Geometrical Analysis of Solutions of Two Dimensional Periodic Dynamical Systems

Bruno ROSSETTO[1] and Jean-Marc GINOUX[1]

[1]Toulon University Institute of Technology, BP 20132, 83957, LA GARDE Cedex (France)

rossetto@univ-tln.fr, ginoux@univ-tln.fr

Abstract.

The aim of this paper is to bring to light the properties provided to the phase plane by a generic two dimensional periodic autonomous dynamical systems (PADS) vectorfield. An associated periodic parameters linear equation (APPLE) is defined in each point of the phase plane. It is shown that the local behavior of the initial PADS trajectories is related to the value the Floquet - Liapunov exponents of this APPLE. A method to compute the Floquet - Liapunov exponent value without integration is used. So, it is possible to predict some characteristic patterns of trajectories as funneling, resonance, period doubling, sensitivity to initial conditions. Moreover, the equation of a manifold periodically crossed by the solutions is carried out. The method is applied to periodic Van der Pol and Duffing equations.

1. Introduction

1.1. General presentation. Most of dynamical systems studied since several decades have constant parameters and it is well known that some simple and generic differential equations exhibit very complex solutions including chaotic attractors. The simplest and the most famous example leading to very different and very complex situations is the Chua model, (E. Bilotta and P. Pantano [2008]). The introduction of time varying coefficients provides an additional degree of complexity. Nevertheless it is impossible to avoid periodic parameters to study, for example, the population evolution of species interacting in an open environment, because particularly the birthrates depend on temperature and photometry which evolve daily and yearly. Other models used in biology to study natural rhythms, or in meteorology to take in account the time variations of physical parameters, or in electronics, involve periodic coefficients.

This paper deals on generic two dimensional parametric autonomous dynamical systems (PADS). Tracking the trajectory would need more and more precise numerical and analytical tools and does not help so much to understand the global solution behavior. The method proposed in this work is based on the properties conferred by the velocity vectofield to each point of the phase plane. At first, some features of the solutions are deduced from the values of the Floquet-Liapunov exponents of an associated periodic parameter linear equation (APPLE). In some domain of the phase plane, the solutions exhibits characteristic patterns which are related to the value of Floquet-Liapunov exponents, as funneling, resonance, period doubling, sensitivity to initial conditions. Then, an associated constant coefficient equivalent equation (ACCES) is defined so as its characteristic exponents are the Floquet-Liapunov of the APPLE. Former works are used to establish the equation of an invariant manifold of the ACCES which is periodically crossed by the trajectories of the initial PADS. The last part is devoted to applications of the method to Van der Pol and Duffing equations.

1.2. **The model.** This work deals on periodic autonomous dynamical system (PADS) :

$$\begin{cases} \dot{x} = f(x,y,t) \\ \dot{y} = g(x,y,t) \end{cases} \tag{1}$$

where the mappings $f(x,y,t): \mathbb{R}^3 \to \mathbb{R}, (x,y,t) \mapsto f$ and $g(x,y,t): \mathbb{R}^3 \to \mathbb{R}, (x,y,t) \mapsto g$, are continuous and derivable with respect to x, y and t in $\mathbb{R}^3$, and are T-periodic with respect to t with the same period T :

$$f(x,y,t+nT) = f(x,y,t) \text{ and } g(x,y,t+nT) = g(x,y,t) \text{ , } \forall t \in \mathbb{R} \text{ and } n \in \mathbb{Z} \tag{2}$$

The mappings $f(x,y,t)$ and $g(x,y,t)$ are supposed to satisfy the condition of existence and unicity of solutions of (1).

2. Local properties of solutions

2.1. **Associated periodic parameter linear equation (APPLE)**. Let $\delta x(t)$ and $\delta y(t)$ be small spatial variations around $x(t)$ and $y(t)$. The locally associated periodic parametric linear equation (APPLE) is defined as

$$\begin{bmatrix} \dot{\delta x} \\ \dot{\delta y} \end{bmatrix} = J_t(x,y) \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} \tag{3}$$

where $J_t(x,y): \mathbb{R}^3 \to \mathbb{R}^4, (x,y,t) \mapsto J_t(x,y)$ is the Jacobian matrix of the PADS in (x,y). $J_t(x,y)$ is as well T-periodic with respect to t.

