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DESIGN OF ACTIVE SUSPENSION CONTROL SYSTEM WITH THE USE OF KALMAN FILTER-BASED DISTURBANCES ESTIMATOR

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Abstract

A solution to the problem of active control and disturbances compensation in vehicles suspensions is proposed. It is shown that the suspension model satisfies differential flatness properties and the associated flat output is a weighted sum of the system's state vector elements. Differential flatness of the suspension's model enables transformation into a linear canonical form for which it is possible to design a state feedback controller. Kalman filtering is used for estimating the non-measurable elements of the suspension's transformed state vector through the processing of measurements provided by a small number of on-board sensors. Moreover, by reformulating the Kalman Filter as a disturbance observer it is possible to simultaneously estimate the external disturbances and the system's transformed state vector. The inclusion of an additional control term based on the disturbances estimation enables to compensate for the disturbances' effects and to attenuate vibrations. The performance of the proposed Kalman Filter-based active suspension control scheme has been tested through numerical simulation experiments.

Key words

Vehicle suspensions, active control, flatness-based control, canonical form, disturbance observer, Kalman Filtering, disturbances compensation.

1 Introduction

In the recent years there has been systematic effort towards designing vehicles of improved safety and comfort and to this end the development of active suspension control systems has been an important research topic. One can note several results about active suspension control systems exhibiting robustness to external disturbance forces and being capable of efficiently suppressing the vibrations induced to the vehicle by these disturbances. H_∞ controllers have been developed taking into account worst case disturbances on the suspension models [Yamashita et al., 1994; Du and Zhang, 2007; Gao *et al.*, 2010]. Moreover, there have been results on operating the suspension's control loop under limited information provided by a small number of on-board sensors. This can be seen in the case of developing some type of state estimator or statistical filter to approximate the nonmeasurable elements of the suspension's state vector and the unknown disturbance forces. Particularly, one can note the use of Linear Quadratic Gaussian (LQG) control where Kalman Filtering is combined with an optimal controller [Marzbanrada *et al.*, 2004; Harrison, 1994; Hrovat, 1990; Lee and Kim, 2010; He and McPhee, 2005]. Moreover in [Hsiao *et al.*, 2011] the application of a sliding-mode controller together with Kalman Filtering has been proposed for implementing state estimation-based control of the suspension's model. Additionally, disturbance observers have been used for simultaneous estimation of the suspension's state vector and of the unknown external disturbances. The suitability of disturbance observers for vibration control problems has been shown in [Bagordo *et al.*, 2011], while the efficiency of disturbance estimators in vehicle control loops and especially in the suspension control problem has been demonstrated in [Beltran-Carbajal *et al.*, 2011; Sira-Ramirez *et al.*, 2011; Koch *et al.*, 2010a; Koch *et al.*, 2010b; Delvecchio *et al.*, 2010; Delvecchio *et al.*, 2011]. Finally, a scheme of distributed Kalman Filtering has been applied to disturbances and state vector estimation for the suspension's mechanism in [Lee *et al.*, 2012].

In this research work an approach to solve the problem of active control of vehicle suspensions is developed with the use of flatness-based controller and a Kalman Filter-based disturbances estimator. The suitability of the Kalman Filter for state estimation in dynamic systems exhibiting vibrations has been shown in [Rigatos, 2012a]. In this paper, dynamic analysis for the vehicle's suspension model is first provided. Active vehicle suspension control systems are underactuated and the efficient suppression of disturbance inputs (e. g. due to rough road surface) is important for attaining the performance objectives of the control loop. The elements of the state vector are variables denoting the displacement of the sprung and unsprung masses from their zero position and variables denoting the linear velocities of these masses. The control inputs to the model are the force generated by the actuator placed between the two masses (which aims at the suppression of vibrations) and the unknown disturbance force that is generated due to contact of the tire with the road surface. The model can take the form of a linear state space equation. Moreover, by assuming nonlinearities in the spring and damper terms of the suspension a nonlinear dynamical model is obtained.

Next, it is shown how a controller for the aforementioned suspension model can be obtained through the application of the differential flatness theory [Fliess and Mounier, 1999; Rudolph, 2003; Villagra et al., 2007; Tang *et al.*, 2011]. The flat output for the suspension's model is a scalar variable which is equal to the weighted sum of the elements of the suspensions state vector. By expressing all state variables and the control input of the suspension model as functions of the flat output and its derivatives the system's dynamic model is transformed into the linear Brunovksy (canonical) form [Sekhavat *et al.*, 2001; Lu *et al.*, 2008]. For the latter model it is possible to design a state feedback controller that enables accurate tracking of the vehicle's velocity set-points. However, since measurements are available only for certain elements of the transformed state vector, to implement a state feedback control loop the rest of the elements of the suspension's transformed state vector have to be estimated with the use of an observer or filter. To this end the concept of *Derivative-free nonlinear Kalman Filtering* is introduced. By avoiding linearization approximations, the proposed filtering method improves the accuracy of estimation of the system's state variables [Rigatos, 2008; Rigatos, 2010; Rigatos, 2011].

A particular difficulty, in the case of the suspension model state estimation is the existence of the unmodeled disturbance forces. It is shown that it is possible to redesign the Kalman Filter in the form of a disturbance observer and using the estimation of the disturbance to develop an auxiliary control input that compensates for the disturbances effects [Cortesao *et al.*, 2005; Cortesao, 2006; Chen *et al.*, 2000; Gupta and Malley, 2011; Miklosovic *et al.*, 2006]. In this way the suspension's control system can become robust with respect to uncertainties in the model's parameters or uncertainties about external forces. It is also noted that in terms of computation speed the proposed Kalman Filter-based disturbance estimator is faster than disturbance estimators that may be based on other nonlinear filtering approaches (e. g. Extended Kalman Filter, Unscented Kalman Filter or Particle Filter) thus becoming advantageous for the real-time estimation of the unknown suspension dynamics. The efficiency of the proposed control and Kalman Filter-based disturbances estimation scheme is evaluated through simulation tests. It has been shown that the accurate estimation of the disturbance forces which are exerted on the suspension enables their efficient compensation. This is succeeded by introducing an additional element in the controller that produces a counter-disturbance input based on the estimated value for the disturbance variable. This control scheme finally results in minimizing the effects of the disturbances on the vehicle's parts.

The structure of the paper is as follows: in Section 2 the dynamic model of the vehicle's suspension is analyzed and state space representation is provided for both the linear and the nonlinear case. In Section 3 it is proven that the vehicle suspension model is differentially flat and this property is used to write the model into a linear canonical form. Based on this latter transformation an active suspension control system is designed. In Section 4 the Kalman Filter is introduced as an estimation approach suitable for reconstructing the suspension's transformed state vector using measurements from a small number of sensors. In Section 5 the problem of state estimation under disturbances is analyzed. The concept of the disturbance observer is explained. In Section 6 it is shown how the Kalman Filter can be redesigned in the form of a disturbance observer so as to succeed simultaneous estimation of the suspension's transformed state vector and estimation of the unknown disturbance forces exerted on the suspension's mechanism. In Section 7 evaluation tests are provided about the suspension's control scheme and about the performance of the state estimator that aims at real-time identification of uncertainty and disturbances in the suspension's dynamics. Finally, in Section 8 concluding remarks are provided.

2 Dynamic Model of Vehicle Suspension

2.1 Dynamics of the 2-DOF Suspension

The suspension system is depicted in Fig. 1 and its dynamics is written as

$$\begin{array}{c} m_1\ddot{x}_1 + c_1(\dot{x}_1 - \dot{x}_2) + k_1(x_1 - x_2) = f \\ m_2\ddot{x}_2 + c_1(\dot{x}_2 - \dot{x}_1) + k_1(x_2 - x_1) + \\ +c_2(\dot{x}_2 - \dot{\zeta}) + k_2(x_2 - \zeta) = -f \end{array} \quad (1)$$

Variable x_1 denotes the sprung mass displacement while variable x_2 denotes the unsprung mass displacement. Tyre's deflection ζ and its time derivatives

$\zeta^{(i)}$. $i = 1, 2, \cdots$ represent unknown external disturbance inputs due to road surface roughness and are assumed to be bounded. The mass that needs regulation is the sprung mass m_1 which is also considered to be larger than m_2. The control force f is generated by an actuator placed between the two masses (see Fig. 1).

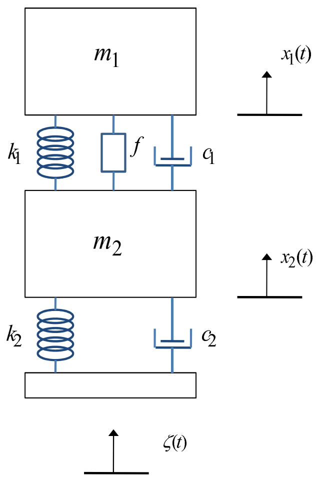


Figure 1. An active vehicle suspension system where both the sprung and the unsprung mass are connected to a spring and a damper.

