{\frac{1}{2}} * (1-t)^{\frac{1}{2}} t^{\frac{3}{2}} *$$

$$*[A_{-1} t^{\frac{7}{2}} + A_0 t^{\frac{9}{2}} + A_1 t^{\frac{11}{2}}]\Delta H(t).$$

In the above calculations we have applied formulas (2.1.3) and (2.9.6) from [Kilbas and Saigo, 2004].

4 Analytical results for the iterated fractional integrals

We have discussed Meijer G-functions series as solutions of the variational equation for fractional oscillator of the order $\alpha \in (n-1, n)$ - being an arbitrary irrational number fulfilling $\alpha > 1/2$ or respectively a rational number fulfilling $\alpha > 1$. The same problem was solved in [Klimek, 2007; Klimek, 2008b] as a special case using Banach fixed point theorem.
The following series was obtained in this procedure for the fractional oscillator (16) on a finite time interval $[0,b]$ for arbitrary real order $\alpha \in (n-1, n)$:

$$f_0(t) = \phi_0(t) + \sum_{m=1}^{\infty} \lambda^m [I^{\alpha}_{0+} I^{\alpha}_{b-}]^m \phi_0(t). \quad (61)$$

As Banach theorem guarantees that the above solution is unique in $C[0,b]$ we can compare results (29, 34) and (47) with (61) from [Klimek, 2007]. Thus in addition to explicit solution of the simple variational problem we arrive at the new class of analytical properties of fractional integrals.
In the case when $\alpha \in (n-1, n)$ is an irrational number, $\alpha > 1/2$ and $k = -n+1, ..., n-1$ we obtain:

$$[I^{\alpha}_{0+} I^{\alpha}_{b-}]^m t^{\alpha+k} \Delta H(t) = \quad (62)$$

$$= G^{m,m}_{2m,2m}\left[\begin{array}{c} \vec{a}_m \\ \vec{b}_m \end{array} \mid t\right] * t^{(2m+1)\alpha+k} \Delta H(t).$$

After evaluating the Melin convolution we arrive at the interesting compact version of the above formula:

$$[I^{\alpha}_{0+} I^{\alpha}_{b-}]^m t^{\alpha+k} \Delta H(t) = \quad (63)$$

$$= b^{(2m+1)\alpha+k} G^{m+1,m}_{2m+1,2m+1}\left[\begin{array}{c} \vec{A}_{m,k} \\ \vec{B}_{m,k} \end{array} \mid \frac{t}{b}\right].$$

In the case when $\alpha \in (n-1, n)$ is a rational number, $\alpha > 1$ and $k = -n+1, ..., n-1$ we have the relation:

$$[I^{\alpha}_{0+} I^{\alpha}_{b-}]^m t^{\alpha+k} \Delta H(t) = G^{0,m}_{m,m}\left[\begin{array}{c} \vec{a'}_m \\ \vec{b'}_m \end{array} \mid t\right] * \quad (64)$$

$$*G^{m,0}_{m,m}\left[\begin{array}{c}\vec{a''}_m\\ \vec{b''}_m\end{array}\;\middle|\; t\right]*t^{(2m+1)\alpha+k}\Delta H(t).$$

The above formulas for the iterated fractional integrals are valid for $t\in[0,b]$ and can be applied in solving other, more complicated variational fractional problems [Klimek, 2008b].

5 Final remarks

We have presented application of the Mellin transform to the derivation of explicit solutions for the variational fractional oscillator equation. The exact solution appears to be Meijer G-functions series provided the order α is a irrational number greater than 1/2. For α rational and greater than 1 we obtain correct Mellin transform but in general the terms in the series are the Mellin convolutions of respective Meijer G-functions. These functions require further study.

Let us however notice that the results obtained in this simple variational problem can be applied to a wider class of fractional equations with mixed derivatives. We have applied the solutions in an example presenting both exact and approximate results. In addition the obtained form of solution yields a class of exact expressions for the iterated mixed fractional integrals which should provide solutions for more general fractional equations.

References

Agrawal O.P. (2002). *J. Math. Anal. Appl.*, **272**, p. 368.

Agrawal O.P., Tenreiro Machado J.A., Sabatier J., (Eds.) (2004). *Fractional Derivatives and Their Application: Nonlinear Dynamics* **38** Berlin, Springer-Verlag.

Agrawal O.P. (2006). *J. Phys. A*, **39**, p. 10375.

Agrawal O.P. (2007). *J. Phys. A*, **40**, p. 5469.

Baleanu D., Avkar T. (2004). *Nuovo Cimento*, **119**, p. 73.

Baleanu D., Muslih S.I. (2005). *Czech. J. Phys.* , **55**, p. 633.

Baleanu D. (2006). *Signal Processing*, **86**, p. 2632.

Cresson J. (2007). *J. Math. Phys.*, **48**, p. 033504.

Glaeske H.-J., Prudnikov A.P., Skórnik K.A. (2007). *Operational Calculus and Related Topics*, Boca Raton, Chapman & Hall/CRC.

Herrmann R. (2007). *J. Phys. G: Nuc. Phys.*, **34**, p. 607.

Hilfer R., (Ed.) (2000). *Applications of Fractional Calclus in Physics* , Singapore, World Scientific.

Kilbas A.A., Saigo M. (2004). *H-Transforms. Theory and Applications*, Boca Raton, Chapman & Hall/ CRC.

Kilbas A.A., Srivastava H.M., Trujillo J.J. (2006). *Theory and Applications of Fractional Differential Equations* , Amsterdam, Elsevier.

Klimek M. (2001). *Czech. J. Phys.*, **51**, p. 1348.

Klimek M. (2002). *Czech J. Phys.*, **52**, p. 1247.

Klimek M. (2007). ”Solutions of Euler-Lagrange equations in fractional mechanics”. In: AIP Conference Proceedings 956, XXVI Workshop on Geometric Methods in Physics, Białowieża 2007, p. 73.

Klimek M. (2008a). ”G-Meijer functions series as solutions for certain fractional variational problem on a finite time interval”, to appear in Journal Europeén des Systèmes Automatisés. **42**.

Klimek M. (2008b). ”Analytical solutions for a class of Euler-Lagrange fractional equations”. In preparation.

Magin R.L. (2006). *Fractional Calculus in Bioengineering*, Redding, Begell House Publisher.

Metzler R., Klafter J. (2004). *J. Phys A* **37**, R161.

Miller K.S., Ross B. (1993). *An Introduction to the Fractional Calculus and Fractional Differential Equations*, New York, Wiley and Sons.

Podlubny I. (1999). *Fractional Differential Equations*, San Diego, Academic Press.

Riewe F. (1996). *Phys. Rev.*, **E 53**, p. 1890.

Riewe F. (1997). *Phys. Rev.*, **E 55**, p. 3581.

Sabatier J., Agrawal O.P., Tenreiro Machado J.A., (Eds.) (2007). *Advances in Fractional Calculus. Theoretical Developments and Applications in Physics and Engineering*. Berlin, Springer- Verlag.

Samko S.G., Kilbas A.A., Marichev O.I. (1993). *Fractional Integrals and Derivatives*, Amsterdam, Gordon & Breach.

West B.J., Bologna M., Grigolini P. (2003). *Physics of Fractional Operators*, Berlin, Springer-Verlag.