2.2. **The Floquet theory**. According to the Floquet - Liapunov theorem, the solution of APPLE is a linear combination of two *modes* :

$$\begin{bmatrix} \delta x(t) \\ \delta y(t) \end{bmatrix} = e^{\mu_1 t} \begin{bmatrix} \chi_1(t) \\ \psi_1(t) \end{bmatrix} + e^{\mu_2 t} \begin{bmatrix} \chi_2(t) \\ \psi_2(t) \end{bmatrix} \tag{4}$$

where $\mu_k \in \mathbb{C}$ is the Floquet - Liapunov exponent and $\chi_k(t)$ and $\psi_k(t)$: $\mathbb{R} \to \mathbb{R}, t \mapsto \chi_k(t)$ and $\psi_k(t)$, are T-periodic mappings, continuous and derivable in $\mathbb{R}^3$ with respect to t, k = 1, 2. Of course, $\mu_k, \chi_k(t), \psi_k(t)$ depend on (x, y). There are values of (x, y) for which one of the Floquet - Liapunov exponent, for example μ_2, has a larger negative real part. Now, the variation of the real part of the Floquet-Liapunov exponent is smooth with respect to the coordinates (x, y) of the location where they are computed. Then, in the corresponding domain of the phase plane, the associated mode vanishes and the APPLE has therefore a *monomodal* solution related to μ_1. The first aim of this work is to find the locus in the phase plane (x, y) where the behavior of the solution of the PADS tends to the remaining mode of APPLE.

2.3. Proposition 1. Let $[\delta x(t), \delta y(t)]^T$ such a monomodal solution of APPLE related to the Floquet - Liapunov exponent μ :

$$\begin{bmatrix}\delta x(t)\\ \delta y(t)\end{bmatrix} = e^{\mu t}\begin{bmatrix}\chi(t)\\ \psi(t)\end{bmatrix} \tag{5}$$

For any point (x, y) of the phase plane, we consider the time t_0, if it exists, such as the corresponding solutions follow the initial conditions

$$\dot{\chi}(t_0) = 0 \text{ and } \dot{\psi}(t_0) = 0 \tag{6}$$

Then, μ is an eigenvalue of $J_{t_0}(x, y)$ and the related eigenvector is

$$\begin{bmatrix}\chi(t_0)\\ \psi(t_0)\end{bmatrix} = \begin{bmatrix}\chi(t_0 + nT)\\ \psi(t_0 + nT)\end{bmatrix} \tag{7}$$

In other words, the following relationship holds

$$J_{t_0}\begin{bmatrix}\chi(t_0)\\ \psi(t_0)\end{bmatrix} = \mu\begin{bmatrix}\chi(t_0)\\ \psi(t_0)\end{bmatrix} \tag{8}$$

Sketch of proof. The derivative of (5) consists of two terms. According the hypothesis (6), one of them is null. For a more complete demonstration, see B. Rossetto and Y. Zhang, 2009.

2.4. Different patterns of PADS solutions in the phase plane. Let us consider the domain of the phase plane where the solution of APPLE is monomodal. Such a domain can be large because the spatial variation of the Floque-Liapunov exponent is smooth. The solutions are funneled if its real part is negative, sensitive to initial conditions if positive. According the value of the imaginary part, period doubling or resonances can be observed.

The exact value of the Floquet-Liapunov exponent of the APPLE is computed using a fast algorithm, without integration (B. Rossetto, 2006)

3. Associated constant coefficient equivalent system (ACCES)

3.1. Definition. The ACCES is the non linear dynamical system with constant coefficients defined on $t = t_0$

$$\begin{cases} \dot{x} = f(x, y, t_0) \\ \dot{y} = g(x, y, t_0) \end{cases} \tag{9}$$

where the mappings $f(x, y, t_0)$ and $g(x, y, t_0)$: $\mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto f$ and g, are the same as in the definition of the initial periodic dynamical system (1) for $t = t_0$.

3.2. Proposition 2. Invariant manifold of the ACCES. The manifold $\phi(x, y) = 0$ of the ACCES is locally defined as the locus in the phase plane $(0, x, y)$ where we can find a value of t_0 such that the Floquet - Liapunov exponent of the monomodal solution of APPLE μ is a *real* eigenvalue of the Jacobian matrix $J_{t_0}(x, y)$. Then the equation of the manifold $\phi(x, y) = 0$ is given by

$$\frac{f(x,y,t_0)}{g(x,y,t_0)} = \frac{\chi(t_0)}{\psi(t_0)} \tag{10}$$

Moreover, on this manifold, the periodic part $\chi_k(t)$ and $\psi_k(t)$ of monomodal solutions of APPLE verify $\dot{\chi}(t_0) = 0$ and $\dot{\psi}(t_0) = 0$.

Proof. This proposition uses former results to work out the manifold equation of autonomous dynamical systems (Rossetto & *al.* [1998] and B. Rossetto and Y. Zhang, 2009).

