CONTROL OF A CART WITH A DISSIPATIVE OSCILLATOR

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Abstract

A control problem for a system, consisting of a rigid body with a viscoelastic link is considered. Such a system is modelled as a cart with a linear dissipative oscillator attached to it. The cart moves along a horizontal line under the action of a control force and unknown disturbance, for example, dry friction, the parameters of which are unknown and impermanent. The phase state of the oscillator is assumed to be not available for measuring. A bounded feedback control which brings the cart to a prescribed terminal state in a finite time is proposed.

Key words

Linear controllable system, observability, disturbance, feedback control

1 Introduction

We consider a control problem for a system representing a simplified model of a precision platform carrying a viscoelastic link or a vessel with a viscous liquid. Precise positioning of the platform is hampered by dry friction acting between the platform and the surface along which it moves, as well as disturbance from the viscoelastic link. The parameters of friction are unknown beforehand and may change in the process of motion. The current state of the viscoelastic link is not available for measuring. A control algorithm which stops the platform in a prescribed terminal position in a finite time is proposed. The state of the viscoelastic link at the final moment is unimportant.

The proposed algorithm consists of two stages. At the first stage, using the available information on the motion of the platform, we restore unknown phase variables characterizing the dynamics of the viscoelastic link, and estimate the error in calculating these variables. For the control law applied at the first stage, this error is not significant, while the system is far from the terminal state.

At the second stage, when the energy of viscoelastic link, as well as the perturbations caused by the oscillations of the link are sufficiently small, a control law is applied, which depends only on the coordinates and velocities of the platform.

At both stages, for constructing the control we use the approach proposed in [Ovseevich, 2015] and developed in [Anan'evskii, 2016].

2 Statement of the Problem

Consider a cart with mass m_0 moving along a straight line on a rough horizontal surface with linear dissipative oscillator attached to it. Oscillator is assumed to be horizontally oscillating particle with mass m_1, connected to the cart via spring with stiffness κ. The cart is acted by a control force u_0 and unknown disturbance v_0.

The system is governed by the equations

$$\begin{aligned} m_0\ddot{\xi} &= \kappa\varphi + \gamma\dot{\varphi} + u_0 + v_0, \\ m_1(\ddot{\xi} + \ddot{\varphi}) &= -\kappa\varphi - \gamma\dot{\varphi}, \end{aligned} \quad (1)$$

where ξ describes the position of the cart on the horizontal line, φ is the elongation of the spring of the oscillator, and $\gamma > 0$ is a coefficient of viscous friction.

The control force u_0 is bounded and exceeds the disturbance v_0, that is,

$$|u_0| \le U_0, \quad U_0 > 0; \quad (2)$$

$$|v_0| \le \rho U_0, \quad 0 < \rho < 1. \quad (3)$$

We assume that variables $\xi, \dot{\xi}$ that describe the phase state of the cart are known at every instant of time, while the phase coordinates and velocities $\varphi, \dot{\varphi}$ of the oscillator are not available for measuring.

The problem is to design a feedback control law that brings the cart to the origin in a finite time. The state of the oscillator is unimportant at the instant when the cart reaches the origin.

3 The System in Canonical Form

Following [Ovseevich, 2015], we transform system (1) to canonical form. First, we introduce the dimensionless time

$$\tau = \frac{\kappa t}{\gamma}, \quad \gamma \neq 0, \tag{4}$$

and denote

$$\begin{gathered}\bar{\xi} = \frac{\kappa^2 m_0}{\gamma^2 U_0}\xi, \quad \bar{\varphi} = \frac{\kappa}{U_0}\varphi, \quad u_1 = \frac{u_0}{U_0}, \\ v_1 = \frac{v_0}{U_0}, \quad a = \frac{\gamma^2}{\kappa m}, \quad b = \frac{\gamma^2}{\kappa m_0}.\end{gathered} \tag{5}$$

