SOLID CYLINDER WITH VISCOUS FILLING ON ROUGH PLANE

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Abstract

We constructed the mathematical model of motion of a cylinder that has axisymmetric cavity filled by viscous filling. The cylinder leans by circle edge upon rough plane. For modeling the internal tangential friction we used the Helmholtz equations for the vortex components for the uniform vortex motion of ideal fluid. The terms corresponding to internal friction moment are phenomenologically introduced into right-hand sides of the equations expressing the change of momentum for solid body and filling.

The examination of dissipativity of the system is performed.

Results of numerical simulation of model equations are in qualitative agreement with known experimental data.

We suggested the procedure for establishing correspondence between model internal friction coefficient σ and fluid filling viscosity μ. Study testifies the necessity of having sufficient experimental data for establishing quantitative correspondence between μ and σ.

Key words

Nonlinear dynamics, phenomenological modeling, viscous filling, internal friction.

1 Introduction

The problem of motion of a solid body with cavities filled with liquid is a classical problem of theoretical mechanics. Its study began in 19th century by Stokes and continued by Helmholtz, Lamb and others by performing experiments as well as by constructing more and more complicated theories. Russian scientist N.E Zhukovski developed theory of motion of body with cavity filled with ideal fluid performing potential motion. Some types of motion of bodies with viscous fluid were analyzed by F.L. Chernousko with the help of asymptotic methods. In the same time, the problems of stability of motion of bodies with viscous filling began to be investigated. Theory and experiment supplement with each other but are developed not evenly. For example, experiments carried out in the LMSU Institute of Mechanics [Zhestkov, Samsonov, 1983] for measuring the moment of friction between liquid filling and precessing walls of vessel need to be interpreted. One cannot expect obtaining the required formulae by means of solving the corresponding problem of hydromechanics.

At the same time, phenomenological models can be useful for engineering practice. The main requirement for such models is that they would adequately describe the qualitative characteristics of the original problem.

Such finite-dimension approach for studying motion of incompressible fluid was used, for example, in meteorological systems of hydrodynamic type [Gledzer, Dolzhanskij, Obukhov 1981].

The phenomenological model for moment of interaction between walls of vessel and filling was proposed in [Savchenko, Samsonov, Sudakov, 1988].

The method for introducing this model into dynamic system describing motion of solid body with cavity filled by liquid filling was suggested in [Dosaev, Samsonov, 2002]. The Helmholtz equations for the vortex components for the uniform vortex motion of ideal fluid are used for modeling the interaction between vessel walls and filling. The terms corresponding to internal friction moments are phenomenologically introduced to right-hand sides of the equations expressing the change of momentum for solid top and filling.

This approach was used in [Karapetyan, Sumin, 2008] for solving a problem of stability of permanent rotation of a rod-suspended body with a viscous filling.

In present paper we applied this technique for the problem of motion along rough plane of an axisymmetric solid cylinder (solid body) having axisymmetric cavity filled with uniform incompressible liquid.

The cylinder motions are studied for wide range of initial conditions and values of parameters. Numerical simulation results are in a good qualitative agreement with experiment results described in [Samsonov, 1982]. If initial spin of cylinder is relatively small, that cylinder monotonically inclines up to the determined limit nutation angle. If initial spin is big enough that cylinder first inclines down to bigger nutation angle and then ascends to the same limit nutation angle.

Matching the model coefficient of internal friction with liquid viscosity is discussed using particular solutions of [Kazmerchuk, Samsonov, 1995].

2 Problem Statement

Consider a motion of axisymmetric solid cylinder with mass M_1 along rough plane. The cylinder has the axisymmetric cavity filled with uniform incompressible liquid with mass M_2. For simplicity it's assumed that the cavity center coincides with the center of mass C of the body (Fig. 1).

