SCENARIOS OF TRANSITION TO CHAOS IN VIBRO-IMPACT SYSTEMS

Sergey Kryzhevich
Faculty of mathematics and mechanics
Saint-Petersburg State University
Russia
kryzhevitz@rambler.ru, kryzhevich@hotmail.com

Abstract
We consider the dynamical systems, described by a system of ordinary differential equations and impact conditions of impulse type. Two mechanisms of formation of chaotic invariant sets are being studied. Some popular numerical methods are justified. We present some conditions, sufficient for a chaotic behavior of the vibro-impact system and give sketches of proofs.

Key words
Chaos, hyperbolicity, Smale horseshoe, impact, grazing, chatter.

1 Introduction

The vibro-impact systems appear in different mechanical problems (modeling of clock mechanisms, immersion of constructions, etc.). All the impact systems are strongly nonlinear. Their properties resemble a lot ones of classical nonlinear systems. Particularly, the chaotic dynamics is possible.

There is a big number of publications, devoted to bifurcations, proper to vibro-impact systems. It was noted (see the list of references) that the chaotic regimes may appear for different particular cases. For example, the dynamics of a ball, jumping over a harmonically oscillating surface has been studied in the paper [Holmes, 1982]. These oscillations were modeled by the following discrete system. Let t_k be the moments of impacts, y_k be the corresponding velocities and T be the period of surface oscillations. The considered dynamical system was described by a particulary continuous mapping which transfers the couple $(y_k, \tau_k = t_k \mod T)$, corresponding to the impact number k to one of the impact number $k+1$. It was shown that if the period T is big enough, the considered dynamical system has a hyperbolic invariant set, similar to the ”Smale horseshoe”.

One of the mechanisms of chaos formation called grazing has been introduced by Nordmark [Nordmark, 1991] (see also [Budd, 1995] and [Ivanov, 1996]).

The s.d.f. motion of a point mass under the action of the restoring force, external force, friction and elastic impacts has been studied in the paper [Kryzhevich and Pliss, 2004]. It was supposed that the period of the external force is big. The conditions sufficient for chaotic dynamics have been given. Also, the opposite case of the unique and stable periodic solution has been studied. The systems with nonelastic impact were studied in the paper [Gorbikov and Men'shenina, 2007]. These systems may have solutions with infinitely many impacts over a finite period of time (so-called chatter). It was shown that in the neighborhood of chatter the invariant sets, similar to the ”Smale horseshoe” (but maybe not hyperbolic) may appear.

2 The mathematical model of vibro-impact system.

Consider the mechanical system with several degrees of freedom, given on free flight intervals by following equations

$$\begin{gathered}\dot{x}_k = y_k; \quad \dot{y}_k = F_k(t, x_1, y_1, \dots, x_n, y_n, \mu), \\ k = 1, \dots, n.\end{gathered} \tag{2.1}$$

Here F is a C^2 – smooth function, defined on the set

$$\mathbb{R} \times [0, +\infty) \times \mathbb{R}^{2n-2} \times [0, \mu^*].$$

Suppose it is T – periodic to the respect of the variable t. Introduce the notations

$$\begin{gathered} x = (x_1, \dots, x_n) = (x_1, \bar{x}), \\ y = (y_1, \dots, y_n) = (y_1, \bar{y}), \\ z = (x, y), \quad \bar{z} = (\bar{x}, \bar{y}) \\ F(t, z, \mu) = (F_1(t, z, \mu), \dots, F_n(t, z, \mu)), \\ F^0(t, \bar{z}, \mu) = (F_1^0(t, \bar{z}, \mu), \dots, F_n^0(t, \bar{z}, \mu)) = F(t, 0, 0, \bar{z}, \mu) \end{gathered}$$

Suppose that the considered system satisfies the following impact conditions.

1. If $x_1(t_0) = 0$ then $y_1(t_0 + 0) = -r_1 y_1(t_0 - 0)$, $\bar{y}(t_0 + 0) = \bar{r}\bar{y}(t_0 - 0)$ Here

$$r_1 = r_1(\bar{x}(t_0), y(t_0 - 0), \mu) \in (0, 1]$$

and $\bar{r} = \bar{r}(\bar{x}(t_0), y(t_0 - 0), \mu) \in (0, 1]$ are C^2 – smooth functions of their arguments.
2. Let a solution $z(t)$ of the vibro-impact system is such that

$$x_1(t_0) = y_1(t_0) = 0,$$

$\bar{z}(t_0) = \zeta_0$. Denote by $\zeta(t)$ the solution of the system

$$\dot{x}_k = y_k; \quad \dot{y}_k = F_k(t, 0, 0, x_2, y_2, \ldots, x_n, y_n, \mu),$$

with the initial data $\bar{z}(t_0) = \zeta_0$. Suppose there is such $t_1 > t_0$, that $F_1(t, 0, 0, \zeta(t), \mu) \leqslant 0$ for all $t \in [t_0, t_1]$, then $x_1(t) = y_1(t) = 0$ and $\bar{z}(t) = \zeta(t)$ for all $t \in [t_0, t_1]$.

