The autoresonance threshold in a system of weakly coupled oscillators *

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We investigate the system of two weakly coupled nonlinear oscillators under a small perturbation

$$
\begin{aligned}
x'' + \omega^2 x &= \varepsilon\alpha_1 xy + \varepsilon(\gamma \exp\{i\varphi\} + c.c.), \\
y'' + (2\omega)^2 x &= \varepsilon\alpha_2 x^2,
\end{aligned}
\tag{1}
$$

here $\varphi = (\omega + \varepsilon\alpha\tau)\theta,\ \tau = \varepsilon\theta$, ε is a small positive parameter, ω, α_1, α_2 and γ are constants.

Our goal is to obtain a solution with an increasing amplitude due to a small oscillating perturbation when $\gamma \neq 0$. There is a standard way to obtain oscillations with an increasing amplitude. We have obtain a new result for two-dimensional system of primary resonance equations. It was shown that the autoresonant phenomenon appears when the amplitude of the perturbations is greater than the threshold value. This threshold value of the perturbation was found explicitly. The mathematical statement of the problem on the existence of threshold is reduced to an existence of asymptotic solutions in the form of power series with respect to t^{-1} as $t \to \infty$. Nonlinear systems with three oscillators were studied in [1] early and constructed bounded solutions. The autoresonant solutions were found for two coupled oscillators [2] and for three ocsillator-interactions [3].

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1 Asymptotic reduction to the system of primary resonance equations

Let us construct the solution for (1) in the complex form

$$x = \mathcal{A}(\tau)\exp\{i\omega\theta\} + c.c., \qquad y = \mathcal{B}(\tau)\exp\{2i\omega\theta\} + c.c.$$

Substitute (**??**) into (1) and average over fast variable θ. It yields

$$\begin{aligned}
\mathcal{A}'(\tau) &= -\frac{i\alpha_1}{2\omega}\mathcal{A}^*\mathcal{B} - \frac{i\gamma}{2\omega}\exp\{i\alpha\tau^2\} + i\varepsilon\mathcal{A}'', \\
\mathcal{B}'(\tau) &= -\frac{i\alpha_2}{4\omega}\mathcal{A}^2 + i\varepsilon\mathcal{B}''.
\end{aligned}$$

Neglecting the terms of order ε and substituting

$$\mathcal{A} = a(\tau)\exp\{i\alpha\tau^2\} + c.c., \quad \mathcal{B} = b(\tau)\exp\{2i\alpha\tau^2\} + c.c.$$

we obtain

$$\begin{aligned}
a'(\tau) &= -2i\alpha\tau a - \frac{i\alpha_1}{2\omega}a^*b - \frac{i\gamma}{2\omega}, \\
b'(\tau) &= -4i\alpha\tau b - \frac{i\alpha_2}{4\omega}a^2.
\end{aligned}$$

Let us simplify system (**??**). Substitution $a(\tau) = \lambda A(t), \quad b(\tau) = \kappa B(t), \quad \tau = \chi t$, where

$$\kappa = \frac{\omega\sqrt{\alpha}}{\alpha_1}, \quad \lambda = \omega\sqrt{\frac{\alpha}{\alpha_1\alpha_2}}, \quad \chi^2 = \frac{1}{\alpha}$$

gives system (2) with $f = \dfrac{\sqrt{\alpha_1\alpha_2}\gamma}{\alpha\omega^2}$.

2 Statement of the problem and result

The long time evolution of amplitudes of oscillating solutions of (1) is reduced to the system of primary resonance equations:

$$\begin{aligned}
A'(t) &= -i\left(2tA + \frac{1}{2}A^*B + f\right), \\
B'(t) &= -i\left(4tB + \frac{1}{4}A^2\right).
\end{aligned} \tag{2}$$

Our goal is to study the behaviour of the solutions of (1) when t approaches infinity.

It is shown there are increasing and bounded solutions. The bounded solutions are studied in detail. It has been determined that periodic perturbation of a system of oscillators leads to the capture into resonance. The asymptotic description and numerical simulations of the phenomenon are presented [2]. An explicit formula for the threshold value of the perturbation has been found. It was found there exist solutions related to autoresonance phenomenon when $|f| \geq 12$.

Algebraic asymptotic solutions of system (2) in the form

$$A(t) = \sum_{k=-1}^{\infty} a_k t^{-k}, \quad B(t) = \sum_{k=-1}^{\infty} b_k t^{-k}, \quad t \to \infty.$$

Theorem 1 *When $t \to \infty$ there exists the solution of system (2) of the form*

$$\begin{aligned} A_2(t) &= -\frac{f}{2}t^{-1} + \frac{if}{4}t^{-3} + \Big(\frac{3f}{8} - \frac{f^3}{512}\Big)t^{-5} + O(t^{-7}), \\ B_2(t) &= -\frac{f^2}{64}t^{-3} + \frac{7if^2}{256}t^{-5} + O(t^{-7}), \end{aligned}$$

When $|f| \geq 12$ and $t \to \infty$ there exist solutions of (2) of the form

$$\begin{aligned} A_1(t) &= -8\Big(\cos(\Psi) + i\sin(\Psi)\Big)t + \frac{f}{4}t^{-1} + O(t^{-3}), \\ B_1(t) &= -4\Big(\cos(2\Psi) + i\sin(2\Psi)\Big)t + (-\frac{f}{4} - 2i)t^{-1} + O(t^{-3}), \end{aligned}$$

here $\sin(\Psi) = \dfrac{12}{f}$.

$$\begin{aligned} A_3(t) &= 8\Big(\cos(\Psi) + i\sin(\Psi)\Big)t + \frac{f}{4}t^{-1} + O(t^{-3}), \\ B_3(t) &= -4\Big(\cos(2\Psi) + i\sin(2\Psi)\Big)t + \Big(-\cos(\Psi)[\frac{f}{4} + \frac{24}{f}] + 2i[1 + \sin^2(\Psi)]\Big)t^{-1} \\ &+ O(t^{-3}), \end{aligned}$$

here $\sin(\Psi) = -\dfrac{12}{f}$.

An oscillating asymptotic solution of (2) in the neighborhood of bounded solution has the form

$$A(t)=a(t)\exp\{-it^2\},\quad B(t)=b(t)\exp\{-2it^2\}.$$

The substitution gives

$$\begin{aligned} ia'(t) &= \frac{1}{2}a^*b+f\exp\{it^2\},\\ ib'(t) &= \frac{1}{4}a^2, \end{aligned}$$

Here we study the behavior of solution of the system in the neighborhood of bounded asymptotic solution (A_2,B_2). Substitution

$$a=A_2\exp\{it^2\}+\alpha,\quad b=B_2\exp\{2it^2\}+\beta, \tag{3}$$

gives a system for α,β

$$i\alpha'=\frac{1}{2}\alpha^*\beta+\frac{1}{2}(A_2^*\beta\exp\{-it^2\}+\alpha^*B_2\exp\{2it^2\}),$$

$$i\beta'=\frac{1}{4}\alpha^2+\frac{1}{2}A_2\alpha\exp\{it^2\}.$$

Theorem 2 *There exists a formal asymptotic solution of (2) in the form of (3) with*

$$\alpha=\sum_{k=0}^{\infty}\alpha_k t^{-k},\quad \beta=\sum_{k=0}^{\infty}\beta_k t^{-k}.$$

This asymptotic solution depends on four real parameters. The leading-order terms of the expansion are determined in terms of elliptic functions.

References

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