3.3. The flow curvature theory. According another simple and general way, called the flow curvature method [Ginoux, 2009], the manifold $\phi(x, y) = 0$ of ACCES is defined as the location of points on which the curvature is null. This theory applies for n^{th}-order dynamical systems. In this case, if $V(t)$ is the velocity and $\Gamma(t)$ the acceleration of the ACCES system, according the flow curvature method, the invariant manifold $\Phi(x, y) = 0$ of this system has a null local curvature. For second order systems, this implies:

$$\Gamma(\mathrm{t}_0) \wedge V(\mathrm{t}_0) = 0 \tag{11}$$

Thus, the acceleration of the motion along the manifold is collinear to the eigenvector (7). The attractive part of this manifold, defined by $V(\mathrm{t}_0).grad(\phi) > 0$, is an *invariant manifold* of ACCES.

3.4. Proposition 3. The manifold crossing problem. The trajectories of PADS cross the invariant manifold $\phi(x, y) = 0$ of ACCES at $t_0 + nT$ with a null transversal acceleration and, therefore, a maximal transversal velocity.

Proof. The acceleration at t_0 or $t_0 + nT$ is collinear to the velocity, itself collinear to the eigenvector. Thus the transverse component of the acceleration is null (B. Rossetto and Y. Zhang, 2009).

3.5. Proposition 4. The trajectory inflexion problem. If $Re\{\mu_2\} < 0$ and $Im\{\mu_1\} \neq 0$, then the local curvature of the trajectory of PADS in the phase plane changes on the manifold defined by $Re\{\mu_1\} = 0$. In other words, there is an inflexion point on this manifold while the sign of $Re\{\mu_1\}$ changes.

Sketch of proof. If $Re\{\mu_1\} > 0$ (resp. < 0), the curvature of the trajectory is positive (resp. negative). There is an inflexion point for $Re\{\mu_1\} = 0$.

4. Examples

4.1. The Fig. 1 (parametric resonance), Fig. 2 (funneling and sensitivity to initial conditions) and Fig. 3 (periodic solution) concerns the **Van der Pol parametric periodic system** :

$$\begin{cases} \dot{x} = f(x,y,t) = 20(-\frac{x^3}{3} + x + y\ (1 + a_1 \cos\left(2\pi \frac{t}{T}\right))\ ) \\ \dot{y} = g(x,y,t) = -x \end{cases} \qquad (12)$$

4.2. The Fig. 4 shows the inflexion locus of the **parametrically driven double-well Duffing equation** :

$$\begin{cases} \dot{x} = f(x,y,t) = y \\ \dot{y} = g(x,y,t) = x - x^3 - 0.2\ y + a_1 x \cos\left(2\pi \frac{t}{T}\right) \end{cases} \qquad (13)$$

5. References

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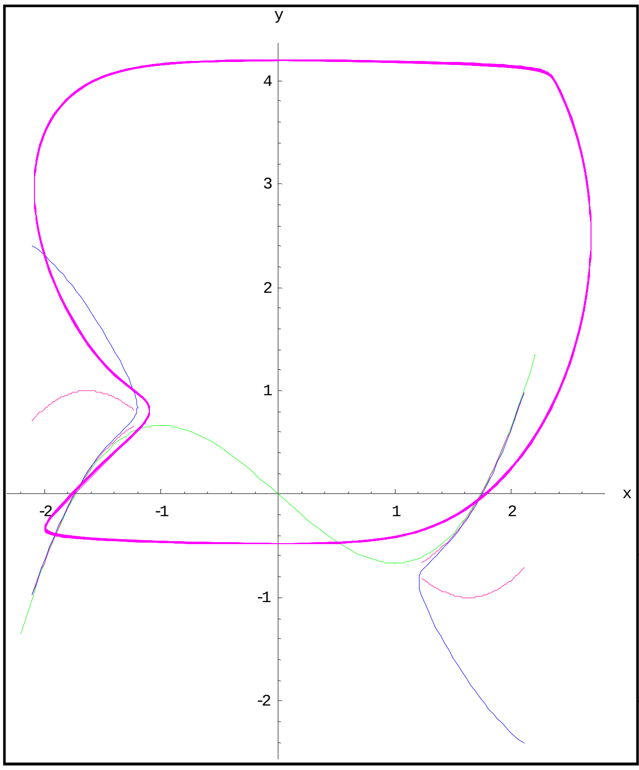

Fig. 1. Periodic Van der Pol equation $a_1 = 1.333\,333\,33$; $\mathrm{T} = 5$ (resonance). In green : the cubic, blue : the slow manifold, red : the manifold periodically crossed by the trajectories.

