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RESONANCE INSTABILITY OF NONLINEAR VIBRATIONS OF A STRING UNDER HARMONIC HEATING

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Abstract
The existence of parametric resonance and transient phenomena in nonlinear systems under the action of an external force is an important characteristic of dynamical systems. Nonlinear vibrations of a thin stretched string, with an alternating electric current passing through, in a non-uniform magnetic field are described by complicated equations of motion. The general mathematical model involves modes coupling by means of the intrinsic and improper nonlinearities; the string also suffers Joule heating. The purpose of the work is to study the combined effect of the intrinsic (geometrical) nonlinearity and Joule heating on the elastic string oscillation in the frame of a simplified model. We use a combined analytical-numerical approach in studying the dynamics of the proposed model. First, we solve our model analytically by iterations; then we solve it numerically. Both analytical and numerical results show a good agreement almost everywhere but in small intervals near resonant frequencies of different modes. It was found that numerical solutions show instabilities near resonant frequencies in contrast to that of the approximate analytical solutions by iterations. We explain those instabilities using the theory of Mathieu equations.

Key words
Nonlinear string vibration, parametric resonance, instabilities, Mathieu equation.

1 Introduction

Elastic string is a basic and fundamental system in the theory of wave propagation. In particular, a string conducting electric current in a magnetic field shows complex nonlinear oscillations [Kourmychev, 1998]. This system with spatially distributed parameters is described by nonlinear models, even if the amplitude of string vibrations is small. The current carrying string shows nonlinearities of the two types, the intrinsic and the improper. The intrinsic nonlinearity observed in any string is due to the variation of tension caused by elongation of the string at the transverse oscillation [Armstrong, 1982; Elliot, 1980; Tufillaro, 1989]. The improper nonlinearity is a specific characteristic of a current carrying string due to the interaction between electric current and magnetic field. Nonlinearities cause coupling between transverse modes [Kourmychev, 1998].

2 Mathematical Model

The intrinsic nonlinearity in oscillation of the string is the result of alternating increase of string tension due to the elongation of the string. On the other hand, the Joule heating by the electric current causes dilatation of the string, and consequently the alternating decrease of tension. To include the two opposite effects in the model we proceed the following way. At a constant magnetic field the driving force of oscillations is governed by the current, $I(t) = I_0 \cos(\Omega t)$ that causes Joule heating, $Q = I^2 R$. Temperature variation on a string is taken to be, $\Delta T(t) = \Delta T_0 + \delta \cos^2(\Omega t - \varphi)$, φ is a phase shift. After some algebra using the Hooke's law we obtain time variation in the tension of string, $F_T = F_{st} - \lambda\alpha\delta \cos^2(\Omega t - \varphi)$, $F_{st} = F - \lambda\alpha\Delta T_0$. Tension is decreased by the heating of string. Harmonic heating varies the transverse wave velocity, $\left(C_t^2\right)_T = (F/\rho)_T = \left[F_{st} - \lambda\alpha\delta\cos^2(\Omega t - \varphi)\right]/\rho$, longitudinal wave velocity remains the same, $\left(C_l^2\right)_T = (\lambda/\rho)_T \cong \lambda/\rho$, ρ is the linear density of the string. The equation of small amplitude transverse vibrations in a thin string of longitude L in the xz-plane follows from [Watzky, 1992]:

$$\ddot{x} + 2\beta\dot{x} - \left[\breve{C}_t^2 - \frac{\lambda\alpha\delta}{\rho}\cos^2(\Omega t - \varphi)\right]x'' - C_l^2 x'' \frac{1}{2L}\int_0^L (x')^2\, dz = \tilde{f}(z)\cos(\Omega t) \quad (1)$$

where $x(t, z)$ is a transverse displacement of the string in the point z; $\dot{x}$ and x' are the time and z partial derivatives; β is a damping coefficient; Ω is the frequency of external force $\tilde{f}(z)\cos(\Omega t)$, C_t and C_l are the velocities of transverse and longitudinal waves. From (1) we conclude that under the accepted approximations the variation of temperature in an elastic string is manifested through the variation of the transverse wave velocity, linearly; while the elongation of the string during its oscillation is expressed nonlinearly through the integral (geometric, intrinsic nonlinearity).

