Chaotic attractors and bifurcations in double-diffusive convection

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Abstract. Transition to chaotic regimes and development of stochastic motion in plain layer of a salt solution are investigated. Sequence of bifurcations from stationary motions to stochastic motions is demonstrated. We found that the attractor has the structure of a Mobius band in chaotic regimes. With the help of Poincare sections and Poincare maps we show modifications of the attractor with increase of supercriticality. First, Poincare map can be represented as a one-valued function. With the growth of supercriticality Poincare map remains one-dimensional but now it has many minima and self-intersections so it can't be approximated by some function. With the help of Lyapunov exponents we show the divergence of trajectories on the attractors. Relative residual of the initial Navier-Stokes equations is calculated for all the numerical solutions, so we can affirm that the numerical solutions almost exactly represent the genuine solution (the third order of accuracy) and properties of the attractor adequately correspond to the initial model. The convergence of Bubnov-Galerkin method is demonstrated with the help norms of kinetic energy and dissipation function. Influence of boundary conditions and differences in two-dimensional and three-dimensional cases are discussed.
Keywords: Strange attractors, Convection, Double-diffusive convection, Stochastic regimes.

At present time there are many works devoted to transition to chaos in dynamical systems. A strong impulse to this domain was made by fundamental works by Ruelle and Takens [1] and famous example of Lorenz [2], which is obtained by approximation of Navier-Stockes equations with the first harmonics after application of Bubnov-Galerkin method. Despite the beauty and importance of Lorenz attractor, it has nothing to do with the initial physical problem, all the properties where were thrown out with the higher harmonics. Till today we don't have many investigations of attractors in hydrodynamic systems with full Navier-Stockes equations. In the present work properties of the attractors are investigated for double-diffusive convection. During the calculations we estimated the relative residual after substitution of the solution to the initial system of equations. The relative residual was about 10^{-3}. The problem considered here can have application not only to the salt solution but also to some other problem with two types of diffusivities (as was discussed in [3]).

The traditional geometry in which convective motions have been quantitatively analyzed [3] confines the fluid between two infinite horizontal planes, heated, and in this case also salted, from below. Let us consider a fluid which occupies the space between two infinite horizontal planes separated by distance H, heated and salted from below. The upper plane is maintained at temperature T_0 and salinity S_0 the lower plane at temperature T_1 and salinity S_1. Both planes are assumed to be stress free and perfect conductors of heat and salt. We restrict attention to two dimensional motion, dependent only upon one horizontal co-ordinate x and the vertical co-ordinate z. Let us non-dimensionalize all lengths with respect to H and time with respect to H^2/k_T, where k_T is the thermal diffusivity. For the components of velocity the dimensionless stream function ψ can be introduced: $v_x = \frac{\partial\psi}{\partial z}, v_z = -\frac{\partial\psi}{\partial x}$; $(rot\bar{v})_y = \psi_{xx} + \psi_{zz} = \Delta\psi$. Hereafter coordinates x and z and time t are supposed to be dimensionless. Instead of temperature and salinity, dimensionless variables t and s are introduced: $T = T_0 + (T_1 - T_0)(1 - z + \tau)$, $\quad S = S_0 + (S_1 - S_0)(1 - z + s)$.

So, we can write the governing Boussineq equations of motion as:

$$\frac{1}{\sigma}\Delta\frac{\partial\psi}{\partial t} + R_T\frac{\partial\tau}{\partial x} - R_S\frac{\partial s}{\partial x} - \Delta^2\psi = \frac{1}{\sigma}J(\psi, \Delta\psi), \tag{1}$$

$$\frac{\partial\tau}{\partial t} + \frac{\partial\psi}{\partial x} - \Delta\tau = J(\psi, \tau), \tag{2}$$

$$\frac{\partial s}{\partial t} + \frac{\partial\psi}{\partial x} - \kappa\Delta s = J(\psi, s). \tag{3}$$

In this system four dimensionless parameters are introduced: thermal Rayleigh number $R_T = \frac{\alpha g h^3 (T_0 - T_1)}{k\nu}$, salinity Rayleigh number, $R_S = bg\frac{h^3(S_1 - S_0)}{k_s \nu}$, the ratio of the coefficients of salt diffusion and temperature $k = k_S/k_T$, where k_S is saline diffusivity, and the Prandtl number $\sigma = \nu/k$. Jacobian is determined by formula $J(f,g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}$. For $x = 0$ and $z = 0$ the Rayleigh boundaries conditions of absence of tangential viscosity forces are taken. So, the full system of boundary conditions has the form: $\psi = \frac{\partial^2 \psi}{\partial z^2} = \tau = s = 0 \quad (z = 0, 1)$ at $z = 0, 1$. Results of some numerical computations of the boundary problems, periodic with respect to x, can be found in fundamental work [3].