A normalization is performed to the model through the following procedure: (i) the normalized time is defined as $\tau = t\sqrt{\frac{k_1}{m_1}}$, (ii) the normalized input force is $u = \frac{f}{k_1}$. The system constant coefficients are redefined as $\epsilon = \frac{m_2}{m_1}$, $\gamma_1 = \frac{c_1}{m_1}\sqrt{\frac{m_1}{k_1}}$, $\gamma_2 = \frac{c_2}{m_1}\sqrt{\frac{m_1}{k_1}}$, $\kappa = \frac{k_2}{k_1}$. Thus, the dynamics model of the vehicle's suspension can be rewritten as

$$\begin{array}{c} \ddot{x}_1 + \gamma_1(\dot{x}_1 - \dot{x}_2) + (x_1 - x_2) = u \\ \epsilon\ddot{x}_2 + \gamma_1(\dot{x}_2 - \dot{x}_1) + (x_2 - x_1) + \\ +\gamma_2(\dot{x}_2 - \dot{\zeta}) + \kappa(x_2 - \zeta) = -u \\ y = x_1 \end{array} \quad (2)$$

The model of Eq. (2) can be also written is state-space form after defining the state variables $z_1 = x_1$, $z_2 = \dot{x}_1$, $z_3 = x_2$, $z_4 = \dot{x}_2$. Thus one has

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -\gamma_1 & 1 & \gamma_1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{\epsilon} & \frac{\gamma_1}{\epsilon} & -\frac{1+\kappa}{\epsilon} & -\frac{\gamma_1+\gamma_2}{\epsilon} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} + \\ + \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{1}{\epsilon} \end{pmatrix} u + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{\kappa}{\epsilon} & \frac{\gamma_2}{\epsilon} \end{pmatrix} \begin{pmatrix} \zeta \\ \dot{\zeta} \end{pmatrix} \quad (3)$$

where all terms associated with disturbance ζ can be represented by the new variable Δ. Thus one obtains

$$\frac{d}{d\tau}x_s = Ax_s + Bu + \Delta. \quad (4)$$

2.2 A Nonlinear Model of Vehicle Suspension Dynamics

The dynamical model of the two-degrees of freedom vehicle suspension (see Fig. 2) is given as follows [Sira-Ramirez *et al.*, 2011]

$$\begin{array}{c} m_s\ddot{z}_s + F_{s_c} + F_{s_k} = F_A \\ m_u\ddot{z}_u - F_{s_k} - F_{s_c} + k_t(z_u - z_r) = -F_A \end{array} \quad (5)$$

where F_A is the force generated by the actuator, F_{s_k} is the force associated with the suspension's spring term, F_{s_c} is the force associated with the suspension's damper term and $F_t = k_t(z_u - z_r)$ is a spring force associated with elasticity coefficient k_t and denoting the spring-type behavior of the wheel when in contact with the road's surface.

It holds that

$$F_{s_k}(z_s, z_u) = k_s(z_s - z_u) + k_{n_s}(z_s - z_u)^3 \quad (6)$$

$$\begin{array}{c} F_{s_c}(z_s, z_u) = b_s(\dot{z}_s - \dot{z}_u) + \\ +b_{ns}(\dot{z}_s - \dot{z}_u)^2 sgn(\dot{z}_s - \dot{z}_u) \end{array} \quad (7)$$

Denoting the state variables $x_1 = z_s$, $x_2 = \dot{z}_s$, $x_3 = z_u$, $x_4 = \dot{z}_u$ one has

$$\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{1}{m_s}[k_s(x_1 - x_3) + k_{n_s}(x_1 - x_3)^3 + \\ +b_s(x_2 - x_4) + b_{ns}(x_2 - x_4)^2 sgn(x_2 - x_4)] + \frac{1}{m_s}u \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = -\frac{k_t}{m_u}x_3 + \frac{1}{m_u}[k_s(x_1 - x_3) + \\ +k_{n_s}(x_1 - x_3)^3 + b_s(x_2 - x_4) + \\ +b_{ns}(x_2 - x_4)^2 sgn(x_2 - x_4)] - \frac{1}{m_u}u + \frac{k_t}{m_u}z_r \end{array} \quad (8)$$

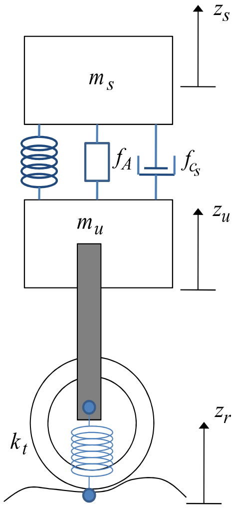


Figure 2. An active suspension system.

where the term $\frac{1}{m_u}z_r$ can be considered as a disturbance term. Denoting the nonlinear functions

$$f_1(x,t) = -\frac{1}{m_s}[k_s(x_1-x_3) + k_{n_s}(x_1-x_3)^3 + \\ +b_s(x_2-x_4) + b_{ns}(x_2-x_4)^2 sgn(x_2-x_4)] \tag{9}$$

$$g_1(x,t) = \frac{1}{m_s} \tag{10}$$

$$f_2(x,t) = -\frac{k_t}{m_u}x_3 + \frac{1}{m_u}[k_s(x_1-x_3) + k_{n_s}(x_1-x_3)^3 +$$

$$b_s(x_2-x_4) + b_{ns}(x_2-x_4)^2 sgn(x_2-x_4)] \tag{11}$$

$$g_2(x,t) = -\frac{1}{m_u} \tag{12}$$

one has the following state-space description

$$\begin{array}{c} \dot{x}_1 = x_2 \\ \dot{x}_2 = f_1(x,t) + g_1(x,t)u \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = f_2(x,t) + g_2(x,t)u \end{array} \tag{13}$$

3 Flatness-Based Control for a Suspension Model
3.1 Differential Flatness Theory

Differential flatness theory can be applied to the generic class of systems $\dot{x} = f(x,u)$. In this study, the interest is in dynamic models of the form of Eq. (14).

$$\dot{x} = f(x,t) + g(x,t)u \tag{14}$$

The principles of differential flatness theory have been extensively studied in the relevant bibliography [Rudolph, 2003; Rigatos, 2010; Villagra et al., 2007]: A finite dimensional system is considered. This can be written in the form of an ordinary differential equation (ODE), i. e. $S_i(w, \dot{w}, \ddot{w}, \cdots, w^{(i)})$, $i = 1, 2, \cdots, q$. The term w denotes the system variables (these variables are for instance the elements of the system's state vector and the control input) while $w^{(i)}$, $i = 1, 2, \cdots, q$ are the associated derivatives. Such a system is said to be differentially flat if there exists a set of m functions $y = (y_1, \cdots, y_m)$ of the system variables and of their time-derivatives, i.e. $y_i = \phi(w, \dot{w}, \ddot{w}, \cdots, w^{(\alpha_i)})$, $i = 1, \cdots, m$ satisfying the following two conditions [Fliess and Mounier, 1999; Rigatos, 2008]:

1. There does not exist any differential relation of the form $R(y, \dot{y}, \cdots, y^{(\beta)}) = 0$ which implies that the derivatives of the flat output are not coupled in the sense of an ODE, or equivalently it can be said that the flat output is differentially independent.

2. All system variables (i. e. the elements of the system's state vector w and the control input) can be expressed using only the flat output y and its time derivatives $w_i = \psi_i(y, \dot{y}, \cdots, y^{(\gamma_i)})$, $i = 1, \cdots, s$. An equivalent definition of differentially flat systems is as follows:

Definition: The system $\dot{x} = f(x,u)$, $x \in R^n$, $u \in R^m$ is differentially flat if there exist relations

$$\begin{array}{l} h:\ R^n \times (R^m)^{r+1} \to R^m, \\ \phi:\ (R^m)^r \to R^n \text{ and} \\ \psi:\ (R^m)^{r+1} \to R^m \end{array} \tag{15}$$

such that

$$\begin{array}{l} y = h(x, u, \dot{u}, \cdots, u^{(r)}), \\ x = \phi(y, \dot{y}, \cdots, y^{(r-1)}), \text{ and} \\ u = \psi(y, \dot{y}, \cdots, y^{(r-1)}, y^{(r)}). \end{array} \tag{16}$$

This means that all system dynamics can be expressed as a function of the flat output and its derivatives, therefore the state vector and the control input can be written as

$$\begin{array}{l} x(t) = \phi(y(t), \dot{y}(t), \cdots, y^{(r)}(t)), \text{and} \\ u(t) = \psi(y(t), \dot{y}(t), \cdots, y^{(r+1)}(t)) \end{array} \tag{17}$$

3.2 Classes of Differentially Flat Systems

For certain classes of dynamical systems it has been proven that they satisfy differential flatness properties. The following classes of nonlinear differentially flat systems are defined [Martin and Rouchon, 1999; Boudouen *et al.*, 2011]:

1. Affine in-the-input systems: The dynamics of such systems is given by:

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i. \quad (18)$$

From Eq. (18) it can be concluded that the above state equation can also describe MIMO dynamical systems. Without loss of generality it is assumed that $G = [g_1, \cdots, g_m]$ is of rank m. In case that the flat outputs of the aforementioned system are only functions of states x, then this class of dynamical systems is called 0-flat. It has been proven that a dynamical affine system with n states and $n-1$ inputs is 0-flat if it is controllable.