Denote the obtained vibro-impact system by (A).

It is natural from the physical point of wiev to assume that $r_1(\bar{x}, y, \mu)$ as well as $r_1(\bar{x}, y, \mu)$

This type of impact dynamics has been studied by different authors (see the reference list). The results on main properties like existence, uniqueness, boundness of solutions can be found in [Babitskiy, 1998] and [Schatzmann, 1998]. The description of bifurcations, proper to impact systems can be found in the articles [Fredriksson, Nordmark, 1997] and [Ivanov, 1996]. Different results on chaotic dynamics, arising in impact systems have been established in the referred papers (of course, the list is not complete). Here we consider the so-called Devaney chaos [Devaney, 1987].

Definition. Let S be a diffeomorphism of a Euclidean space or of a smooth manifold. We say, the hyperbolic invariant set K is chaotic if

1. periodic points of S are dense in K;
2. K is transitive, i.e. there is a point p, whose orbit $\{S^n(p) : n \in \mathbb{Z}\}$ is dense in K.

3 Grazing bifurcation.

Condition 3.1. Suppose there exists a continuous family of T – periodic solutions of the system (A)

$$\phi(t, \mu) = (\phi_{x1}(t, \mu), \phi_{y1}(t, \mu), \ldots, \phi_{xn}(t, \mu), \phi_{yn}(t, \mu).$$

1. If $\mu > 0$, the normal component $\phi_{x1}(t)$ has exactly $N + 1$ zeros

$$\tau_0(\mu) < \ldots < \tau_N(\mu)$$

over the period, such that all impact instants $\tau_j(\mu)$ and velocities

$$Y_j = -\phi_{y1}(\tau_j(\mu) - 0, \mu)$$

are C^2 – smooth functions of μ.
2. The velocities $Y_j(\mu)$ are such that $Y_j > 0$ if $\mu > 0$ or $j > 0$ and $Y_0(0) = 0$.

Without loss of generalty suppose that $\tau_0(\mu)^{\cdot} = 0$ for all μ. Let $\theta > 0$ be a small parameter. Consider the shift mapping $S_{\mu,\theta}$ defined by the formula $S_{\mu,\theta}(z_0) = z(T_\theta + 0, -\theta, z_0, \mu)$ and the matrix

$$A = \lim_{\mu,\theta \to 0+} \frac{\partial z}{\partial z_0}(T - \theta, \theta, z_0, \mu).$$

Let a_{ij} $(i, j \in \{1, \ldots, 2n\}$ be the elements of the matrix A and α_{ij} be ones of the matrix A^{-1}). Suppose $\sigma_j = 1$ if j is odd and $\sigma_j = r(\bar{\phi}_x(0-), \phi_y(0-), 0)$ if j is even. Consider the $2n-2 \times 2n-2$ matrices $\bar{A} = (\bar{a}_{ij})$ and $\widetilde{A} = (\widetilde{a}_{ij})$ defined by formulae $\bar{a}_{ij} = a_{i+2,j+2}\sigma_j$, $\widetilde{a}_{ij} = \alpha_{i+2,j+2}\sigma_j^{-1}$.

Condition 3.2. Suppose that either

1. $a_{12} > 0$, $\sum_{k=1}^{2n} a_{1k}a_{k2} < 0$ and $n = 1$ or all eigenvalues of the matrix $\bar{A}$ are out of the unit circle or
2. $\alpha_{12} > 0$, $\sum_{k=1}^{2n} \alpha_{1k}\alpha_{k2} < 0$ and $n = 1$ or all eigenvalues of the matrix $\widetilde{A}$ are out of the unit circle.

Theorem 3.1. *Let the considered system satisfy the conditions 3.1 and 3.2. Then there exist such positive values μ_0 and θ that for all $\mu \in [0, \mu_0]$, there exists a chaotic invariant set $K_{\mu,\theta}$ of the mapping $S_{\mu,\theta}$.*

4 Lienard equation with a long period right hand side.

Consider an s.d.f. mechanical system, described by the equation

$$\ddot{x} + p(x)\dot{x} + q(x) = f(t, T), \quad x \geqslant 0, \qquad (4.1)$$

provided $p, q \in C^2([0, +\infty) \to [0, +\infty))$. We suppose that $f(t, T) = f_0(t/T)$ is a C^2 – smooth periodic function of the period T, which is supposed to be a big parameter. Assume that a basic function $f_0(t)$ has exactly two zeros $t = 0$ and $t = \tau_1$ over the period $[0, 1)$ and, moreover, $f_0'(0) > 0$, $f_0'(\tau_1) < 0$. Let $M_f = \max |f(t)|$. Suppose, there is such $x_m > 0$, that $q(x_m) = M_f$ and

$$p(x) > 0, \qquad q'(x) > p^2(x)/4 \qquad \forall x \in [0, x_m]. \qquad (4.2)$$

Assume that the elastic impact conditions take place.