Introduce $CXYZ$ as principal axes of inertia. For convenience introduce two moving coordinate systems: $Cxyz$, $C\xi\eta\zeta$. Axis $C\zeta$ is vertical, axis Cz is directed along axis of body symmetry. We assign the body orientation by three Euler angles: ϑ is nutation angle, the angle between axes Cz and $C\zeta$; ψ is the angle of precession of axis Cz around axis $C\zeta$; φ is the angle of body rotation about axis Cz. Coordinates ψ and φ are obviously cyclic ones.

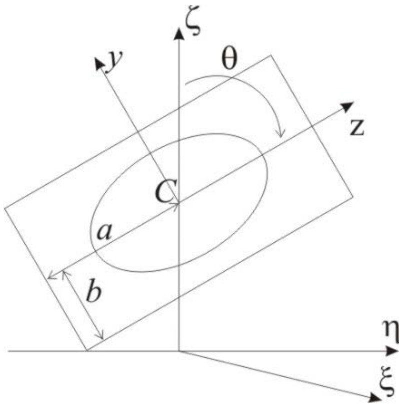

Figure 1. Cylinder with cavity on a plane.

Redefine coordinate systems $Cxyz$ and $C\xi\eta\zeta$ in such a manner that axes $C\xi$ and Cx are directed along the line of nodes. Height h of center of mass C is related with the nutation angle ϑ:

$$h = a\cos\vartheta + b\sin\vartheta$$

Two other coordinates of C are cyclic, so we introduce projections V_ξ and V_η of center mass velocity onto axes $C\xi$ and $C\eta$, correspondingly.

Suppose that external forces: gravity $M\vec{g}$ and supporting force $\vec{R}$, and resulting moment $\vec{M}_{ext}$ act directly upon the solid body. The filling experiences influence of only the walls of the cavity. As the center of mass of the filling does not move relatively to the body, the interaction between the filling and the body can be reduced to resulting couple of forces with the moment $\vec{M}_{\text{int}}$.

To describe the motion of center of mass C we use the principle of momentum. The corresponding equation looks as following in axes $C\xi\eta\zeta$

$$\left.\begin{aligned} M\ddot{h} &= -Mg + N \\ M\dot{V}_\xi &= -M\dot{\psi}V_\eta - F_{fr\xi} \\ M\dot{V}_\eta &= M\dot{\psi}V_\xi - F_{fr\eta} \end{aligned}\right\}$$

Note that first equation is required only for determination of vertical component N of supporting force $\vec{R}$ (F_{fr} is the horizontal component of $\vec{R}$)

$$N = M\left[g + b\left(\cos\vartheta\ddot{\vartheta} - \sin\vartheta\dot{\vartheta}^2\right) - -a\left(\cos\vartheta\dot{\vartheta}^2 - \sin\vartheta\ddot{\vartheta}\right)\right]$$

Define the friction force F_{fr} as a dry friction force directed against velocity of point P of contact:

$$\vec{F}_{fr} = -f\left|N\right|\vec{V}_P / V_P$$

To describe the motion of the system around the center of mass we use the classical angular momentum theorem.

Angular momentum of cylinder $\vec{G}_1$ is related with angular speed as follows:

$$\vec{G}_1 = \Theta_1\vec{\omega}$$

here Θ_1 is a tensor of inertia of body. Denote the index 1 for solid body and index 2 for liquid filling.

We assumed that state of filling can be described by its instantaneous vortex vector $\vec{\Omega}$. Then it takes sense to consider the motion of the filling as combination of two motions: potential motion with some potential and vortex rotation. In this case the angular momentum of filling G_2 can be presented as follows:

$$\vec{G}_2 = \Theta^*\vec{\omega} + \Theta'\vec{\Omega}$$

here Θ^* is a diagonal tensor of inertia of so called equivalent body, $\Theta' = \Theta_2 - \Theta^*$ is a difference between tensor of inertia of liquid filling and tensor of inertia of equivalent body.