2. Driftless systems: These are systems of the form

$$\dot{x} = \sum_{i=1}^{m} f_i(x)u_i \quad (19)$$

For driftless systems with two inputs, i. e.

$$\dot{x} = f_1(x)u_1 + f_2(x)u_2 \quad (20)$$

the flatness property holds, if and only if the rank of matrix $E_{k+1} := \{E_k, [E_k, E_k]\},\ k \geq 0$ with $E_0 := \{f_1, f_2\}$ is equal to $k+2$ for $k = 0, \cdots, n-2$. It has been proven that a driftless system that is differentially flat, is also 0-flat.

Moreover, for flat systems with n states and $n-2$ control inputs, i. e.

$$\dot{x} = \sum_{i=1}^{n-2} u_i f_i(x)\ x \in R^n \quad (21)$$

the flatness property holds, if controllability also holds. Furthermore, the system is 0-flat if n is even.

3.3 A Flatness-Based Controller for the Vehicle Suspension Model

It can be proven that the suspension model is differentially flat by defining the following flat output [Sira-Ramirez *et al.*, 2011]

$$F = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} C_o^{-1} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \quad (22)$$

i. e. $F = \frac{\epsilon}{\kappa}x_1 - \frac{\epsilon\gamma_2}{\kappa^2}\dot{x}_1 + \frac{\epsilon\kappa - \gamma_2^2\epsilon}{\kappa^2}x_2 - \frac{\epsilon^2\gamma_2}{\kappa^2}\dot{x}_2$. where C_o stands for the system's controllability matrix.

$$C_o = [B, AB, A^2B, A^3B] \quad (23)$$

It can be shown that all state variable of the system and the control input can be written as functions of the flat output and its derivatives. Indeed for the unperturbed system, i. e. the model obtained from Eq. (22) if the disturbance input ζ and its derivatives $\dot{\zeta}$ are ignored, it holds

$$\begin{array}{l} F = \frac{\epsilon}{\kappa}x_1 - \frac{\epsilon\gamma_2}{\kappa^2}\dot{x}_1 + \frac{(\epsilon\kappa - \gamma_2^2)\epsilon}{\kappa^2}x_2 - \frac{\epsilon^2\gamma_2}{\kappa^2}\dot{x}_2 \\ \dot{F} = \frac{\epsilon}{\kappa}\dot{x}_1 + \frac{\epsilon\gamma_2}{\kappa}x_2 + \frac{\epsilon^2}{\kappa}\dot{x}_2 \\ \ddot{F} = -\epsilon x_2 \\ F^{(3)} = \epsilon\dot{x}_2 \end{array} \quad (24)$$

while one also has

$$\begin{array}{l} x_1 = \frac{\kappa}{\epsilon}F + \frac{\gamma_2}{\epsilon}\dot{F} + \ddot{F} \\ x_2 = \frac{\kappa}{\epsilon}\dot{F} + \frac{\gamma_2}{\epsilon}\ddot{F} + F^{(3)} \\ x_3 = \frac{-1}{\epsilon}\ddot{F} \\ x_4 = \frac{-1}{\epsilon}F^{(3)} \end{array} \quad (25)$$

and with the use of $u = \ddot{x}_1 + \gamma_1(\dot{x}_1 - \dot{x}_2) + (x_1 - x_2)$ it can be also concluded that the control input is a function of the flat output and its derivatives.

Taking also into account the effects of the disturbance input ζ and of its derivative $\dot{\zeta}$ the flat output and its derivative are formulated as follows

$$\begin{array}{l} F = \frac{\epsilon}{\kappa}x_1 - \frac{\epsilon\gamma_2}{\kappa^2}\dot{x}_1 + \frac{\epsilon\kappa - \gamma_2^2\epsilon}{\kappa^2}x_2 - \frac{\epsilon^2\gamma_2}{\kappa^2}\dot{x}_2 \\ \dot{F} = \frac{\epsilon}{\kappa}x_1 - \frac{\epsilon\gamma_2}{\kappa^2}x_2 + \frac{\epsilon^2}{\kappa}\dot{x}_2 - \frac{\epsilon\gamma_2}{\kappa}\zeta(\tau) - \frac{\epsilon\gamma_2^2}{\kappa^2}\dot{\zeta}(\tau) \\ \ddot{F} = -\epsilon x_2 + \epsilon\zeta(\tau) + \epsilon\gamma_2(1 - \frac{1}{\kappa})\dot{\zeta}(\tau) - \frac{\epsilon\gamma_2^2}{\kappa^2}\ddot{\zeta}(\tau) \\ F^{(3)} = -\epsilon\dot{x}_2 + \epsilon\dot{\zeta}(\tau) + \epsilon\gamma_2(1 - \frac{1}{\kappa})\ddot{\zeta}(\tau) - \frac{\epsilon\gamma_2^2}{\kappa^2}\zeta^{(3)}(\tau) \end{array} \quad (26)$$

Differentiating one more time the flat output one obtains

$$\begin{array}{l} F^{(4)} = -x_1 - \gamma_1\dot{x}_1 + (1+\kappa)x_2 + \\ +(\gamma_1 + \gamma_2)\dot{x}_2 + u - \kappa\zeta(\tau) - \gamma_2\dot{\zeta}(\tau) + \\ +\epsilon\ddot{\zeta}(\tau) + \epsilon\gamma_2(1 - \frac{1}{\kappa})\zeta^{(3)(\tau)} - \\ -\epsilon\frac{\gamma_2^2}{\kappa^2}\zeta^{(4)}(\tau) \end{array} \quad (27)$$

Aggregating all terms other than u into one variable

$$\begin{array}{l} \phi(\tau) = x_1 - \gamma_1\dot{x}_1 + (1+\kappa)x_2 + (\gamma_1 + \gamma_2)\dot{x}_2 - \\ -\kappa\zeta(\tau) - \gamma_2\dot{\zeta}(\tau) + \epsilon\ddot{\zeta}(\tau) + \\ +\epsilon\gamma_2(1 - \frac{1}{\kappa})\zeta^{(3)}(\tau) - \frac{\epsilon\gamma_2^2}{\kappa^2}\zeta^{(4)}(\tau) \end{array} \quad (28)$$

one has the system dynamics

$$F^{(4)} = u + \phi(\tau) \quad (29)$$

or equivalently, in state-space form

$$\begin{pmatrix} \dot{F}_1 \\ \dot{F}_2 \\ \dot{F}_3 \\ \dot{F}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \big(u + \phi(\tau)\big) \tag{30}$$

with the state variable $F_i(\tau),\ i = 1, \cdots, 4$ to stand for the the $(i-1)$-th order derivative $F^{(i-1)}(t)$ of the flat output. The estimation of the term $\phi(\tau)$ by a disturbance observer enables to design a controller for the vehicle's suspension model as follows

$$\begin{aligned} u(\tau) = F_d^{(4)}(\tau) - k_1(F^{(3)}(\tau) - F_d^{(3)}(\tau)) - \\ -k_2(\ddot{F}(\tau) - \ddot{F}_d(\tau)) - k_3(\dot{F}(\tau) - \dot{F}_d(\tau)) - \\ -k_4(F(\tau) - F_d(\tau)) - \hat{\phi}(\tau). \end{aligned} \tag{31}$$

4 State Estimation with the Kalman Filter

4.1 The Continuous-Time Kalman Filter for the Linear State Estimation Model

In the dynamic model of the suspension described in Eq. (30) it is assumed that the measurable element of the transformed state vector is F_1, which can be computed from the displacement and velocity of the sprung and unsprung masses. Therefore, to implement the state feedback control of Eq. (31) it is necessary to estimate the non-measurable state elements of the transformed state vector through some filtering/estimation procedure. Moreover, the filtering/estimation will be useful for identifying unknown forces and torques exerted on the suspension.