Let us seek the solution of the system (1-3) by Bubnov-Galerkin method in the form satisfying boundary conditions:

$$\Psi = \sum sin(j\pi z)\Psi_j(x,t);$$
$$T = \sum sin(j\pi z)T_j(x,t);$$
$$S = \sum sin(j\pi z)S_j(x,t);$$

The length of the period along axis x corresponds to the length of the wave for which the static solution loses stability with the minimal Rayleigh number $R_T = 27p^4/4$. Corresponding dynamic systems (accordingly to the terminology of the authors [4]) belong to the systems of the hydrodynamic type possessing the property of decreasing of the phase volume with time growth.

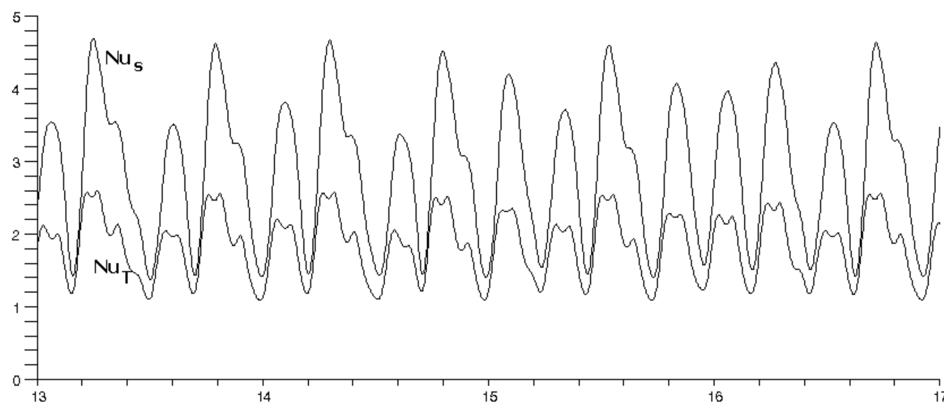

Fig. 1. Saline and thermal Nusselt numbers for $R_T = 11000$; $R_S = 10000$; $\sigma = 1$.

On Figure 1 stochastic regime for moderate supercriticality is demonstrated using Nusselt numbers (flow of temperature and salt through the surface). This figure is given here to provide insight for what kind of chaotic motion we will present on pictures of attractors. One of the most important characteristics of turbulent motions and fundamental notion in dynamical systems is sensitivity on the initial values. Figure 2 shows how two close trajectories on the attractor diverge in a stochastic regime.

Bubnov-Galerkin method has a great benefit over finite element method for this type of problems. We can estimate relative residual of the initial model after the substitution of the numerical solution to Boussineq equations of motion so we can speak about proximity of the numerical solution to the genuine solution. For finite elements method this is not easy because we must calculate time derivatives for evaluation of the residual and this calculation increases the inaccuracy of the residual. On figure 4 we can see an example of absolute residual distribution over physical space. The main forces which serve for the normalization are about 6000. On figure 5 we can see absolute and relative residual behavior with time growth. Figure 6 shows the dependency of the residual on the number of harmonics (maximum of the residual over space and time).

Absolute and relative residual behavior in time Figures 4 and 5 give insight on the structure of the attractor in stochastic regime for $R_T = R_S = 15000$. By investigation of the transversal section we showed

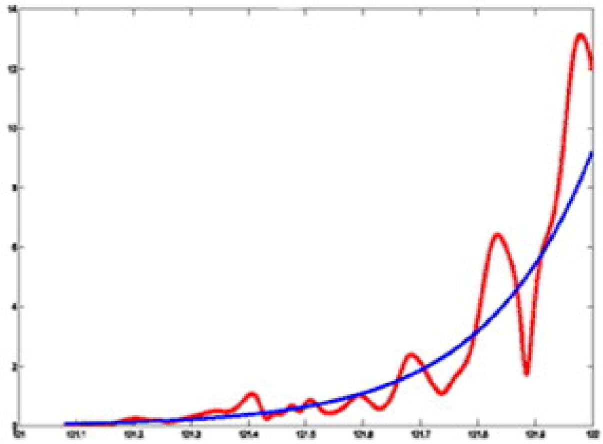

Fig. 2. Lyapunov exponent $\lambda = 5.3276$; $R_T = 15000$; $R_S = 15000$; $\sigma = 1$.