The equation for description of system rotation about center of mass looks as following:

$$\begin{aligned} &\left(\Theta_1 + \Theta^*\right)\frac{d\vec{\omega}}{dt} + \Theta'\frac{d\vec{\Omega}}{dt} + \\ &+\omega \times \left[\left(\Theta_1 + \Theta^*\right)\vec{\omega} + \Theta'\vec{\Omega}\right] = \vec{M}_{ext} \end{aligned} \qquad (1)$$

State of the filling is described by the vortex components satisfying the Helmholtz equations.

$$\begin{aligned} \dot{\Omega}_1 &= \left(1 - \varepsilon_3\right)\omega_3\Omega_2 - \left(1 + \varepsilon_3\right)\omega_2\Omega_3 + \\ &+\left(\varepsilon_2 + \varepsilon_3\right)\Omega_2\Omega_3, \quad (123) \end{aligned} \qquad (2)$$

Parameters ε_i characterize the cavity geometry.

We converted the system (1-2) into the system of two vector equations expressing the change of angular momentum for solid body and filling correspondingly.

$$\left.\begin{aligned} \frac{d\vec{G}_1}{dt} + \vec{\omega} \times \vec{G}_1 &= \vec{\xi} + \vec{M}_{ext} \\ \frac{d\vec{G}_2}{dt} + \vec{\omega} \times \vec{G}_2 &= -\vec{\xi} \end{aligned}\right\} \qquad (3)$$

$$\begin{aligned} \xi_1 &= A_1\left[B'\omega_3\Omega_2 - C'\omega_2\Omega_3 - A'\delta_1 + \right. \\ &\left. +\left(B^* - C^* + B_1 - C_1\right)\omega_2\omega_3\right] / \left(A_1 - A^*\right) + \\ &+\left(C_1 - B_1\right)\omega_2\omega_3 - A^*M_{ext1} / \left(A_1 + A^*\right) \end{aligned}$$

here $A_1 = B_1, C_1$; $A^* = B^*, C^*$; and $A' = B', C'$ are diagonal elements of Θ_1; Θ^*; and Θ' correspondingly in axes $CXYZ$.

Evidently, there exist two mechanisms of interaction between filling and solid body. The first mechanism is related with the pressure of the viscous filling on walls of the cavity. The second mechanism is related with tangential stresses or, in other words, with the friction between the filling and walls of the cavity. Vector $\vec{\xi}$ is responsible for first mechanism of interaction due to the fact that it was received from equations for motion of ideal liquid.

Introduce the vector $\vec{L}_{fr}$ which is responsible for second mechanism of interaction. Assume that the moment $\vec{L}_{fr}$ depends linearly on the difference between the vortex vector of the filling and angular velocity of the body

$$\vec{L}_{fr} = \sigma\left(\vec{\Omega} - \vec{\omega}\right)$$

Here σ is a tensor which looks as follows in axes $CXYZ$:

$$\sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

Then the terms corresponding to internal friction moments are phenomenologically introduced to right-hand sides of the equations. After that, system (3) looks as follows

$$\left.\begin{aligned} &\frac{d\vec{G}_1}{dt} + \vec{\omega} \times \vec{G}_1 = \vec{\xi} + \vec{L}_{fr} + \vec{M}_{ext} \\ &\frac{d\vec{G}_2}{dt} + \vec{\omega} \times \vec{G}_2 = -\vec{\xi} - \vec{L}_{fr} \end{aligned}\right\}$$

As a result, we received a system of 9 equations with 9 variables: $\vartheta, \omega_x, \dot{\psi}, \dot{\varphi}, \Omega_x, \Omega_y, \Omega_z, V_\xi, V_\eta$.

Assume that cavity is ellipsoidal and its equation in axes $CXYZ$ looks as following

$$\left(x_1^2 + x_2^2\right)/a_1^2 + x_3^2/a_3^2 = 1$$

Then we have a set of 11 dimensional parameters: $M_1, a, b, A_1, C_1, M_2, a_1, a_3, \sigma_1, \sigma_3, f$.