In the continuous-time representation of the system's dynamics, the continuous-time Kalman Filter stands for a state estimator of optimal accuracy. The following continuous-time dynamical system is assumed [Kamen and Su, 1999; Rigatos and Tzafestas, 2007]:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + w(t),\ t \geq t_0 \\ z(t) = Cx(t) + v(t),\ t \geq t_0 \end{cases} \tag{32}$$

where again $x \in R^{m \times 1}$ is the system's state vector, and $z \in R^{p \times 1}$ is the system's output. Matrices A,B and C can be time-varying and $w(t)$,$v(t)$ are uncorrelated white Gaussian noises. The covariance matrix of the process noise $w(t)$ is $Q(t)$, while the covariance matrix of the measurement noise is $R(t)$. Then the Kalman Filter is again a linear state observer which is given by

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + K[z - C\hat{x}],\ \hat{x}(t_0) = x_0 \\ K(t) = PC^TR^{-1} \\ \dot{P} = AP + PA^T + Q - PC^TR^{-1}CP \end{cases} \tag{33}$$

where $\hat{x}(t)$ is the optimal estimation of the state vector $x(t)$ and $P(t)$ is the covariance matrix of the state vector estimation error with $P(t_0) = P_0$. It can be seen that as in the case of the Luenberger observer, the Kalman Filter consists of the system's state equation plus a corrective term $K[z - C\hat{x}]$. The associated Riccati equation for calculating the covariance matrix $P(t)$ has the form given in the last row of Eq. (33).

4.2 The Discrete-Time Kalman Filter for the Linear State Estimation Model

In the discrete-time case the dynamical system is assumed to be expressed in the form of a discrete-time state model:

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)u(k) + w(k) \\ z(k) = Cx(k) + v(k) \end{cases} \tag{34}$$

where the state $x(k)$ is a m-vector, $w(k)$ is a m-element process noise vector and A is a $m \times m$ real matrix. Moreover the output measurement $z(k)$ is a p-vector, C is an $p \times m$-matrix of real numbers, and $v(k)$ is the measurement noise. It is assumed that the process noise $w(k)$ and the measurement noise $v(k)$ are uncorrelated.

Now the problem of interest is to estimate the state $x(k)$ based on the measurements $z(1), z(2), \cdots, z(k)$. The initial value of the state vector $x(0)$, the initial value of the error covariance matrix $P(0)$ is unknown and an estimation of it is considered, i. e. $\hat{x}(0) =$ a guess of $E[x(0)]$ and $\hat{P}(0) = a$ guess of $Cov[x(0)]$.

For the initialization of matrix P one can set $\hat{P}(0) = \lambda I$ with $\lambda > 0$. The state vector $x(k)$ has to be estimated taking into account $\hat{x}(0)$, $\hat{P}(0)$ and the output measurements $Z = [z(1), z(2), \cdots, z(k)]^T$, i. e. there is a function relationship:

$$\hat{x}(k) = \alpha_n(\hat{x}(0), \hat{P}(0), Z(k)) \tag{35}$$

Actually, this is a linear minimum mean squares estimation problem (LMMSE) which is solved recursively, through the function relationship

$$\hat{x}(k+1) = a_{n+1}(\hat{x}(k), z(k+1)) \tag{36}$$

The process and output noise are white and their covariance matrices are given by: $E[w(i)w^T(j)] = Q\delta(i-j)$ and $E[v(i)v^T(j)] = R\delta(i-j)$.

Using the above, the discrete-time Kalman Filter can be decomposed into two parts: i) time update, and ii) measurement update. The first part employs an estimate of the state vector $x(k)$ made before the output measurement $z(k)$ is available (a priori estimate). The second part estimates $x(k)$ after $z(k)$ has become available (a posteriori estimate).

- When the set of measurements $Z^{-} = \{z(1), \cdots, z(k-1)\}$ is available. From Z^{-} an a priori estimation of $x(k)$ is obtained which is denoted by $\hat{x}^{-}(k)$ = the estimate of $x(k)$ given Z^{-}.
- When $z(k)$ becomes available, the set of the output measurements becomes $Z = \{z(1), \cdots, z(k)\}$ where $\hat{x}(k)$ = the estimate of $x(k)$ given Z.

The associated estimation errors are defined by

$$\begin{array}{c} e^{-}(k) = x(k) - \hat{x}^{-}(k) = \text{the a priori error} \\ e(k) = x(k) - \hat{x}(k) = \text{the a posteriori error} \end{array} \quad (37)$$

The estimation error covariance matrices associated with $\hat{x}(k)$ and $\hat{x}^{-}(k)$ are defined as [Kamen and Su, 1999]

$$\begin{array}{c} P^{-}(k) = Cov[e^{-}(k)] = E[e^{-}(k)e^{-}(k)^{T}] \\ P(k) = Cov[e(k)] = E[e(k)e^{T}(k)] \end{array}$$

The mean square error of the estimates can be found by computing the trace of the estimation error covariance matrices, i. e.

$$\begin{array}{c} MSE(\hat{x}^{-}(k)) = tr(P^{-}(k)) \\ MSE(x(k)) = tr(P(k)) \end{array}$$

Finally, the linear Kalman filter equations in cartesian coordinates are

measurement update:

$$\begin{array}{c} K(k) = P^{-}(k)C(k)^{T}[C{\cdot}P^{-}(k)C(k)^{T} + R(k)]^{-1} \\ \hat{x}(k) = \hat{x}^{-}(k) + K(k)[z(k) - C\hat{x}^{-}(k)] \\ P(k) = P^{-}(k) - K(k)C(k)P^{-}(k) \end{array} \quad (38)$$

time update:

$$\begin{array}{c} P^{-}(k+1) = A(k)P(k)A^{T}(k) + Q(k) \\ \hat{x}^{-}(k+1) = A(k)\hat{x}(k) + B(k)u(k) \end{array} \quad (39)$$

The schematic diagram of the Kalman Filter (KF) loop is given in Fig. 3.

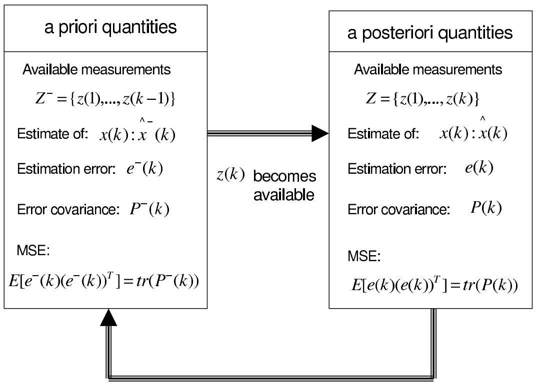


Figure 3. Schematic diagram of the Kalman Filter loop.

4.3 The Extended Kalman Filter

State estimation can be also performed for nonlinear dynamical systems using the Extended Kalman Filter (EKF) recursion. The following nonlinear model is considered [Rigatos, 2010; Rigatos and Tzafestas, 2007]:

$$\begin{array}{c} x(k+1) = \phi(x(k)) + L(k)u(k) + w(k) \\ z(k) = \gamma(x(k)) + v(k) \end{array} \quad (40)$$

where $x{\in}R^{m \times 1}$ is the system's state vector and $z{\in}R^{p \times 1}$ is the system's output, while $w(k)$ and $v(k)$ are uncorrelated, zero-mean, Gaussian zero-mean noise processes with covariance matrices $Q(k)$ and $R(k)$ respectively. The operators $\phi(x)$ and $\gamma(x)$ are vectors defined as

$$\begin{array}{c} \phi(x) = [\phi_1(x), \phi_2(x), \cdots, \phi_m(x)]^{T} \\ \gamma(x) = [\gamma_1(x), \gamma_2(x), \cdots, \gamma_p(x)]^{T}, \end{array} \quad (41)$$

respectively. It is assumed that ϕ and γ are sufficiently smooth in x so that each one has a valid series Taylor expansion. Following a linearization procedure, ϕ is expanded into Taylor series about $\hat{x}$:

$$\phi(x(k)) = \phi(\hat{x}(k)) + J_{\phi}(\hat{x}(k))[x(k) - \hat{x}(k)] + \cdots \quad (42)$$

where $J_{\phi}(x)$ is the Jacobian of ϕ calculated at $\hat{x}(k)$:

$$J_{\phi}(x) = \frac{\partial \phi}{\partial x}|_{x=\hat{x}(k)} = \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_m} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \cdots & \frac{\partial \phi_2}{\partial x_m} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \phi_m}{\partial x_1} & \frac{\partial \phi_m}{\partial x_2} & \cdots & \frac{\partial \phi_m}{\partial x_m} \end{pmatrix} \quad (43)$$

Likewise, γ is expanded about $\hat{x}^{-}(k)$

$$\gamma(x(k)) = \gamma(\hat{x}^{-}(k)) + J_{\gamma}[x(k) - \hat{x}^{-}(k)] + \cdots \quad (44)$$

where $\hat{x}^{-}(k)$ is the estimation of the state vector $x(k)$ before measurement at the k-th instant to be received and $\hat{x}(k)$ is the updated estimation of the state vector after measurement at the k-th instant has been received. The Jacobian $J_{\gamma}(x)$ is

$$J_{\gamma}(x) = \frac{\partial \gamma}{\partial x}|_{x=\hat{x}^{-}(k)} = \begin{pmatrix} \frac{\partial \gamma_1}{\partial x_1} & \frac{\partial \gamma_1}{\partial x_2} & \cdots & \frac{\partial \gamma_1}{\partial x_m} \\ \frac{\partial \gamma_2}{\partial x_1} & \frac{\partial \gamma_2}{\partial x_2} & \cdots & \frac{\partial \gamma_2}{\partial x_m} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \gamma_p}{\partial x_1} & \frac{\partial \gamma_p}{\partial x_2} & \cdots & \frac{\partial \gamma_p}{\partial x_m} \end{pmatrix} \quad (45)$$