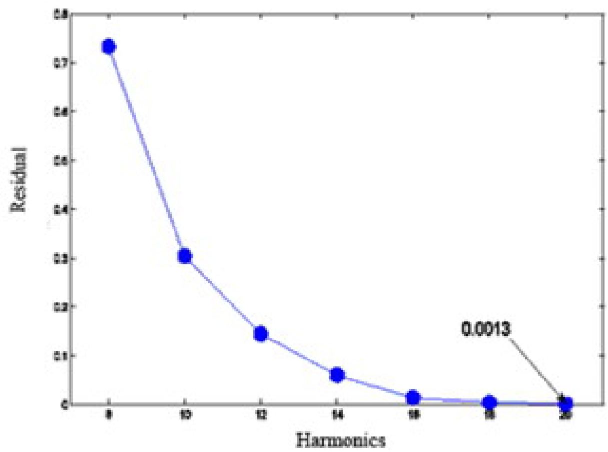

Fig. 3. Dependence of the relative residual on number of harmonics.

that attractor always has the structure of a Mobius band. (Attractor is not exactly two-dimensional and has fractal dimension) Let us consider the evolution of the flow with the growth of thermal Rayleigh number for a fixed salinity Rayleigh number. First we have static distribution of temperature and salinity. Then static regime loses stability and we have the regime of stationary convective rolls. With the further growth of Rayleigh number after loosing of stability of stationary motion limit cycle appears. Then, with the further increase of supercriticality, periodical motion loses stability and we will have the bifurcation of the doubling of the periodical motion.

The projection of limit cycle on plane ψ_{11}, τ_{11} appears to be closed not self-intersected curve, diffeomorphic to a circle up to values of parameter $R_T = 8870$. For this value of parameter bifurcation of period doubling of limit cycle takes place. The cascade of bifurcations can be understood better with the help of succession mapping (mapping of Poincare). Let us transversally (locally) intersect the limit cycle by hyperplane and points of intersection denote M_j, $j = 1, 2, \dots$. For the small perturbations of the dynamic system around the limit cycle we will get linear system with periodic coefficients. Accordingly to the Floquet theorem there exist a linear substitution of variables with periodic matrix which reduces the system with periodic coefficients to the system of linear equations with constant coefficients. Roots of corresponding characteristic equation will be eigenvalues (multiplicators) of the matrix of monodromy of system with periodic coefficients. If all the multiplicators are different and less than one in absolute value, then limit cycle is stable. If one of the multiplicators with the parameter growth traverses -1, then locally monodromy transformation in transversal plane (using the possibility of linear transformation of parameter and variable

) may be represented as follows

$$x_{j+1} = -x_j(1+\lambda) + x_j^2 + \beta x_j^3.$$

For $\lambda < 0$ mapping of succession converges to zero, corresponding to stable limit cycle. For $\lambda > 0$ sequence converges to cycle of two points $x_{1,2} = \pm\sqrt{\lambda/(1+\beta)}$. So for $\lambda > 0$ stable limit cycle of doubled period appears.

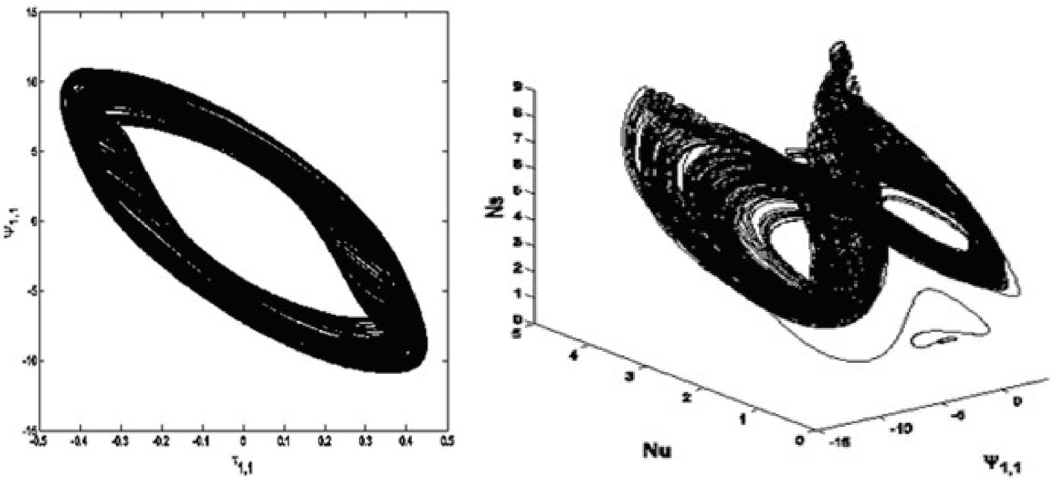

Fig. 4. Strange attractor in (ψ_{11}, τ_{11}) phase plane and in (Nu_S, Nu_T, ψ_{11}) phase plane. $R_T = 15000$; $R_S = 15000$; $\sigma = 1$.