Substituting these expressions into (1) and using the dots to designate derivatives with respect to the dimensionless time give

$$\begin{aligned}\ddot{\bar{\xi}} &= \bar{\varphi} + \dot{\bar{\varphi}} + u_1 + v_1, \\ \ddot{\bar{\varphi}} &= -a(\bar{\varphi} + \dot{\bar{\varphi}}) - b(u_1 + v_1), \\ |u_1| &\leq 1, \quad |v_1| \leq \rho.\end{aligned} \tag{6}$$

We turn to the vector variables $x \in R^4$:

$$x_1 = \bar{\xi}, \quad x_2 = \dot{\bar{\xi}}, \quad x_3 = \bar{\varphi}, \quad x_4 = \dot{\bar{\varphi}}, \tag{7}$$

to obtain

$$\dot{x} = A_0 x + B_0(u_1 + v_1). \tag{8}$$

Here

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -a & -a \end{pmatrix}, \; B_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -b \end{pmatrix}. \tag{9}$$

The controllability matrix $F = [B_0 \; A_0B_0 \; A_0^2B_0 \; A_0^3B_0]$ is nondegenerate because $\det F = b^2(a-b)^2$ and $a-b, b > 0$. Therefore, when $v_1 \equiv 0$, the Kalman controllability condition is met.

A measured output $y \in R^2$ is given by

$$y = C_0 x, \quad C_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{10}$$

When $v_1 \equiv 0$, system (8) is also observable because the observability matrix $[C_0^\top \; A_0^\top C_0^\top \; {A_0^\top}^2 C_0^\top \; {A_0^\top}^3 C_0^\top]$ has full rank 4.

We denote $c = (a-b)^{-1}$ and introduce the new variables $z = Sx, \; z \in R^4$, where

$$S = \begin{pmatrix} 0 & 0 & 0 & -1/b \\ 0 & 0 & 1/b & 0 \\ 0 & 2c & 2/b & 2c/b \\ -6c & 6c & 6(a-b-1)c/b & 6c/b \end{pmatrix}. \tag{11}$$

In the variables z system (8) takes the form

$$\begin{aligned}\dot{z} &= Az + B(u + v_1), \\ y &= Cz,\end{aligned} \tag{12}$$

with

$$\begin{gathered}A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\ C = C_0 S^{-1} = \begin{pmatrix} 0 & -c & 1/2 & -1/6 \\ c & -1 & 1/2 & 0 \end{pmatrix},\end{gathered} \tag{13}$$

and

$$u(z) = u_1(z) - a(z_1 + z_2). \tag{14}$$

4 Phase State Estimation

In this section, using the measured output, we find approximately the vector of the current phase state of system (12). Denote by $Z(t)$ the fundamental matrix of (12)

$$Z(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -t & 1 & 0 & 0 \\ t^2 & -2t & 1 & 0 \\ -t^3 & 3t^2 & -3t & 1 \end{pmatrix}. \tag{15}$$

Then, the solution of (12) with the initial state z^0 is

$$z(t) = Z(t)\left(z^0 + \int_0^t Z^{-1}(\tau)B(u(\tau) + v_1(\tau))d\tau\right) \tag{16}$$

and, using the notation $H(t) = CZ(t)$, the measured output $y(t), \; t \in [0, t_1]$, can be written as follows

$$y(t) = H(t)\left(z^0 + \int_0^t Z^{-1}(\tau)B(u(\tau) + v_1(\tau))d\tau\right). \tag{17}$$

Let us find the expected initial state $z^0_{t_1}$ with the same output, but in case when there are no disturbance v. We have

$$H(t)z^0_{t_1} = y(t) - CZ(t)\int_0^t Z^{-1}(\tau)Bu(\tau)d\tau. \quad (18)$$

Multiplying by $H^\top(t)$ from the left and integrating over the interval $[0, t_1]$, we obtain

$$\hat{H}(t_1)z^0_{t_1} = \int_0^{t_1} H^\top(t)y(t)dt - \int_0^{t_1} H^\top(t)H(t)\int_0^t Z^{-1}(\tau)Bu(\tau)d\tau dt, \quad (19)$$

where

$$\hat{H}(t_1) = \int_0^{t_1} H^\top(t)H(t)dt. \quad (20)$$

The difference between the true initial state z^0 and expected one $z^0_{t_1}$ is

$$\hat{H}(t_1)(z^0_{t_1} - z^0) = -\int_0^{t_1} H^\top(t)H(t)\int_0^t Z^{-1}(\tau)Bv_1(\tau)d\tau dt. \quad (21)$$