The resulting expressions create first order approximations of ϕ and γ. Thus the linearized version of the system is obtained:

$$\begin{array}{c} x(k+1) = \phi(\hat{x}(k)) + J_\phi(\hat{x}(k))[x(k)-\hat{x}(k)] + w(k) \\ z(k) = \gamma(\hat{x}^-(k)) + J_\gamma(\hat{x}^-(k))[x(k)-\hat{x}^-(k)] + v(k) \end{array} \quad (46)$$

Now, the EKF recursion is as follows: First the time update is considered: by $\hat{x}(k)$ the estimation of the state vector at time instant k is denoted. Given initial conditions $\hat{x}^-(0)$ and $P^-(0)$ the recursion proceeds as:

Measurement update. Acquire $z(k)$ and compute:

$$\begin{array}{l} K(k) = P^-(k)J_\gamma^T(\hat{x}^-(k))\cdot \\ \cdot[J_\gamma(\hat{x}^-(k))P^-(k)J_\gamma^T(\hat{x}^-(k)) + R(k)]^{-1} \\ \hat{x}(k) = \hat{x}^-(k) + K(k)[z(k) - \gamma(\hat{x}^-(k))] \\ P(k) = P^-(k) - K(k)J_\gamma(\hat{x}^-(k))P^-(k) \end{array} \quad (47)$$

Time update. Compute:

$$\begin{array}{l} P^-(k+1) = J_\phi(\hat{x}(k))P(k)J_\phi^T(\hat{x}(k)) + Q(k) \\ \hat{x}^-(k+1) = \phi(\hat{x}(k)) + L(k)u(k) \end{array} \quad (48)$$

4.4 Compensating for Model Uncertainty with the Use of the H_∞ Kalman Filter

The Kalman Filter can be redesigned to cope with the case of maximum errors of some linear combination of states for worst case assumptions of process noise, measurement noise and disturbances. This can be useful in state estimation for the vehicle suspension model, as a method for model uncertainty compensation. Filters designed to minimize a weighted norm of state errors are called H_∞ or minimax filters [Simon, 2006; Gibbs, 2011].

The discrete time H_∞ filter uses the same state model as the Kalman Filter, which has the form

$$\begin{array}{c} x(k+1) = A(k)x(k) + B(k)u(k) + w(k) \\ z(k) = C(k)x(k) + v(k) \end{array} \quad (49)$$

$E[w(k)] = 0$, $E[w(k)w(k)^T] = Q(k)\delta ij$, $E[v(k)] = 0$, $E[v(k)v(k)^T] = R(k)\delta_{ij}$ and $E(w(k)v(k)^T) = 0$. The update of the state estimate is again given by

$$\hat{x}(k) = \hat{x}^-(k) + K(k)(z(k) - C(k)\hat{x}^-(k)) \quad (50)$$

that minimizes the trace of the covariance matrix of the state vector estimation error

$$J = \tfrac{1}{2}E\{\tilde{x}(k)^T\cdot\tilde{x}(k)\} = \tfrac{1}{2}tr(P^-(k)) \quad (51)$$

where $\tilde{x}^-(k) = x(k) - \hat{x}^-(k)$ and $P^-(k) = E[\tilde{x}^-(k)^T\cdot\tilde{x}^-(k)]$. The H_∞ filtering approach defines first a transformation

$$d(k) = L(k)x(k) \quad (52)$$

where $L(k)\in R^{n\times n}$ is a full rank matrix. The use of the transformation given in Eq. (52) allows certain combinations of states to be given more weight than others. Next, defining the estimation error variable $\tilde{d}_1(i) = d(i) - \hat{d}(i)$, the cost function of the H_∞ filter is initially formulated as

$$\begin{array}{l} J(k) = \sum_{i=0}^{k-1}\tilde{d}(i+1)^T S(i)\tilde{d}(i+1)/b \\ \\ b = \tilde{x}^-(0)^T P^-(0)^{-1}\tilde{x}^-(0) + \\ +\sum_{i=0}^{k-1} w^T(i+1)Q(i+1)^{-1}w(i+1) + \\ +\sum_{i=0}^{k-1} v^T(i)R(i)^{-1}v(i) \end{array} \quad (53)$$

where S_i is a positive-definite symmetric weighting matrix. It can be observed that both matrices $S(k)$ and $L(k)$ appear in the cost function and thus affect the solution $\hat{x}^-(k+1)$ of the optimization problem. The objective is to find state vector estimates $\hat{x}^-(k)$ and $\hat{x}(k)$ that keep the cost function below a given value $1/\theta$ for worst case conditions, i. e.

$$J(k) < \tfrac{1}{\theta}. \quad (54)$$

By rewriting Eq. (53) and substituting Eq. (49) a modified cost functional is obtained

$$\begin{array}{l} J_a(k) = -\frac{1}{\theta}\tilde{x}^-(0)^T P^-(0)\tilde{x}^-(0) + \sum_{i=0}^{k-1}\Gamma(i) \\ \\ \Gamma(i) = \\ (x(i+1) - \hat{x}^-(i+1))^T W_i(x(i+1) - \hat{x}^-(i+1)) - \\ -\frac{1}{\theta}(w^T(i+1)Q(i+1)^{-1}w(i+1) + \\ +(y(i) - C(i)x^-(i))^T R(i)^{-1}(y(i) - C(i)x^-(i))) \end{array} \quad (55)$$

and

$$W(i) = L(i)^T S(i)L(i) \quad (56)$$

This cost function does not include the dynamic model of the system given in Eq. (49) and this is added by using a vector of Lagrange multipliers $\lambda(i+1)$. This gives

$$\begin{array}{l} J(k) = -\frac{1}{\theta}\tilde{x}^-(0)^T P^-(0)\tilde{x}^-(0) + \\ +\sum_{i=0}^{k-1}(\Gamma_i + 2\frac{\lambda(i+1)^T}{\theta})(A(i)\hat{x}(i) + B(i)u(i) + \\ +w(i) - x(i+1)) + \frac{2\lambda(0)^T}{\theta}x(0) - \frac{2\lambda(0)^T}{\theta}x(0) \end{array} \quad (57)$$

The cost function of the filter given in Eq. (57) can be used as the basis for the solution. It is aimed to find equations defining $\hat{x}^-(k+1)$, or equivalently a measurement weighting matrix (similar to the Kalman gain matrix), that minimize the cost for worst case assumptions about $x(0)$, $w(i)$ and $y(i)$. Thus, the optimization objective is formulated as

$$J^*(k) = \min_{x_i}\ \max_{x(0),w(i),y(i)} J(k). \quad (58)$$

It is noted that the estimation algorithm has knowledge of the output measurement $y(i)$ but no knowledge about the initial conditions of the system $x(0)$ and the process noise $w(i)$. Under this assumption, the estimation should be able to compensate for worst case values for the unknown parameters. This is a game theoretic problem that is solved in two steps.

In the first step of optimization, partial derivatives of $J(k)$ with respect to $x(0)$, $w(i)$ and $\lambda(i)$ are set to zero so as to maximize the cost function of Eq. (57), now being dependant only on the terms $\hat{x}^-(k+1)$ and $y(k)$ which are included in Γ_i. In the second step of optimization, the partial derivatives of $J(k)$ with respect to $\hat{x}^-(k+1)$ and $y(k)$ are set to zero, to obtain a condition for the filter's gain matrix that minimizes this cost functional. From the optimization conditions $\partial J(k)/\partial x_0 = 0^T$, $\partial J(k)/\partial w(i) = 0^T$, $\partial J(k)/\partial \lambda(i) = 0^T$ ones obtains an expression of $J(k)$ as function of $\hat{x}^-(k+1)$ and $y(k)$. Next, from the optimization conditions $\partial J(k)/\partial \hat{x}^-(i+1) = 0^T$, and $\partial J(k)/\partial y(i) = 0^T$ one obtains the filter's equations.

The recursion of the H_∞ Kalman Filter can be formulated again in terms of a *measurement update* and a *time update* part

Measurement update:

$$\begin{array}{l} D(k) = [I - \theta W(k)P^-(k) + \\ +C^T(k)R(k)^{-1}C(k)P^-(k)]^{-1} \\ K(k) = P^-(k)D(k)C^T(k)R(k)^{-1} \\ \hat{x}(k) = \hat{x}^-(k) + K(k)[y(k) - C\hat{x}^-(k)] \end{array} \quad (59)$$

Time update:

$$\begin{array}{l} \hat{x}^-(k+1) = A(k)x(k) + B(k)u(k) \\ P^-(k+1) = A(k)P^-(k)D(k)A^T(k) + Q(k) \end{array} \quad (60)$$

where it is assumed that parameter θ is sufficiently small to maintain

$$P^-(k)^{-1} - \theta W(k) + C^T(k)R(k)^{-1}C(k) \quad (61)$$

positive definite. When $\theta = 0$ the H_∞ Kalman Filter becomes equivalent to the standard Kalman Filter. It is noted that apart from the process noise covariance matrix $Q(k)$ and the measurement noise covariance matrix $R(k)$ the H_∞ Kalman filter requires tuning of the weight matrices L and S, as well as of parameter θ.

