NECESSARY OPTIMALITY CONDITION FOR OPTIMAL IMPULSIVE CONTROL PROBLEM FOR ONE CLASS OF LAGRANGIAN SYSTEM

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Abstract
The report is devoted to the impulsive control optimization problem, when the object of control is described by the Lagrangian equation of the second kind. We consider a class of the Lagrangian equations of the second kind when the right part does not contain operations of multiplication of discontinuous functions and distributions. In this case necessary conditions of an optimality in the form of a maximum principle are received.

Key words
Impulsive control, Lagrangian systems, maximum principle.

1 Introduction

Currently, necessary optimality condition for optimal control problem was got, when the control object is describe by nonlinear equations with linearly dependent control [Bressan, A., Rampazzo, F.,1991; Bressan, A., Rampazzo, F., 1994; Dykhta, V.A., Sumsonuk O.N., 1997; Miller, B.M., Rubinovich, E.Y.,2002]. Unfortunately, these conditions gets bulky and inconvenient for application. In particular, it is the reason for that in the right part of the systems is an operation of multiplication of discontinuous functions and distributions [Schwartz, L.,1950-1951]. There are many classes of controlled systems in which right part incorrect operation is not present. In this case necessary optimality conditions are turn out compact and convenient for application. The report is devoted to the impulsive control optimization problem, when the object of control is described by the Lagrangian equation of the second kind, when the right part does not contain operations of multiplication of discontinuous functions and distributions.

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2 Statement of the problem

We shall consider the object of control, which will be described by the Lagrangian equation of the second kind, which after the solving with respect to the second derivative of the vector x will be in the form

$$\ddot{x} = f(t,x) + B(t,x)\,v(t). \qquad (1)$$

Here, x and v are respectively n- and m-vector functions of time $(m \leq n)$, $t \in [t_0, \vartheta]$, $f(t,x)$ is an $n-$ dimensional vector function defined on $[t_0,\vartheta] \times R^n$, $B(t,x)$ is $n \times m$ matrix function. Assume that $f(\cdot,\cdot)$ and $B(\cdot,\cdot)$ are continuous in the totality of variables and Lipschitz continuous in x on the set $\{t \in [t_0,\vartheta],\ ||x|| < \infty\}$, where $||x|| = (\sum_{i=1}^{n} |x_i|^2)^{1/2}$, and satisfy the following standard conditions in the same set:

$$||f(t,x)|| \leq \kappa(1 + ||x||), \ \ ||B(t,x)|| \leq \kappa(1 + ||x||),$$

where κ is some positive constant.

As a admissible impulsive control we shall take an control of the type

$$v(t) = u(t) + \sum_{i=1}^{k} C_i \delta(t - \Theta_i), \qquad (2)$$

where $u(t)$ - is a sectionally continuous vector function, C_i - constant vectors are the impulse function intensity, Θ_i - is the moments of application of impulses from the interval $[t_0,\vartheta]$.

Suppose $u(t) \in U$ at any $t_0 \leq t \leq \vartheta$, and $C_i \in Q$, where U and Q - are some compact sets, Θ_i - is the unfixed moments of applications of impulses.

At the monographs [Zavalishchin, S.T., Sesekin, A.N.,1991; Zavalishchin, S.T., Sesekin, A.N..,1997] defines the notion of the approximable solution of a differential equation with the affine right site as the pointwise limit of the twice differentiable solutions of the equations (1) $x_k(t)$, generated by the smooth approximations $v_k(t)$ of the distribution [Schwartz, L.,1950-1951] $v(t)$ of type (2), the primitives of which converge pointwise to primitives of the distribution $v(t)$, if this x does not depend on the choice of $v_k(t)$. The right part of equation (1) does not contain operations of multiplication of discontinuous functions and distributions. Therefore, the approximable solution of equation (1) will be the solution of the respective integral representation of equation (1).

Problem 1. The functional

$$J[v] = g(x(\vartheta)) + \int\limits_{t_0}^{\vartheta} f_0(s, x(s)\dot{x}(s))\, ds, \qquad (3)$$

should be minimised along the trajectories of the system of differential equations of system (1) by miens of the control $v(t)$ from the described class.