3 The Effect of Heating on the Vibrations of a String

In order to understand better the influence of heating on the oscillation of the string we separate thermal effect neglecting the geometrical nonlinearity expressed by the integral term in equation (1). After studying the heating effect, we incorporate the geometrical nonlinearity into the equation again. This procedure allows us to use a hybrid analytical-numerical technique for the solution of equation (1), because the linear equation with the only thermal effect can be solved analytically by iterations. Iterative procedure differs of the standard multiple scales expansion in perturbation method that was used by [Kidachi and Onogi, 1997] in studying of stability of Mathieu equation. So, neglecting the proper nonlinearity in (1), we obtain the equation

$$\ddot{x}+2\beta\dot{x}-\left[\breve{C}_t^2-\frac{\lambda\alpha\delta}{\rho}\cos^2(\Omega t-\varphi)\right]x''=\tilde{f}(z)\cos(\Omega t) \quad (2)$$

that describes the effect of heating on a string vibration. Separating variables, equation (2) is reduced to the equation for normal modes, $X_n(z) = \sin(k_n z)$. The equation for the temporal part of inhomogeneous equation (2), corresponding to the n-th normal mode, is as follows

$$\frac{d^2T_n}{dt^2}+2\beta\frac{dT_n}{dt}+\left[\breve{C}_t^2-\frac{\lambda\alpha\delta}{\rho}\frac{1+\cos(2\Omega t-2\varphi)}{2}\right]k_n^2T_n=f_n\cos(\Omega t) \quad (3)$$

where $\tilde{f}(z) = \Sigma_n f_n X_n(z)$. Figure 1 shows the numerical solution of (3) for $n = 1$ and different frequencies of driving force. The solution is diverged in the small vicinity of resonant frequency.

Solution by iterations. Equation (3) can be solved by iterations [Kourmychev, 2003]. The approximate solution at the i-th iteration is the solution of (4) for $n = 1, 2, ...$

$$\frac{d^2T_n^{(i)}}{dt^2}+2\beta\frac{dT_n^{(i)}}{dt}+\left[\breve{C}_t^2-\frac{\lambda\alpha\delta}{2\rho}\right]k_n^2T_n^{(i)}= f_n\cos(\Omega t)+\frac{\lambda\alpha\delta}{\rho}\frac{\cos(2\Omega t-2\varphi)}{2}k_n^2T_n^{(i-1)} \quad (4)$$

First iteration analytical solution of (4) is given by the following expression:

$$\begin{aligned}T_n^{(1)}(t) = {} & e^{-\beta t}a_0\cos(\hat{\omega}t-\delta_0)+\\ & e^{-\beta t}a_3\cos[(2\Omega+\tilde{\omega}_n)t-\theta_1-\delta_3]+\\ & e^{-\beta t}a_4\cos[(2\Omega-\tilde{\omega}_n)t-\theta_2-\delta_4]+\\ & a_1\cos(\Omega t-\chi_1-\delta_1)+\\ & a_2\cos(3\Omega t-2\varphi-\chi-\delta_2)\end{aligned} \quad (5)$$

This solution, being plotted for different frequencies of driving force, shows the behavior similar to that of