2.1 Dissipativity condition

Like for any mechanical system with friction (including internal friction), the energy in this system must dissipate. The full mechanical energy of system is given by the following relation:

$$E = M\left(V_\xi^2 + V_\eta^2 + V_\zeta^2\right)/2 + \\ +\left[A\dot{\vartheta}^2 + A\dot{\psi}^2\sin^2\vartheta + C\left(\dot{\varphi} + \dot{\psi}\cos\vartheta\right)^2\right]/2 + \\ +T_{liq} + \Pi_b$$

Here $T_{liq} = \left(\Omega\Theta'\Omega + \omega\Theta^*\omega\right)/2$ is kinetic energy of the liquid filling, $\Pi_b = Mg\left(a\cos\vartheta + b\sin\vartheta\right)$ is potential energy of gravity.

Energy time derivative is

$$\frac{dE}{dt} = -\sigma_1\left[\frac{A_1 + A^*}{A_1}\left[\left(\Omega_1 - \omega_1\right)^2 + \left(\Omega_2 - \omega_2\right)^2\right]\right] - \\ -\sigma_3\frac{C_1 + C^*}{C_1}\left(\Omega_3 - \omega_3\right)^2 + \left(\vec{V}_P, \vec{F}_{fr}\right)$$

Scalar product $\left(\vec{V}_P, \vec{F}_{fr}\right)$ provides the energy dissipation related with the external friction between the body and the rough plane. Necessary and sufficient conditions for dissipativity of the suggested internal friction model are as follows:

$$\sigma_1 > 0, \quad \sigma_3 > 0,$$

2.2 Trajectories of dynamic system

Compare trajectories of developed dynamic system with trajectories of real vessel containing liquid filling. To this effect we performed the computer simulation of system and compared results of numerical simulation with results of natural experiments [Samsonov, 1982].

In [Samsonov, 1982] the motion of aluminium cylinder leaned by circle edge upon plane wooden surface was registered. Cylinder has cylindrical cavity filled by mixture of water and glycerin. Two characteristic types of cylinder behavior were registered during tests. Under relatively small initial spin (about 5000 rpm) cylinder performed monotonic inclining down to some specific angle ϑ_* and then transferred to regime of precession-rolling and then fell down. If initial spin was greater in 1.5-2 times, cylinder first inclined down to angle greater than ϑ_*, then cylinder raised up to the angle ϑ_*, and then fell down. So, under greater initial spin the nutation angle ϑ of the cylinder was non-monotonic time function.

We aimed the numerical calculation of dynamic equations for checking the fact if this system has trajectories with difference depending on initial conditions in monotony of behavior of the nutation angle ϑ.

For computer simulation, the following values of parameters were chosen: a = 2,2 *cm*, b = 1,2 *cm*, C_x = 0,7 *cm*, C_z = 1,4 *cm*, M_1 = 45,9 *g*, A_1, = 164,9 gcm^2, C_1, = 37,2 gcm^2.

Estimations are given for rest of parameters. The coefficient *f* of friction between aluminium and other materials known from literature is from 0.1 to 0.9. For calculation we've chosen *f* = 0.1 taking into account the fact that during observed experiment the permanent contact of cylinder with support surface was not guaranteed. Mass-geometry parameters of the filling are determined from the assumption that the cavity is spherical with radius *R* = 0.88 *cm*.

Model parameter $\sigma_1 = \sigma_3 = \sigma$ depends on glycerin viscosity. The nature of this dependence is unknown. In this context, the purpose of simulation is redefined: to determine the range of σ, for which the increasing initial values of $\dot{\varphi}$ and $\dot{\psi}$ violates the monotony of angle ϑ.

Calculation is carried out for two sets of initial conditions:

I. $\begin{vmatrix} \vartheta_0 = 0.6, \omega_{x0} = \Omega_x = 0, \dot{\psi}_0 = 148\, rad/\sec, \\ \dot{\varphi}_0 = 400\, rad/\sec, \Omega_{20} = \dot{\psi}_0 \sin\vartheta, \\ \Omega_{30} = \dot{\psi}_0 \cos\vartheta + \dot{\varphi}_0, V_{\xi 0} = V_{\eta 0} = 0 \end{vmatrix}$

II. $\begin{vmatrix} \vartheta_0 = 0.6, \omega_{x0} = \Omega_x = 0, \dot{\psi}_0 = 222\, rad/\sec, \\ \dot{\varphi}_0 = 600\, rad/\sec, \Omega_{20} = \dot{\psi}_0 \sin\vartheta, \\ \Omega_{30} = \dot{\psi}_0 \cos\vartheta + \dot{\varphi}_0, V_{\xi 0} = V_{\eta 0} = 0 \end{vmatrix}$

Two ranges of parameters σ are found satisfied to simulation task: 1) $\sigma \approx 50$; 2) $\sigma \approx 450$.