1. If $x(t_0) = 0$ then $\dot{x}(t_0 + 0) = -\dot{x}(t_0 - 0)$.

2. If $x(t_0) = \dot{x}(t_0 - 0) = 0$ and there exists such $t_1 > t_0$ that $f(t) \leqslant 0$ for all $t \in [t_0, t_1]$, then $x(t) \equiv 0$ for all $t \in [t_0, t_1]$.

Consider the shift mapping

$$S(x_0, y_0) = (x(T + 0, 0, x_0, y_0), \dot{x}(T + 0, 0, x_0, y_0)).$$

Theorem 5.1. *Let the coefficients of the equation (4.1) satisfy the conditions (4.2) and the mentioned assumptions on the right hand side f. Then there exists such $\bar{T} > 0$ that for all $T > \bar{T}$ there is a chaotic invariant set of the shift mapping S.*

Acknowledgements

This work was partially supported by the Russian Fund of Basic Researches (08-01-00346-a), by the Grant Council of the President of Russia (MK-4032.2009.1), by the Program of support of the leading scientific schools (954.2008.1) and by the fund of Vladimir Potanin.

References

Babitsky V.I. *Theory of Vibro-Impact Systems and Applications*, Springer, Berlin, 1998.

Banerjee S., Yorke J.A., Grebogi C. (1998) Robust chaos. *Phys. Rew. Letters*. **80**, no.14, 3049–3052.

Budd C. (1995) Grazing in impact oscillators. Branner B. and Hjorth P. *Real and Complex Dynamical Systems*. Kluwer Academic Publishers, 47–64.

Chin W., Ott E., Nusse H.E., Grebogi C. (1995). Universal behavior of impact oscillators near grazing incidence. *Phys. Let. A.* **201**, 197–204.

Devaney R.L. (1987). *An Introduction to Chaotic Dynamical Systems.* Redwood City, CA: Addison-Wesley.

Fredriksson M.H., Nordmark A.B. (1997). Bifurcations caused by grazing incidence in many degrees of freedom impact oscillators. *Proc. Roy. Soc. London Ser. A.* **453**, 1261–1276.

Gorbikov S.P. Men'shenina A.V. (2007). Statistical description of the limiting set for chaotic motion of the vibro-impact system. *Automation and remote control,* **68**, no. 10, 1794–1800.

Holmes P.J. (1982). The dynamics of repeated impacts with a sinusoidally vibrating table. *J. Sound. Vib.* **84**, 173–189.

Ivanov A.P. (1996). Bifurcations in impact systems. *Chaos, Solitons and Fractals,* **7**, 1615–1634.

Kozlov V.V., Treschev D.V. (1991). *Billiards. A genetic introduction to the Dynamics of Systems with Impacts.* Translations of mathematical Monographs, vol. 89. Providence, RI: Amer. Math. Soc.

Kryzhevich S.G. (2006). Properties of solutions of Duffing type equations with impact conditions. *Electronic journal Differential equations and processes of control,* no.2, 1–25.

Kryzhevich S.G., Pliss V.A. (2005). Chaotic modes of oscillations of a vibro-impact system. *J. Appl. Math. Mech.* **69**, no. 1, 15–29.

Kryzhevich S.G. (2008) Grazing bifurcation and chaotic oscillations of single-degree-of-freedom dynamical systems. *J. Appl. Math. Mech.* **72**, no. 4, 539–556.

Molenaar J, van de Water W., de Wegerand J. (2000). Grazing impact oscillations. *Phys. Rev. E*, **62**, no. 2, 2030–2041.

Nordmark A.B. (1991) Non-periodic motion caused by grazing incidence in an impact oscillator. *J. Sound Vib.*, **145**, no 2., 279–297.

Pavlovskaia E. ,Wiercigroch M. (2004). Analytical drift reconstruction for visco-elastic impact oscillators operating in periodic and chaotic regimes. *Chaos, Solitons and Fractals* **19**, 151–161.

Schatzman M. (1998). Uniqueness and continuous dependence on data for one-dimensional impact problem. *Math. Comput. Modelling.* **28**, no. 4–8, 1–18.

Thomson J.M.T., Ghaffari R. (1983) Chaotic dynamics of an impact oscillator. *Phys. Rev. A*, 27, no. 3, 1741–1743.

Whiston G.S. Global dynamics of a vibro-impacting linear oscillator. *J. Sound Vib.* **118**, 395–429.