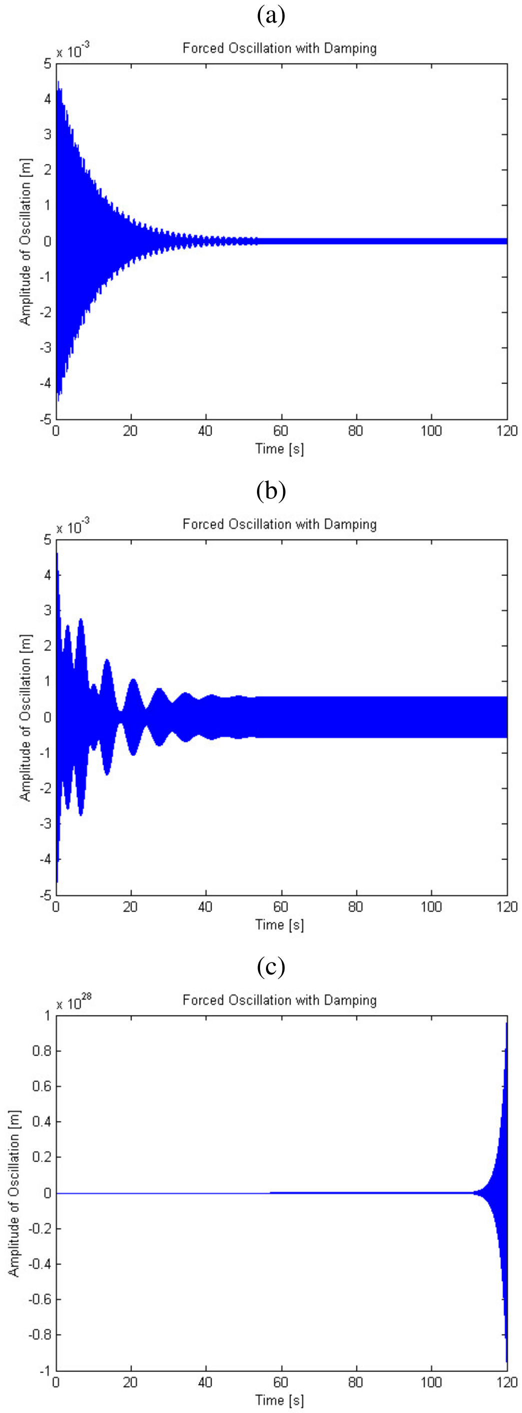

Figure 1. Numerical solution of the nonlinear inhomogeneous equation (3), for the first normal mode $n = 1$ at $\Delta T_0 = 5^\circ C$, $\delta = 0.5^\circ C$, $\varphi = \pi/48$ and $f_n = 0.1$: (a) $\Omega = 30\ Hz$, (b) $\Omega = 30.5\ Hz$, (c) $\Omega = \Omega_r$

figure 1 for $\Omega = 30\ Hz$ and $\Omega = 30.5\ Hz$, but it does not diverge in the vicinity of $\Omega_r = 30.67\ Hz$. When the frequency of driving force is close to the resonant frequency of the m-th normal mode, then the first order approximate solution of equation (3) is

$$x(z,t)=\sum_{i=1}^{\infty}\sin(k_iz)\cdot T_i^{(1)}(t)\ \approx\ \sin(k_mz)\cdot T_m^{(1)}(t) \quad (6)$$

4 Intrinsic Nonlinearity and the Heating of a String

To see the combined effect of intrinsic nonlinearity and the heating, we use the approximate analytical solution $x(z,t) = T_m^{(1)}(t)\sin(k_m z)$, equation (6), to evaluate the integral term of equation (1). Substituting $x' = T_m^{(1)}(t) k_m \cos(k_m z)$ into the integral term of equation (1), integrating with respect to z and separating variables in normal modes $X_n(z) = \sin(k_n z)$ and harmonics $T_n(t)$, we get the equation

$$\frac{d^2 T_{nm}}{dt^2} + 2\beta \frac{dT_{nm}}{dt} + \left\{ \breve{C}_t^2 - \frac{\lambda\alpha\delta}{\rho}\cos^2(\Omega t - \varphi) + C_l^2 \frac{k_m^3}{4}\left[T_m^{(1)}(t)\right]^2 \right\} k_n^2 T_{nm} = f_n \cos(\Omega t) \qquad (7)$$

where the second subindex m in $T_{nm}(t)$ shows the dependence of n-th harmonic on the harmonic m, $T_m^{(1)}(t)$ that was chosen as the resonant mode. Equation (7) is the linearization of equation (1), which describes driven oscillations of a string subject to heating and intrinsic nonlinearity. Plots of numerical solutions of (7) at three frequencies of driven force are presented in figure 2; in small vicinity of resonant frequency of m-th harmonic, the huge (unphysical) increase of amplitude is observed, similar to that of the model without heating, equation (3).