3 Necessary condition of optimality

Let us introduce the Pontryagin function for the problem 1

$$H(t, x, \dot{x}, \psi, \dot{\psi}, v) = (\frac{\partial f_0(t, x, \dot{x}}{\partial \dot{x}} - \dot{\psi})^T \cdot \dot{x}$$

$$+\psi^T \cdot f(t, x) + \psi^T \cdot B(t, x) \cdot v - f_0(t, x, \dot{x}). \qquad (4)$$

The conjugate system for the system (1) will look like

$$\ddot{\psi} = \psi^T \frac{\partial f(t, x)}{\partial x} + \psi^T \frac{\partial}{\partial x}(B(t, x) \cdot v) - \frac{\partial f_0(t, x, \dot{x})}{\partial x}$$

$$+\frac{\partial^2 f_0(t, x, \dot{x})}{\partial t \partial \dot{x}} + \frac{\partial^2 f_0(t, x, \dot{x})}{\partial x \partial \dot{x}} \cdot \dot{x}$$

$$+\frac{\partial^2 f_0(t, x, \dot{x})}{\partial \dot{x}^2} (f(t, x) + B(t, x)\, v(t)) \qquad (5)$$

The boundary conditions for system (5) will be

$$\Big\{ \psi(\vartheta) = 0; \dot{\psi}(\vartheta) = \frac{\partial q(x(\vartheta))}{\partial x}.$$

Then the following will be true:

Theorem 1. Suppose optimal control in the problem 1 takes the following form:

$$v^* = u^*(t) + \sum_{i=0}^{k} C_i^* \delta(t - \Theta_i^*). \qquad (6)$$

Let $x_*(t)$ and $\psi_*(t)$ are the corresponding solution and dual curves, solutions to systems (1) and (5). Then the following conditions will be true:

a) for all $t \in [t_0, \vartheta]$ except for the points of discontinuity $u^*(t)$ and the moments Θ_i^*

$$\psi_*^T \cdot B(t, x_*) \cdot u^*(t) = \max_{u \in U} \psi_*^T \cdot B(t, x_*) \cdot u(t); \quad (7)$$

b) in every point Θ_i^*

$$\psi_*^T(\Theta_i^*) \cdot B(\Theta_i^*, x_*(\Theta_i^*)) = 0; \qquad (8)$$

c) the function $\psi_*^T(t) \cdot B(t, x_*(t))$ is continuous on the interval $[t_0, \vartheta]$ and differentiable everywhere, except for the moments Θ_i^*;

d) the function

$$H_0(t, x, \dot{x}, \psi, \dot{\psi})$$

$$= (\frac{\partial f_0(t, x, \dot{x}}{\partial \dot{x}} - \dot{\psi})^T \cdot \dot{x} + \psi^T \cdot f(t, x) - f_0(t, x, \dot{x}) \quad (9)$$

is continuous at (t_0, ϑ) and differentiable throughout it, except for points Θ_i^*, that

$$\frac{\partial}{\partial t} H_0(t, x_*(t), \dot{x}_*(t), \psi_*(t), \dot{\psi}_*(t))$$

$$= \frac{d}{dt} H_0(t, x_*(t), \dot{x}_*(t), \psi_*(t), \dot{\psi}_*(t)).$$

The latter means in particular that if the problem is autonomous, then

$$H_0(t, x_*(t), \dot{x}_*(t), \psi_*(t), \dot{\psi}_*(t)) = const$$

at (t_0, ϑ).

We shall make some remarks to the theorem 1.

1. The condition (7) allows to find a regular component of the optimum control $v^*(t)$.

2. If $Q = R^m$ then the condition (7) turns to

$$\psi_\alpha^T B(t,x) = 0. \tag{10}$$

3. In case conditions $x(\vartheta) = x_\vartheta$ and $\dot{x}(\vartheta) = \dot{x_\vartheta}$ are set for the right end of the trajectory, no restriction is placed on the conjugate variable.

4. The moments of the action of impulsive control are defined from a condition (8). Unfortunately, the maximum principle does not contain the obvious information on sizes of intensity of impulses. These sizes it is necessary to receive from consequences of conditions a) and b). Here we have analogy with the singular control in the classical theory of optimal control.

Solutions of the impulsive optimal control problem of Lagrangian systems wich was stem of digitization at the time series was considered in [Yunt, K., 2007].

4 Example

Assume that control object is described by equation

$$\ddot{x} = \upsilon; \tag{11}$$

The problem is to minimize functional

$$J[\upsilon] = \int_0^1 (x^2 + k\dot{x}^2)dt$$

along system trajectories (11) starting at the point

$$x(0) = x_0, \dot{x(0)} = \dot{x_0}$$

and to arrive at zero point at the end point.