For system (12), for small t_1, the following approximation is valid

$$z^0_{t_1} - z^0 \approx \rho(p_0 + t_1 p_1), \quad p_0 = \begin{pmatrix} 1 \\ 1-c \\ 2-4c \\ 6-18c+6c^2 \end{pmatrix}, \quad p_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ (1-c) \\ 3(1-2c) \end{pmatrix}. \quad (22)$$

Assuming no disturbance, we expect the following state of the system at the time instant t_1:

$$\hat{z}(t_1) = Z(t_1)\left(z^0_{t_1} + \int_0^{t_1} Z^{-1}(\tau)Bu(\tau)d\tau\right). \quad (23)$$

Thus, due to the disturbance v_1, the accumulated error in the determination of the state of system (12) at the instant t_1 equals

$$z(t_1, z^0_{t_1}) - z(t_1, z^0) = Z(t_1)\left(z^0_{t_1} - z^0 - \int_0^{t_1} Z^{-1}(\tau)Bv_1(\tau)d\tau\right), \quad (24)$$

and the following asymptotic estimate holds

$$\lim_{t_1 \to o} \|z(t_1, z^0_{t_1}) - z(t_1, z^0)\| = \rho\|p_0\|. \quad (25)$$

5 Control Algorithm at the First Stage

Let $z(t)$ be the true current state vector of the system, $\hat{z}(t)$ be its estimate, and

$$\triangle z(t) = \hat{z}(t) - z(t). \quad (26)$$

Substituting $\hat{z}$ into control function (14) gives

$$u_1(z + \triangle z) = u(z + \triangle z) + a(z_1 + z_2) + a(\triangle z_1 + \triangle z_2). \quad (27)$$

Now, system (12) becomes

$$\dot{z} = Az + B(u(z) + v), \quad (28)$$

where

$$v = v_1 + v_2 + v_3, \quad v_2 = u(z + \triangle z) - u(z), \quad v_3 = a(\triangle z_1 + \triangle z_2). \quad (29)$$

Below, following [Ovseevich, 2015], we describe a bounded feedback control, which brings system (28) to zero in the case when the entire vector of phase variables z is known. To this end, we introduce the following scalar function $T(z)$, the diagonal matrices $\delta(T)$ and M, the vector f, and the positive definite matrix Q:

$$\begin{aligned} &\delta(T) = \mathrm{diag}(T^{-1}, T^{-2}, T^{-3}, T^{-4}), \\ &M = \mathrm{diag}(-1, -2, -3, -4), \\ &f^\top = (-10, 90, -210, 140), \\ &Q = \begin{pmatrix} 20 & -180 & 420 & -280 \\ -180 & 2220 & -5880 & 4200 \\ 420 & -5880 & 16800 & -12600 \\ -280 & 4200 & -12600 & 9800 \end{pmatrix}. \end{aligned} \quad (30)$$

We compose the matrix A_3 by filling the top row of the matrix A with the components of the vector f

$$A_3 = \begin{pmatrix} -10 & 90 & -210 & 140 \\ -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix}. \quad (31)$$

The following property holds:

$$QA_3 + A_3^\top Q = QM + MQ = P < 0, \quad (32)$$

where P is the negative definite constant matrix

$$P = \begin{pmatrix} -40 & 540 & -1680 & 1400 \\ 540 & -8880 & 29400 & -25200 \\ -1680 & 29400 & -100800 & 88200 \\ 14000 & -25200 & 88200 & -78400 \end{pmatrix}. \quad (33)$$

We define the function $T(z)$ implicitly by the equation

$$(Q\delta(T)z, \delta(T)z) = 1/5, \quad z \neq 0, \tag{34}$$

(from now on, $(\cdot,\cdot)$ means a scalar product). As it is determined in [Ovseevich, 2015], equation (34) has only one positive solution for T in the whole phase space $z \in R^4$ except zero. This solution is given by an analytic function. Moreover, the function $T(z)$ can be defined at zero as $T(0) = 0$, which preserves the continuity of it.