5 Stability of Oscillations

To explain the origin of divergence in solutions of equations (3) and (7), see figure 1(c) and 2(c), let us consider the homogeneous equation corresponding to equation (3). Substitutions $T_n(t) = e^{-\beta t}u(t)$, $z = t - \varphi$ transform the homogeneous equation into the Mathieu equation,

$$\tilde{u}''(z) + [a - 2q\cos(2z)]\,\tilde{u}(z) = 0 \qquad (8)$$

where $a = \hat{a}/\Omega^2$, $q = \hat{q}/\Omega^2$, $\hat{a} = k_n^2\left(c_t^2 - \lambda\alpha\delta/2\rho\right) - \beta^2$ and $\hat{q} = k_n^2\lambda\alpha\delta/4\rho$; a and q are the characteristic parameters which determine the properties of the system. Figure 3 shows the regions of stability and instability of Mathieu functions. According to [McLachlan, 1947], the solution of (8) is stable if the parametric point (q,a) is in the region between the curves a_m and b_{m+1}; it is unstable if (q,a) is between the curves b_m and a_m; it shows pulsations when the parametric point (q,a) is on the one of the curves separating stable from unstable regions. Parameters of the problem are: $\beta = 0.1$, $\alpha = 17 \cdot 10^{-6}\ 1/{}^\circ C$, $\lambda = 2{,}545\ N$, $\rho = 2 \cdot 10^{-4}\ kg/m$, $F = 0.98\ N$, $L = 1\ m$, $\Delta T_0 = 5, 10\ {}^\circ C$, $\delta = 0.5, 1\ {}^\circ C$. For normal modes $n = 1,2,3$ studied in this work, all the parameters given above are fixed, except the frequency Ω of the external force, which was varied in an interval close to the resonant frequency of each normal mode. As an example, we present the analysis of stability in the case of normal mode $n = 1$ and $\Delta T_0 = 5\ {}^\circ C$. In case of other modes and/or $\Delta T_0 = 10\ {}^\circ C$ the analysis is similar. At $\Omega = 30\,Hz$ the parametric point **A** in figure 4, is located between the characteristic curves a_1 and b_2; in