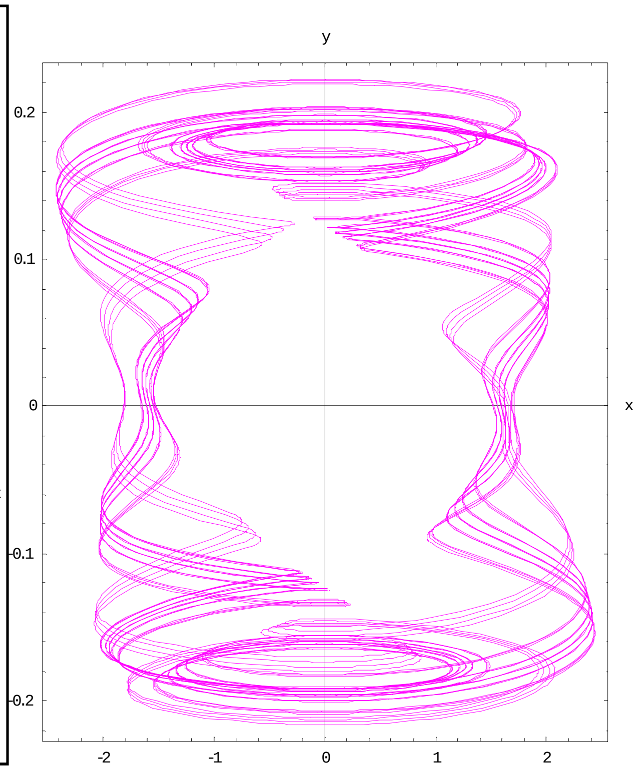

Fig. 2. Periodic Van der Pol equation $a_1 = 30$; $\mathrm{T} = 0.08929$ (funneling).

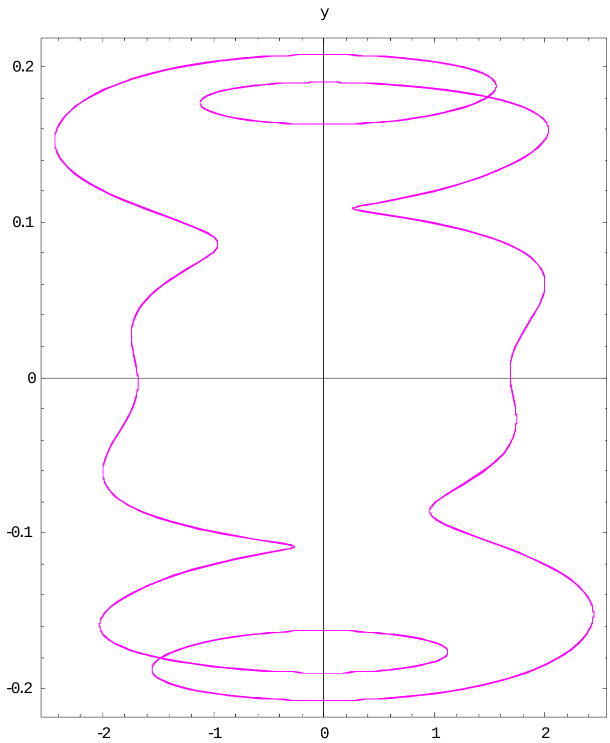

Fig. 3. Periodic Van der Pol equation $a_1 = 30$; $\mathrm{T} = 0.08929$ (periodic solution).

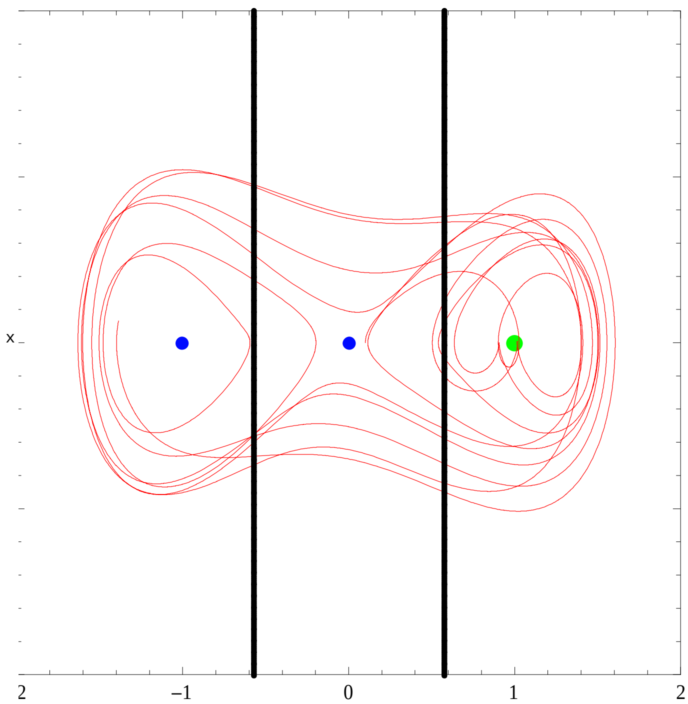

Fig. 4. Parametrically driven double-well Duffing equation. The locus of inflexion points $x = \pm 0.57595$ only depends on x.