We designate the feedback law

$$u(z) = (f, \delta(T)z), \quad z \neq 0. \tag{35}$$

The coefficients of the feedback control function (35) at the phase variables z increase infinitely as z tends to zero. Nevertheless, control (35) meets the constraint $|u(z)| \leq 1$.

Denote $q = \delta(T(z)z), \quad q \in R^4$. Then system (28) becomes

$$\dot{q} = T^{-1}\left(A_3 q + Bv + M\dot{T}q\right). \tag{36}$$

Differentiating the function T by virtue of (28) gives

$$\dot{T} = -\frac{(Pq,q) + 2v(QB,q)}{(Pq,q)}. \tag{37}$$

Theorem 1. *There exists $\rho_1 > 0$ such that if*

$$|v| \leq \rho_1, \tag{38}$$

then the derivative of the function T by virtue of (28) meets the inequality $\dot{T} < -\sigma,\ \sigma > 0$.

Substituting $\hat{z}$ instead of z in expression (35), we come to system (28) where the function v is given by (29).

Theorem 2. *For a given $\delta > 0$ there exists such ρ, introduced in (3), that outside the neighborhood*

$$G = \{z \in R^4\colon\ \|z\| < \delta\} \tag{39}$$

inequality (38) holds.

It follows from Theorem 1, that every trajectory of (28) reaches the neighborhood G in a finite time.

6 Control Algorithm at the Second Stage

At the second stage, when the system moves within the neighborhood G, we consider the first equation of system (1) separately:

$$m_0\ddot{\xi} = u_0 + F, \quad F = \kappa\varphi + \gamma\dot{\varphi} + v_0. \tag{40}$$

Now F is treated as an uncertain disturbance.

Theorem 3. *The number δ in (39) can be chosen so that, in the neighborhood G, the following inequality holds:*

$$|F| \leq \frac{3-\sqrt{3}}{6}U_0. \tag{41}$$

In the neighborhood G we use the control function

$$u_0(\xi,\dot{\xi}) = -\frac{6m_0\xi}{T_1^2(\xi,\dot{\xi})} - \frac{3m_0\dot{\xi}}{T_1(\xi,\dot{\xi})}. \tag{42}$$

Here, the function $T_1(\xi,\dot{\xi})$ is implicitly defined by the equation

$$dT_1^4 - 6\dot{\xi}^2T_1^2 - 24\xi\dot{\xi}T_1 - 36\xi^2 = 0, \quad d > 0. \tag{43}$$

Similarly to the function T, the function T_1 is analytic and positive in R^2 except zero, and can be defined at zero as $T(0,0) = 0$, which preserves the continuity of it.

As it is shown in [Anan'evskii, 2016], the derivative of T_1 according to equation (40), under condition (41), meets the inequality $\dot{T}_1 < -\sigma_1,\ \sigma_1 > 0$.

Thus, the function T_1 vanishes to zero in a finite time, i.e. every trajectory of equation (40) reaches the origin of the phase space $\xi, \dot{\xi}$ in a finite time. This means that the cart will be stopped in the origin by control (42).

7 Conclusion

The proposed approach is effective due to the fact that, at the first stage, far from the terminal state, the feedback control used is insensitive to inaccurate knowledge of current phase variables. At the second stage, in the vicinity of the terminal state, where the vibrations of the viscoelastic link are small, the control copes with perturbations caused by these vibrations.

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