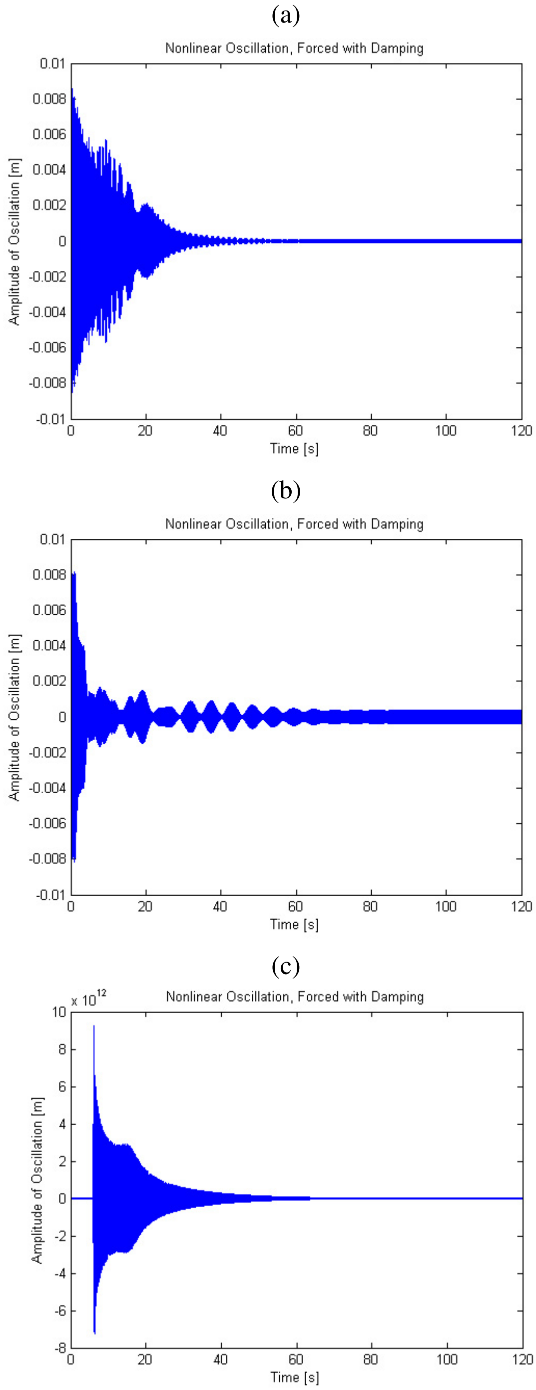

Figure 2. Numerical solution of the linearized Eq. (7), for the first normal mode $n = 1$ and first iteration $T_1^{(1)}(t)$ at $\Delta T_0 = 5°C$, $\delta = 0.5°C$, $\varphi = \pi/48$ and $f_n = 0.1$: (a) $\Omega = 30\,Hz$, (b) $\Omega = 30.5\,Hz$, (c) $\Omega = \Omega_r$

this case the solution is a stable periodic Mathieu function. Similar behavior is observed for the point **D** at $\Omega = 30.7\,Hz$. At $\Omega = 30.5\,Hz$ the point **B** in figure 4 is localized near the curve that separates the regions of stability and instability. So either numerical or approximate analytical solution shows pulsations (see figures 1(b) and 2(b)). When $\Omega = \Omega_r = 30.67\,Hz$, the point **C** in figure 4 is localized in the region of instability between the curves b_1 and a_1, see figure 1(c). Analytical solution, equation (5) shows notable increase of ampli-

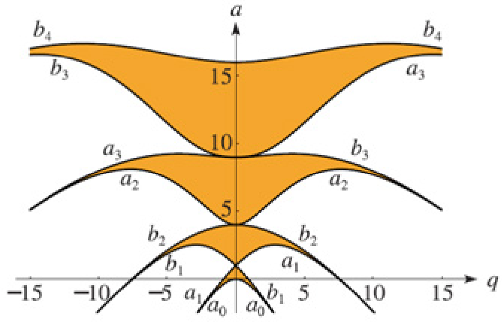

Figure 3. Stability regions for the Mathieu functions, reproduced from [NIST, 2010]

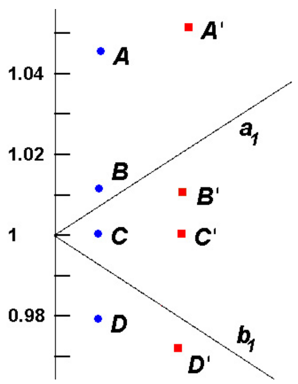

Figure 4. Position of the parametric point (q, a) for the first normal mode, $n = 1$, equation (8). Points **A**, **B**, **C** and **D** correspond to the $\Delta T_0 = 5°C$, and the points **A'**, **B'**, **C'** y **D'** corresponds to the $\Delta T_0 = 10°C$.

tude but not the divergence at $\Omega \approx \Omega_r$.

6 Conclusions

A model of oscillations that includes both the heating and intrinsic nonlinear effects in vibrating string was proposed. A combined analytical-numerical technique was used to study the dynamics of oscillations. The comparing of numerical and approximate analytical solutions permitted us to establish the range of validity for the approximate analytical solutions. Parametric resonance in vibrating string subject to the harmonic heating was found to cause the divergence of solutions in an interval of frequencies close to the resonant one; this was not observed in the case of approximate analytical solutions. The observed instability of solutions was found to be of the Mathieu type. We established the range of parameters at which the solutions are diverged; the range of near resonant frequencies where solutions are unstable, increases with the increase of heating. Because heating and intrinsic nonlinearity are opposite acting effects, it was expected that the intrinsic nonlinearity would suppress the divergence caused by the parametric resonance. This was not confirmed in the linearized version of equation (1): numerical solution of (7) also shows the divergence at the frequencies nearby the resonant one of each normal mode. In the frame of the proposed model, the intrinsic nonlinearity is not able to suppress the divergence of solutions caused by harmonic heating at frequencies close to the resonant. Modified model is in the process.

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