For convenience of comparison of calculations with results of other works we project trajectories onto phase plane *(u,v)*, where

$$u(t) = \partial T / \partial \dot{\varphi} = C_1 \dot{\varphi} + C_1 \dot{\psi} \cos\vartheta$$

$$v(t) = \partial T / \partial \dot{\psi} = \dot{\psi}(A_1 \sin^2\vartheta - C_1 \cos^2\vartheta) + C_1 \dot{\varphi} \cos\vartheta + eA' \sin\vartheta + A^* \dot{\psi} \sin^2\vartheta$$

u(t) is projection of the angular momentum of system onto axis *Cz*, *v(t)* is projection of the angular momentum of system onto axis C_ζ.

Curves *u(v)* are shown in Fig. 2 for values $\sigma = 50$ and $\sigma = 450$. Curves appearance qualitatively agrees with curves predicted in [Samsonov, 1985].

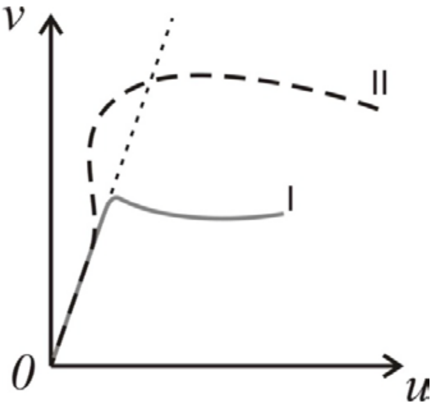

Figure 2. Phase trajectories of dynamic system

For small values of parameter σ ($\sigma \ll 50$) the curve *u(v)* looks as type I both for the first and the second set of initial conditions. For $50 \ll \sigma \ll 450$ curves *u(v)* look as type II for both sets of initial conditions. For σ essentially exceeding 450, curves *u(v)* look as type I for both cases. But in neighborhood of $\sigma = 50$ and $\sigma = 450$ curves for first set of initial conditions look as type I, and for second set of initial conditions, as type II.

Initial spin of liquid during nature experiments is unknown value. We carried out additional special calculations for wide range of initial filling spin value that showed that this value does not influence significantly the behavior of trajectories.

Thus, numerical simulation indicates two ranges of values of parameter σ for which there exists qualitative correspondence between behavior of the modeling dynamic system and of the physical object. It is necessary to carry out additional study to determine, which range really corresponds to viscosity of liquid filling used in natural experiment

2.3 Connection between model coefficient and filling viscosity

The internal friction coefficient is undoubtedly related with the viscosity μ of the filling. It would be rather useful to describe this relationship. Evidently, both the coefficient and viscosity influence on the moment that acts from the filling upon walls of the cavity. Since the liquid performing a regular precession is not at rest relative to the body, it is convenient to use this steady motion for construction of the relationship between the moment and the viscosity μ. From the results of experiments [Mark, 1974; Miller, 1981] and theoretical analysis [Kazmerchuk, Samsonov, 1995] it is known that the axial projection of the moment is a non-monotonic function of viscosity, and each moment value corresponds to two viscosity values.

Now we describe the procedure for making the relation between the internal friction coefficient and the filling viscosity, using formulae from [Kazmerchuk, Samsonov, 1995] for the axial projection M_z of the moment $\vec{M}_{\text{int}}$ of interaction between the filling and the cavity walls.