Figure 9 illustrates periodic regime after two bifurcations of period doubling and Figure 10 represents the corresponding Poincare map. In Figure 11 we have Poincare map for stochastic regime after the sequence of bifurcations. In [5] you can find more details on sequence of bifurcations. With the further increase of R_T the reverse process begins, the Mobius band is cut along itself (by cutting the Mobius band we again obtain the Mobius band due to non-orientability of the surface). Finally we get periodic solution.

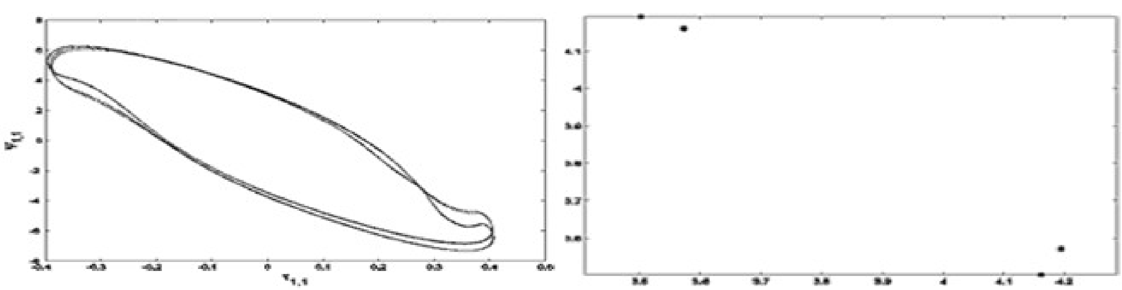

Fig. 5. Periodic regime after two bifurcations of period doubling and corresponding Poincare map. $R_T = 9090$; $R_S = 8000$; $\sigma = 1$.

Appearance of periodic solutions is close to the assertion of the Scharkowsky theorem (see the discussion in [6]). After loosing of stability of this periodic solution the new sequence of bifurcations begins which again leads to attractor in the form of Mobius band, but this time Poincare map can't be represented in the form of a one-valued function and has many minima and self-intersection, so Feigebaum theory [7] can't be applicable any more (Figure 12).

Structure and development of attractors for double-diffusive convection were demonstrated. Attractor has the form of a Mobius band. Positive Liapunov exponent on the attractors was demonstrated. Modifications of Poincare map with the growth of supercriticality were shown. The convergence of Bubnov-Galerkin

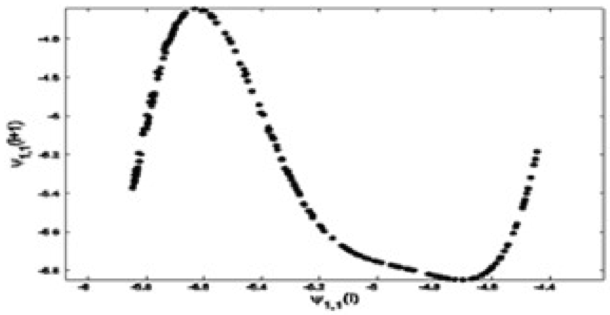

Fig. 6. Poincare map for stochastic regime after the sequence of bifurcations. $R_T = 9110$; $R_S = 8000$; $\sigma = 1$.

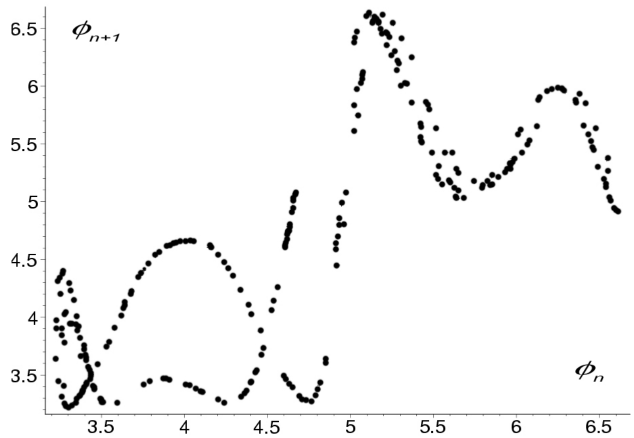

Fig. 7. Poincare map for stochastic regime after the sequence of bifurcations. $R_T = 10850$; $R_S = 10000$; $\sigma = 1$.

method was demonstrated in norms of kinetic energy and dissipation function. Dependence of relative residual on the number of harmonics was shown, so the described properties of the solution adequately correspond to the initial model of double-diffusive convection. Influence of boundary conditions and differences in two-dimensional and three-dimensional cases are discussed.

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