5 Robust State Estimation with the Use of Disturbance Observers

5.1 Unknown Input Observers

To account for model uncertainties and external disturbances, observer-based estimation has been proposed, enabling to solve the problem of model accuracy in reverse [Cortesao *et al.*, 2005; Cortesao, 2006; Chen *et al.*, 2000; Gupta and Malley, 2011; Miklosovic *et al.*, 2006]. This is done by modeling the mechatronic or robotic system with an equivalent input disturbance that includes unmodeled dynamics. An observer is then designed to estimate the disturbance in real time and provide feedback to cancel it.

The Unknown Input observer is applied to dynamical systems of the form

$$\begin{array}{c} \dot{x} = Ax + B(u + w_e) \\ z = Cx \end{array} \quad (62)$$

while the disturbance dynamics is given by

$$\begin{array}{c} \dot{d} = A_f d \\ w_e = C_f d \end{array} \quad (63)$$

Then, the unknown input observer provides a state estimate of the extended state vector

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{d}} \end{pmatrix} = \begin{pmatrix} A & BC_f \\ 0 & A_f \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{d} \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + K(z - C\hat{x}) \quad (64)$$

In the generic case one can assume that the disturbances vector w_e varies dynamically in time. However, in several cases it suffices to assume a constant or piecewise constant disturbance $\dot{w}_e(z) = 0$ where $A_f = 0$ and $C_f = 1$. The observer's gain can be obtained through the standard Kalman Filter recursion.

5.2 Perturbation Observer

The perturbation observer is an extension of the unknown inputs observer which takes into account not only external disturbances but also parametric uncertainties. In discrete-time form, the system dynamics is given by

$$\begin{array}{c} x_{k+1} = Ax_k + Bu_k + w_f \\ z = Cx_k \end{array} \quad (65)$$

while the disturbance dynamics is given by

$$\begin{array}{c} d_k = A_f d_{k-1} + B_f(B^+(\hat{x}_k - A\hat{x}_{k-1}) - u_{k-1}) \\ \hat{w}_{f_k} = C_f d_k \\ \hat{x}_{k+1} = A\hat{x}_k + B(u_k + \hat{w}_{f_k}) + L(z_k - C\hat{x}_k) \end{array} \quad (66)$$

where B^+ is the Moore-Penrose pseudo-inverse of matrix B. The unknown input can represent external disturbances and model uncertainties, i. e.

$$w_f = w_e + \Delta A x_k + \Delta B u_k.$$

5.3 Extended State Observer

The Extended State Observer uses a canonical form so the unmodelled dynamics appear at the disturbance estimation part. The system's description in the canonical form is given by

$$\begin{array}{c} x_1^{(n)} = f(x,t,u,w_f) + b_m u \\ z = x_1 \\ x = \left(x_1 \; \dot{x}_1 \; \cdots \; x_1^{(n-1)}\right)^T \end{array} \tag{67}$$

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \cdots \\ \dot{\hat{x}}_{n-1} \\ \dot{\hat{x}}_n \\ \dot{\hat{f}} \end{pmatrix} = \begin{pmatrix} \hat{x}_2 \\ \cdots \\ \hat{x}_n \\ \hat{f} + b_m u \\ 0 \end{pmatrix} + K(x_1 - \hat{x}_1) \tag{68}$$

The Extended State Observer can be also modified to take into account derivatives of the disturbance

$$\begin{array}{c} x_1^{(n)} = f(x,t,u,w_f) + b_m u \\ z = x_1 \\ x = \left(x_1 \; \dot{x}_1 \; \cdots \; x_1^{(n-1)}\right)^T \\ F = \left(f \; \dot{f} \; \cdots \; f^{(h-1)}\right)^T \end{array} \tag{69}$$

and now the state and disturbance observer takes the form

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \cdots \\ \dot{\hat{x}}_{n-1} \\ \dot{\hat{x}}_n \\ \dot{\hat{F}}_1 \\ \cdots \\ \dot{\hat{F}}_{h-1} \\ \dot{\hat{F}}_h \end{pmatrix} = \begin{pmatrix} \hat{x}_2 \\ \cdots \\ \hat{x}_n \\ \hat{f} + b_m u \\ \hat{F}_2 \\ \cdots \\ \hat{F}_h \\ 0 \end{pmatrix}^T + K(x_1 - \hat{x}_1) \tag{70}$$

The latter form of the Extended State Observer described in Eq. (70) enables to track various types of disturbances. For example, $h = 1$ allows estimation of disturbance dynamics defined by its first order derivative, and $h = 2$ allows estimation of disturbance dynamics defined by its second order derivative.

6 Estimation of Suspension Disturbance Forces with Kalman Filtering

6.1 State Estimation with the Derivative-Free Nonlinear Kalman Filter

Previous results about state estimation through transformation to linear canonical forms can be found in [Marino, 1990; Marino and Tomei, 1992; Rigatos, 2012b; Rigatos, 2012c; Rigatos, 2012d]. It was shown that the dynamical model of the suspension can be written in the MIMO canonical form of Eq. (30). Thus one has a MIMO linear model of the form

$$\begin{array}{c} \dot{y}_f = A_f y_f + B_f v \\ z_f = C_f y_f \end{array} \tag{71}$$

where $y_f = [F_1, F_2, F_3, F_4]^T$ and matrices A_f,B_f,C_f are in the form

$$A_f = \begin{pmatrix} 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \\ 0\,0\,0\,0 \end{pmatrix} \quad B_f = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad C_f^T = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{72}$$

where the measurable variables $y_1 = F$ is associated with the displacement of the sprung and unsprung mass in the suspension model. For the aforementioned model, and after carrying out discretization of matrices A_f, B_f and C_f with common discretization methods one can apply linear Kalman filtering using Eq. (38) and Eq. (39). This is *Derivative-free nonlinear Kalman filtering* for the model of the suspension which is performed without the need to compute Jacobian matrices and does not introduce numerical errors due to approximative linearization with Taylor series expansion.

6.2 Kalman Filter-Based Estimation of Suspension Disturbance Forces

Considering the effects of disturbances on the suspension's model and after applying a transform on the system's state variables according to the differential flatness theory it has been shown that the suspension model is described by

$$F^{(4)} = u + \phi(\tau). \tag{73}$$

The suspension's state space model of Eq. (30) will be extended to take into account also the dynamics and the effects of the disturbance input $\phi(t)$. The extended state vector of the suspension model is defined as $z \in R^{8\times 1}$ with $z_1 = F$, $z_2 = \dot{F}$, $z_3 = \ddot{F}$, $z_4 = F^{(3)}$, $z_5 = \phi$, $z_6 = \dot{\phi}$, $z_7 = \ddot{\phi}$, $z_8 = \phi^{(3)}$. The dynamics of the disturbance is assumed to be defined by its fourth order derivative, i.e. $\phi^{(4)} = f_d(F, \dot{F}, \ddot{F}, F^{(3)})$. Thus one has the extended state-space model

$$\begin{array}{c} \dot{z} = \tilde{A} \cdot z + \tilde{B} \cdot \tilde{v} \\ q = \tilde{C} z \end{array} \tag{74}$$

with

$$\tilde{A} = \begin{pmatrix} 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \tilde{B} = \begin{pmatrix} 0&0 \\ 0&0 \\ 0&0 \\ 1&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&1 \end{pmatrix}, \tilde{C}^T = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (75)$$

where the measurable variable is z_1 and the control input is

$$\tilde{v} = \left(u, \phi^{(4)}\right)^T. \quad (76)$$

The disturbance estimator has the following structure

$$\begin{matrix} \dot{\hat{z}} = \tilde{A}_o \cdot \hat{z} + \tilde{B}_o \cdot \tilde{v} + K(z_1 - \hat{z}_1) \\ \hat{q} = \tilde{C}_o \hat{y} \end{matrix} \quad (77)$$

where the estimator's gain $K \in R^{8\times1}$ and matrices $\tilde{A}_o$, $\tilde{B}_o$ and $\tilde{C}_o$ are defined as

$$\tilde{A}_o = \begin{pmatrix} 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \tilde{B}_o = \begin{pmatrix} 0&0 \\ 0&0 \\ 0&0 \\ 1&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \end{pmatrix}, \tilde{C}_o^T = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (78)$$

The disturbance estimator's gain $K \in R^{8\times1}$ will be computed through the Kalman Filter recursion.