In this work there were obtained explicit formulae for the particular cases of small and large Reynolds numbers:

$$\text{Re} \gg 1 \quad |M_z| = \left|8\sqrt{2}\pi\rho\dot{\psi}^2 C_x^4 C_z \sin^2\vartheta / \sqrt{\text{Re}}\right| \tag{4}$$

$$\text{Re} \ll 1 \quad |M_z| = \left|\pi\rho\dot{\psi}^2 C_x^4 C_z \sin^2\vartheta\, \text{Re}/6\right| \tag{5}$$

here $\text{Re} = \dot{\varphi} C_x^2 \rho / \mu$, ρ is the liquid density, C_x and C_z are the dimensions of the cavity.

Consider regular precession of the body-vessel with spherical cavity filled with fluid. In this case the equations describing the change of the vortex vector components can be solved analytically, and a finite formula can be obtained for the axial projection of moment $\vec{M}_{\text{int}}$. Expectedly, the axial projection of the moment is a non-monotonic function of the friction parameter, and each moment value (except the maximal one), corresponds to two friction parameter values.

The expression for the axial projection of $\vec{M}_{\text{int}}$ looks as follows:

$$M_z = \sigma_3(\Omega_z - \omega_z) = = -2M_2 R^2 \sigma\dot{\varphi}\sin^2\vartheta\dot{\psi}^2 / 5(\sigma^2 + \dot{\psi}^2) \tag{6}$$

Two formulae (for small and large Reynolds numbers) follow from (4-6) for relation between viscosity μ and internal friction parameter σ

For small Re (curve l_1, Fig. 3):

$$\mu = \pi\rho^2 R^7 (25\sigma_3^2 / 4M_2^2 / R^4 + \dot{\psi}^2) / 6\sigma_3$$

For large Re (curve l_2, Fig. 3):

$$\mu = \sigma_3{}^2 \dot{\varphi}^3 / \left[128\pi^2 \rho C_x^6 C_z^2 (\sigma^2 + \dot{\psi}^2)^2\right]$$

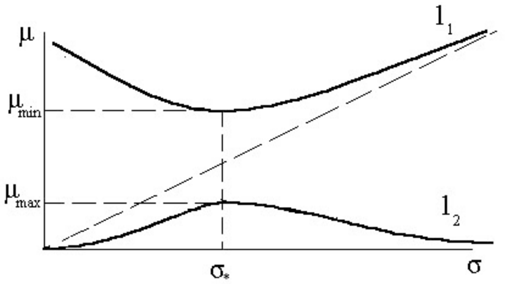

Figure 3. Relation between filling viscosity and model coefficient of internal friction.

In both cases each value of μ corresponds to two values of σ. These curves attain extrema $\mu_{\min}=5\rho R^2\dot{\psi}/8$, $\mu_{\max}=\rho R^2\dot{\varphi}^3/\left(900\dot{\psi}^2\right)$ at one specific value $\sigma_*=2/5\dot{\psi}M_fR^2$. It would be natural to suppose that there exists a monotonic dependence between the parameter σ and the viscosity. In [Savchenko, Samsonov, Sudakov, 1988] the experiments [Miller, 1981] were evaluated, and it was shown that for $\sigma\to\infty$ the relationship between σ and viscosity is linear, while for small values of the viscosity $\sigma\sim\sqrt{\mu}$.

As the relationship between σ and μ is linear for σ tending to infinity ($\sigma\to\infty$), only the curve part l_1 does for values of σ that are much greater than σ_*. So, we should not consider the curve part l_2 for such σ. But for small values of μ we have to use the curve part l_2 for σ much smaller than σ_*.

3 Conclusion

The mathematical model of motion of cylinder with axisymmetric cavity filled by viscous filling and leaned by circle edge onto rough plane is constructed. The verification of dissipativity of system is performed.

Results of numerical simulation of model equations are in qualitative agreement with known experimental data.

The procedure is developed for establishing correlation between the model internal friction coefficient σ and the fluid filling viscosity μ. Study testifies that it is necessary to obtain experimental data that would be sufficient for establishing quantitative correspondence between μ and σ.

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