OUTPUT-FEEDBACK GLOBAL STABILIZATION OF UNCOUPLED PVTOL AIRCRAFT WITH BOUNDED INPUTS

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Abstract

An output-feedback scheme for the global stabilization of uncoupled PVTOL aircraft with bounded inputs is proposed. The control objective is achieved avoiding input saturation and through the exclusive consideration of system positions in the feedback. To cope with the lack of velocity measurements, the proposed algorithm involves a finite-time observer. With respect to previous approaches, the developed finite-time-observer-based scheme guarantees the global stabilization objective disregarding velocity measurements in a bounded input context. Simulation results corroborate the analytical developments.

Key words

Global stabilization, output feedback, PVTOL aircraft, bounded inputs, finite-time observers

1 Introduction

Since the publication of [Hauser, Sastry and Meyer, 1992], stabilization of the Planar Vertical Take-off and Landing (PVTOL) aircraft has been a subject of special interest in the control community. The design of a suitable algorithm, in such analytical framework, has proven to constitute a challenging task. This is mainly due to the complexities involved by the PVTOL dynamics: highly nonlinear, underactuated, signed (thrust) input. The efforts devoted to such a particular study case have given rise to diverse approaches. For instance, in view of the unstable non-zero dynamics obtained through the application of conventional geometric control techniques, an approximate input-output design procedure dealing with non-minimum phase nonlinear systems has been proposed in [Hauser, Sastry and Meyer, 1992]. Applying a decoupling change of coordinates, global stabilization was proven to be achieved through backstepping under the consideration of input coupling (generally neglected) in [Olfati-Saber, 2002]. A globally stabilizing nonlinear feedback control law that casts the system into a cascade structure was proposed in [Wood and Cazzolato, 2007]. By transforming the system dynamics into a chain of integrators with nonlinear perturbations, global stabilization was also proven to be achieved in [Ye, Wang and Wang, 2007] through a control technique that involves saturation functions. Based on partial feedback linearization and optimal trajectory generation to enhance the behavior and the stability of the system internal dynamics, a nonlinear prediction-based stabilization algorithm was proposed in [Chemori and Marchand, 2008]. An open-loop exact tracking scheme ensuring bounded internal dynamics was developed in [Consolini and Tosques, 2007] through a Poincaré map approach. Further, a path following controller with the properties of output invariance of the path and boundedness of the roll dynamics was proposed in [Consolini, Maggiore, Nielsen and Tosques, 2010].

Other works have considered additional constraints that commonly arise in real applications. For instance, bounded inputs have been considered in [Zavala-Río, Fantoni and Lozano, 2003; López-Araujo, Zavala-Río, Fantoni, Salazar and Lozano, 2010; Ailon, 2010]: in [Zavala-Río, Fantoni and Lozano, 2003], global stabilization was achieved, neglecting the lateral force coupling, through the use of embedded (linear) saturation functions; this approach was further proven to achieve the global stabilization objective under the additional consideration of the lateral force coupling in [López-Araujo, Zavala-Río, Fantoni, Salazar and Lozano, 2010] (for sufficiently small values of the parameter characterizing such a coupling); in [Ailon, 2010], a semiglobal tracking controller was developed involving smooth sigmoidal functions. Furthermore, schemes that achieve the control objective without velocity measurements have been proposed in [Do, Jiang and Pan, 2003; Frye, Ding, Qian and Li, 2010]: in [Do, Jiang and Pan, 2003], the design and analysis procedures were developed under the consideration of a Luenberger-type observer, while in [Frye, Ding, Qian and Li, 2010], finite-time observers are involved. However, these output-feedback approaches were developed disregarding input constraints.

Inspired by the techniques involved in [Frye, Ding, Qian and Li, 2010], this work proposes an output-feedback control scheme for the global stabilization of uncoupled PVTOL aircraft with bounded inputs. First, a state feedback algorithm is presented and proven to globally stabilize the closed-loop system avoiding input saturation. Then, the same algorithm is proven to achieve the global stabilization objective, avoiding input saturation, by replacing the velocity variables by auxiliary states coming from a finite-time observer, which turns out to exactly reproduce the aircraft positions and velocities after a finite-time transient during which the system variables are proven to remain bounded. The finite-time stabilizers considered in this work are generalized versions of those involved in [Frye, Ding, Qian and Li, 2010]. This gives rise to an additional degree of design flexibility that has not only permitted to solve the output-feedback stabilization problem in a bounded input context, but may also be used in aid of performance improvements. Simulation results corroborate the efficiency of the proposed scheme.

2 The PVTOL aircraft dynamics

The PVTOL aircraft dynamics is given by [Hauser, Sastry and Meyer, 1992]:

$$\begin{aligned}\ddot{x} &= -u_1 \sin\theta + \varepsilon u_2 \cos\theta \\ \ddot{y} &= u_1 \cos\theta + \varepsilon u_2 \sin\theta - 1 \\ \ddot{\theta} &= u_2\end{aligned} \tag{1}$$

where x and y are the center of mass horizontal and vertical positions, and θ is the roll angle of the aircraft with the horizon. The control inputs u_1 and u_2 are respectively the thrust and the rolling moment. The constant "-1" is the normalized gravity acceleration. The parameter ε is a coefficient characterizing the coupling between the rolling moment and the lateral acceleration of the aircraft. Its value is generally so small that $\varepsilon = 0$ can be assumed [Hauser, Sastry and Meyer, 1992, §2.4]. Thus, under such a common consideration, in this work we consider the (uncoupled) PVTOL aircraft dynamics with $\varepsilon = 0$, *i.e*

$$\ddot{x} = -u_1 \sin\theta \ , \ \ddot{y} = u_1 \cos\theta - 1 \ , \ \ddot{\theta} = u_2 \tag{2}$$

Under the consideration of bounded inputs, *i.e.* $0 \leq u_1 \leq U_1$ and $|u_2| \leq U_2$ for some constants[1] $U_1 > 1$ and $U_2 > 0$, we state the control objective as being the global stabilization of the system trajectories towards $(x, y, \theta) = (0, 0, 0)$, through a bounded control scheme that only feeds back configuration variables from the PVTOL and avoids input saturation *i.e.* such that $0 < u_1(t) < U_1$ and $|u_2(t)| < U_2, \forall t \geq 0$.