In this case adjoined equation is given by:

$$\ddot{\psi} = -x + k\upsilon; \tag{12}$$

Theorem 1 is applied to the problem. In this example no restrictions are placed on control variable υ, i.e. $U = Q = R^1$. Hence, in our case condition (7) will also take the form (10): $\psi(t) \equiv 0$ with $t \in (0,1)$. Then $\ddot{\psi}(t) = 0$ according to (12) condition $-x + k\upsilon = 0$ must also be true for the regular part of optimal control, i.e.

$$\upsilon = \frac{1}{k}x. \tag{13}$$

It is easy to show that the trajectories of the system (11) with control (13) will be given by hyperbola curves (the graph which you can see at Figure 1):

$$x^2 - k\dot{x^2} = C. \tag{14}$$

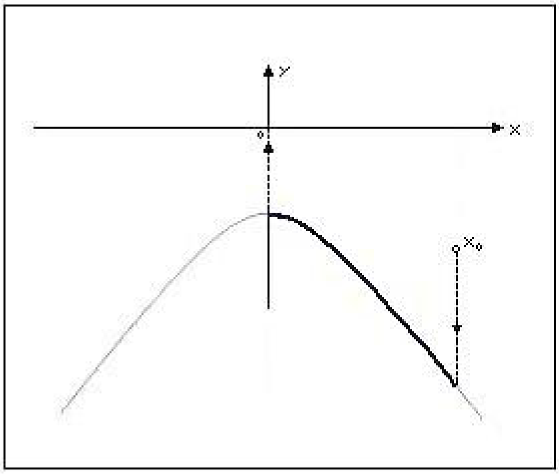

Figure 1.

According to the condition d), function $H_0(t,x,\dot{x},\psi,\dot{\psi})$ must be continuous at $(0,1)$. In this example function must be continuous at $(0,1)$ and

$$H_0(t,x,\dot{x},\psi,\dot{\psi}) = -\frac{1}{2}(x^2 + k\dot{x^2}).$$

Therefore, optimal control has no impulse components on $(0,1)$, which are only possible with $t = 0$ and $t = 1$. As the point must arrive at zero at the end point and function $x(t)$ is continuous on $(0,1)$, $x(1) = 0$ and $\dot{x}$ coordinate is jumps to zero by impulse.

Finally, we can define the value of initial jump by which a phase point arrives at one of hyperbola curves (14). By substituting equation (13) to equation (11) and solving it, we obtain the following:

$$x(t) = C_1 e^{\frac{1}{\sqrt{k}}t} + C_2 e^{-\frac{1}{\sqrt{k}}t}. \tag{15}$$

$$\dot{x}(t) = \frac{1}{\sqrt{k}} C_1 e^{\frac{1}{\sqrt{k}}t} - \frac{1}{\sqrt{k}} C_2 e^{-\frac{1}{\sqrt{k}}t}. \tag{16}$$

It agrees (15) $x(1) = 0$, then

$$C_1 e^{\frac{1}{\sqrt{k}}} + C_2 e^{-\frac{1}{\sqrt{k}}} = 0. \tag{17}$$

Except for that $x(0) = x_0$. Therefore

$$C_1 + C_2 = x_0 \tag{18}$$

From system (17), (18) we find, that

$$C_1 = -\frac{x_0 e^{-\frac{1}{\sqrt{k}}}}{e^{\frac{1}{\sqrt{k}}} - e^{-\frac{1}{\sqrt{k}}}};\ C_2 = \frac{x_0 e^{\frac{1}{\sqrt{k}}}}{e^{\frac{1}{\sqrt{k}}} - e^{-\frac{1}{\sqrt{k}}}}.$$

Therefore it agrees (16)

$$\dot{x}(0+) = -\frac{x_0(e^{\frac{1}{\sqrt{k}}} + e^{-\frac{1}{\sqrt{k}}}}{\sqrt{k}(e^{\frac{1}{\sqrt{k}}} - e^{-\frac{1}{\sqrt{k}}})}$$

With the last formula we can find initial jump value

$$C_0^* = \dot{x}(0+) - \dot{x_0} = -\frac{x_0(e^{\frac{1}{\sqrt{k}}} + e^{-\frac{1}{\sqrt{k}}})}{\sqrt{k}(e^{\frac{1}{\sqrt{k}}} - e^{-\frac{1}{\sqrt{k}}})} - \dot{x_0}.$$

Initial jump which ensures arrival at zero point is given by equation:

$$C_1^* = -\dot{x}(1-0).$$

Thus, extreme control will take the form $\upsilon^* = C_0^*\delta(t) + C_1^*\delta(t-1) + \frac{1}{k}x$. On account of the linear nature of system (11) and convexity of functional $J[\upsilon]$, we can state that obtained extreme control is optimal.

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