Defining as $\tilde{A}_d$, $\tilde{B}_d$, and $\tilde{C}_d$, the discrete-time equivalents of matrices $\tilde{A}_o$, $\tilde{B}_o$ and $\tilde{C}_o$ respectively, a Derivative-free nonlinear Kalman Filter can be designed for the aforementioned representation of the system dynamics [Rigatos, 2012b; Rigatos, 2012c; Rigatos, 2012d]. The associated Kalman Filter-based disturbance estimator is given by

measurement update:

$$\begin{matrix} K(k) = P^-(k)\tilde{C}_d^T[\tilde{C}_d \cdot P^-(k)\tilde{C}_d^T + R]^{-1} \\ \hat{x}(k) = \hat{x}^-(k) + K(k)[z(k) - \tilde{C}_d \hat{x}^-(k)] \\ P(k) = P^-(k) - K(k)\tilde{C}_d P^-(k) \end{matrix} \quad (79)$$

time update:

$$\begin{matrix} P^-(k+1) = \tilde{A}_d(k)P(k)\tilde{A}_d^T(k) + Q(k) \\ \hat{x}^-(k+1) = \tilde{A}_d(k)\hat{x}(k) + \tilde{B}_d(k)\tilde{v}(k). \end{matrix} \quad (80)$$

To compensate for the effects of the disturbance forces it suffices to use in the control loop the modified control input vector $v_1 = u - \hat{\phi}(t)$.

Remark 1: The contribution of the paper is in introducing a Kalman Filter-based disturbance observer within a flatness-based control scheme. The Kalman Filter enables simultaneous estimation of the nonmeasurable elements of the suspension's state vector and of the external disturbances which are associated with the tyre's deflection due to the rough road surface. The use of the Kalman Filter as disturbance estimator has specific advantages: The Kalman Filter is an optimal estimator for linear systems subjected to Gaussian noise since it minimizes the trace of the estimation error's covariance matrix. This results in smooth convergence of the estimated state vector to the real state vector of the system and consequently in state estimation-based control where the variations of the control signal are also smooth. The selection of the observation gain in the Kalman Filter is performed through an adaptive procedure and can also cope with time-varying dynamic models whereas in deterministic state observers there should be explicit recomputation of the observation gain in case of change of the system's dynamic model. Finally, the fast computation features of the Kalman Filter make the method suitable for estimating and compensating in real time the disturbance terms that affect the suspension's model.

Remark 2: The Kalman Filter can give additional robustness with respect to model uncertainties and parametric variations in the monitored system, if redesigned in the form of a disturbance observer. It is also possible to obtain other robust implementations of the Kalman Filter, as for example the H_∞ Kalman Filter. However, when such a filter operates under normal conditions (free of disturbances) its accuracy of estimation may be inferior than the one of the standard Kalman Filter [Gibbs, 2011; Simon, 2006]. The fast recursion of the Kalman Filter makes it suitable for real-time applications. It has been proven that in nonlinear estimation problems, the previously analyzed derivative-free nonlinear Kalman Filter, is faster than other nonlinear filters, such as the Extended Kalman Filter, the Unscented Kalman Filter and the Particle Filter [Rigatos, 2012c].

Remark 3: The problem of estimation and control of unknown system dynamics and the problem of disturbances compensation through the use of a supplementary control term in the control loop can be found in several research articles (see for example [Amelin and Granichin, 2011; Bobtsov *et al.*, 2011; Landau *et al.*, 2005; Landau *et al.*, 2011a; Landau *et al.*, 2011b; Marino *et al.*, 2008; Titov *et al.*, 2011]). The approach analyzed in the previous sections aims at simultaneous estimation of the non-measurable elements of the suspension's state vector and at compensation of the disturbance terms affecting the suspension's model [Cortesao *et al.*, 2005; Delvecchio *et al.*, 2010; Koch *et al.*, 2010b; Lee *et al.*, 2012; Rigatos, 2011]. Moreover, the approach on disturbance estimation and compensation followed in the previous sections is characterized by its efficiency in estimating disturbance signals of multi-frequency content.

7 Simulation Tests

To evaluate the performance of the proposed Kalman Filter-based and disturbances estimation scheme for the vehicle's suspension model simulation tests were carried out. Different disturbance forces were assumed to be exerted on the vehicle's wheel due to its contact with the rough road surface. The disturbances dynamics was completely unknown to the controller and their identification was performed in real time by the disturbance estimator. The parameters of the suspension mechanism were as follows: $m_1 = 2000Kg$, $m_2 = 40Kg$, $K_1 = 1.0 \cdot 10e4\ N/m$, $K_2 = 1.0 \cdot 10e4\ N/m$, $c_1 = 1790\ N{\cdot}s/m$ and $c_2 = 20\ N{\cdot}s/m$. The control loop used for the vehicle's suspension is given in Fig. 4.

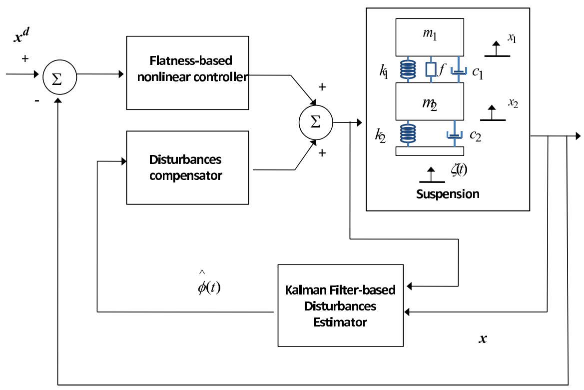
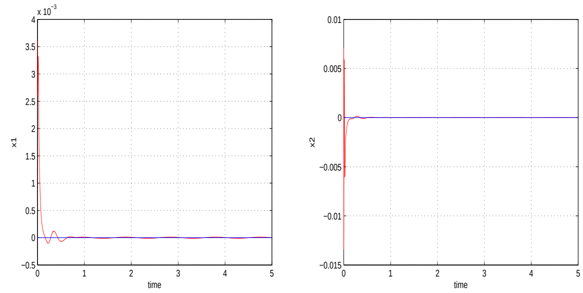


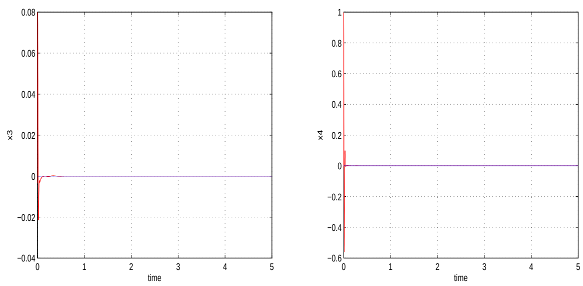
Figure 4. Control loop for the vehicle's suspension comprising a flatness-based nonlinear controller and a Kalman Filter-based disturbances estimator.

The monitored parameters were the state variables of the suspension. The control input was the force generated by the actuator. The measured parameters were the position and velocity of the sprung and unsprung mass. The dynamics of the disturbance force was assumed to be defined by its fourth-order derivative. The extended state vector used by the disturbance observer was of dimension $x \in R^8$, where the first four state variables were describing the suspension's model whereas the rest four state variables were associated with the dynamics of the disturbance force. The real-time estimation of the external disturbance that was provided by the Kalman Filter was used by an additional control term in the control loop to generate a counter disturbance control input. Thus, the disturbance's effects on the vehicle's parts were eliminated and vibrations were efficiently suppressed. As shown in Fig. 5 to Fig. 16 fast stabilization of the suspension's sprung and unsprung masses to the desirable set-points was succeeded and accurate estimation of the unknown disturbances forces was performed. Moreover, is should be taken into account that for common nominal values of k_1 and m_1 one obtains $t < \tau$ i.e. $t = \sqrt{\frac{m_1}{k_1}}\tau$ which finally gives the real time scale of suspension's active control system.



(a) (b)

Figure 5. Suspension control under disturbances profile 1: (a) Convergence of sprung mass position x_1 to the desirable setpoint, (b) Convergence of sprung mass velocity x_2 to the desirable setpoint.

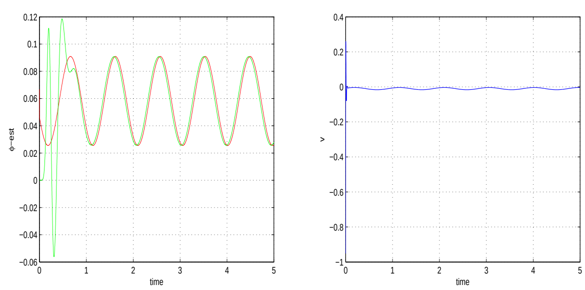


(a) (b)

Figure 6. Suspension control under disturbances profile 1: (a) Convergence of unsprung mass position x_2 to the desirable setpoint, (b) Convergence of unsprung mass velocity x_4 to the desirable setpoint.

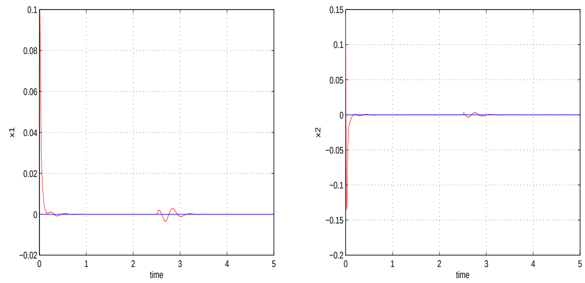
8 Conclusions

An active suspension control system has been designed with the use of Kalman Filtering, aiming at reconstructing the suspension's state vector out of measurements



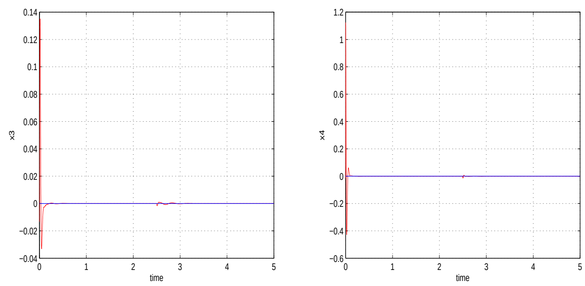
(a) **(b)**

Figure 7. Suspension control under disturbances profile 1: (a) Estimation of the disturbance terms, (b) Control input generated by the actuator.



(a) **(b)**

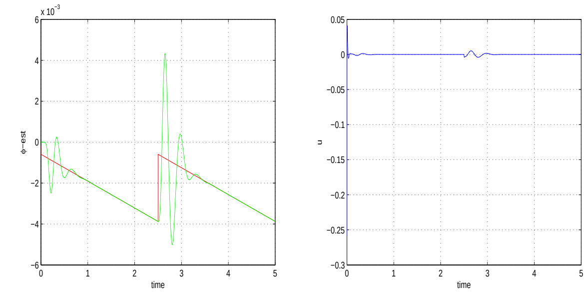
Figure 8. Suspension control under disturbances profile 2: (a) Convergence of sprung mass position x_1 to the desirable setpoint, (b) Convergence of sprung mass velocity x_2 to the desirable setpoint.



(a) **(b)**

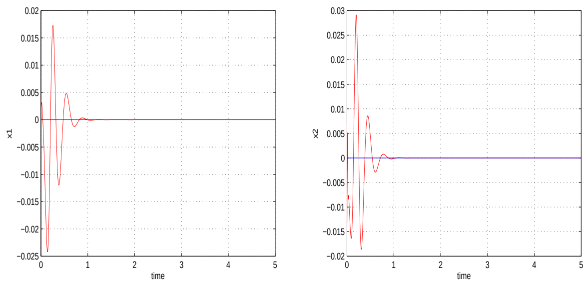
Figure 9. Suspension control under disturbances profile 2: (a) Convergence of unsprung mass position x_2 to the desirable setpoint, (b) Convergence of unsprung mass velocity x_4 to the desirable setpoint.

provided by a limited number of sensors and at estimating the unknown disturbance inputs exerted on the wheel. First, dynamic modeling of the suspension has been provided showing that the initial linear state-space description of the suspension can take a nonlinear form if nonlinearities in the spring and damper elements of the mechanism are taken into account. Next, it has



(a) **(b)**

Figure 10. Suspension control under disturbances profile 2: (a) Estimation of the disturbance terms, (b) Control input generated by the actuator.

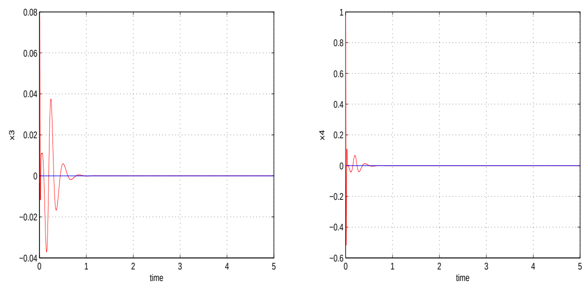


(a) **(b)**

Figure 11. Suspension control under disturbances profile 3: (a) Convergence of sprung mass position x_1 to the desirable setpoint, (b) Convergence of sprung mass velocity x_2 to the desirable setpoint.

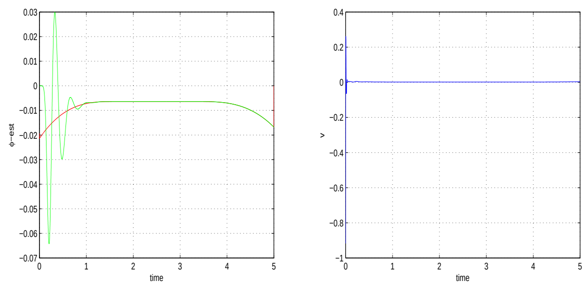
been shown that the suspension's model is differentially flat and that the flat output is a scalar resulting from a weighted sum of the state vector elements. By expressing all state variable and the control input of the suspension model as a function of the flat output and its derivatives, a linear canonical form of the suspension dynamics was obtained. For the latter model the design of state feedback controller becomes possible.

As analyzed, the design of an active controller for the suspension model with the use of state feedback requires knowledge of all elements of the suspension's transformed state vector. The nonmeasurable elements can be estimated through the Kalman Filter-based processing of sensor readings from the measurable elements. An additional difficulty in the estimation problem comes from the fact that unknown disturbance forces are exerted on the suspension and these can cause divergence of the filtering procedure. Filtering under model uncertainties and external disturbances is obtained by redesigning the Kalman Filter in the form of a disturbance observer. This enables simultaneous estimation of the suspension's state vector and identification of the unknown disturbance forces.



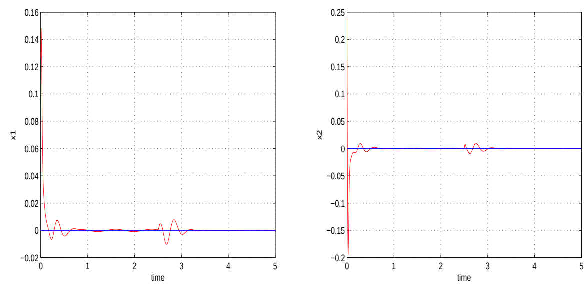
(a) (b)

Figure 12. Suspension control under disturbances profile 3: (a) Convergence of unsprung mass position x_2 to the desirable setpoint, (b) Convergence of unsprung mass velocity x_4 to the desirable setpoint.



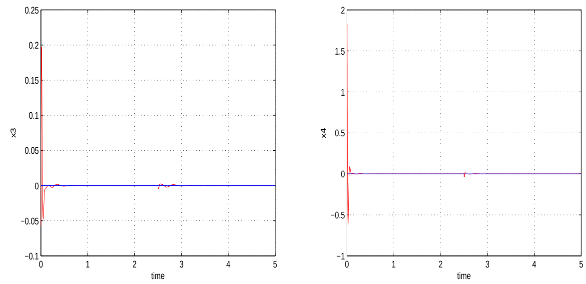
(a) (b)

Figure 13. Suspension control under disturbances profile 3: (a) Estimation of the disturbance terms, (b) Control input generated by the actuator.



(a) (b)

Figure 14. Suspension control under disturbances profile 4: (a) Convergence of sprung mass position x_1 to the desirable setpoint, (b) Convergence of sprung mass velocity x_2 to the desirable setpoint.



(a) (b)

Figure 15. Suspension control under disturbances profile 4: (a) Convergence of unsprung mass position x_2 to the desirable setpoint, (b) Convergence of unsprung mass velocity x_4 to the desirable setpoint.

It has been also shown that once an estimation of the unknown disturbance inputs is obtained their effect can be compensated by an additional element that is included in the control loop. This new control input stands for a counter-disturbances signal that is based on the estimated value of the disturbance forces and which finally enables the elimination of vibrations in the vehicle's parts. The performance of the proposed Kalman Filter-based active control scheme for vehicle suspensions has been tested through numerical simulation experiments. Yet simple in concept the use of a disturbance estimator in the control loop can compensate efficiently for the disturbances and model uncertainties effects, thus improving also significantly the functioning of the suspension's mechanism.

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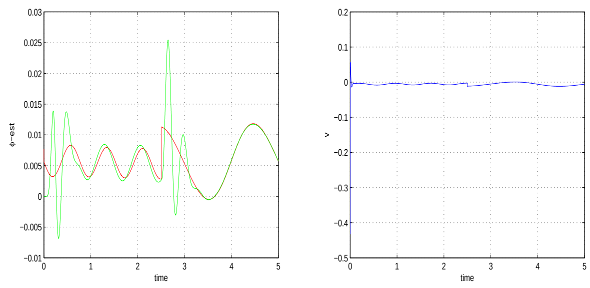

Figure 16. Suspension control under disturbances profile 4: (a) Estimation of the disturbance terms, (b) Control input generated by the actuator.

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