3 Preliminaries

Let $\mathbb{N}$ and $\mathbb{Z}^+$ respectively stand for the set of natural and nonnegative integer numbers. For particular values $m \in \mathbb{N}$ and $n \in \mathbb{Z}^+$, let $\mathbb{N}_m = \{1, \ldots, m\}$ and $\mathbb{Z}_n^+ = \{0, \ldots, n\}$. We denote 0_n the origin of $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, x_i represents its i^{th} element, while $\|\cdot\|$ is used to denote the standard Euclidean vector norm, *i.e.* $\|x\| = \left[\sum_{i=0}^{n} x_i^2\right]^{1/2}$. Let $\mathbb{R}_{>0}^n \triangleq \{x \in \mathbb{R}^n : x_i > 0, \forall i \in \mathbb{N}_n\}$ and $\mathbb{R}_{\geq 0}^n \triangleq \{x \in \mathbb{R}^n : x_i \geq 0, \forall i \in \mathbb{N}_n\}$. Let $\mathcal{A}$ and $\mathcal{E}$ be subsets (with nonempty interior) of some vector spaces $\mathbb{A}$ and $\mathbb{E}$ respectively. We denote $\mathcal{C}^m(\mathcal{A}; \mathcal{E})$ the set of m-times continuously differentiable functions from $\mathcal{A}$ to $\mathcal{E}$, with $\mathcal{C}^0$ the set of continuous functions. Consider a scalar function $\zeta \in \mathcal{C}^m(\mathbb{R}; \mathbb{R})$ with $m \in \mathbb{Z}^+$. The following notation will be used: $\zeta' : s \mapsto \frac{d}{ds}\zeta$, when $m \geq 1$; $\zeta'' : s \mapsto \frac{d^2}{ds^2}\zeta$, when $m \geq 2$; and more generally $\zeta^{(n)} : s \mapsto \frac{d^n}{ds^n}\zeta$, $\forall n \in \mathbb{N}_m$, and $\zeta^{(0)} = \zeta$. We denote $\text{sat}(s)$ the standard saturation function, *i.e.* $\text{sat}(s) = \text{sign}(s) \min\{|s|, 1\}$. In the rest of this section, some definitions and results that underlie the contribution of this work are stated. The proofs of the lemmas and corollaries of this section were thoroughly developed by the authors and will be omitted because of space limitations.

Analogously to the (conventional) case of homogeneous functions and vector fields [Bacciotti and Rosier, 2005; Aeyels and de Leenheer, 2002], the following local homogeneity concept was stated in [Zavala-Río and Fantoni, 2013] in terms of *family of dilations* δ_ε^r defined as $\delta_\varepsilon^r(x) = (\varepsilon^{r_1} x_1, \ldots, \varepsilon^{r_n} x_n)$, $\forall x \in \mathbb{R}^n$, $\forall \varepsilon > 0$, where $r = (r_1, \ldots, r_n)$, with the *dilation coefficients* $r_1, \ldots, r_n$ being positive real numbers.

Definition 3.1. *[Zavala-Río and Fantoni, 2013] Given $r \in \mathbb{R}_{>0}^n$, a neighborhood of the origin $D \subset \mathbb{R}$ is said to be* δ_ε^r-connected *if, for every $x \in D$, $\delta_\varepsilon^r(x) \in D$ for all $\varepsilon \in (0, 1)$. A function $V : \mathbb{R}^n \to \mathbb{R}$, resp. vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, is* locally homogeneous *of degree α with respect to the family of dilations δ_ε^r —or equivalently, it is said to be* locally r-homogeneous *of degree α— if there exists a δ_ε^r-connected open neighborhood of the origin $D \subset \mathbb{R}^n$ —referred to as the* domain of homogeneity*— such that $V(\delta_\varepsilon^r(x)) = \varepsilon^\alpha V(x)$, resp. $f(\delta_\varepsilon^r(x)) = \varepsilon^\alpha \delta_\varepsilon^r(f(x))$, for every $x \in D$ and all $\varepsilon \in \mathbb{R}_{>0}$ such that $\delta_\varepsilon^r(x) \in D$.*

Definition 3.2. *[Bacciotti and Rosier, 2005; Bhat and Bernstein, 2005] Consider an n-th order autonomous system $\Sigma : \dot{x} = f(x)$, where $f : \mathcal{D} \to \mathbb{R}^n$ is continuous on an open neighborhood $\mathcal{D} \subset \mathbb{R}^n$ of the origin and $f(0_n) = 0_n$, and let $x(t; x_0)$ represent the system solution with initial condition $x(0; x_0) = x_0$. The origin is said to be a* finite-time stable *equilibrium of system Σ if it is Lyapunov stable and there exist an*

[1] Notice from the vertical motion dynamics in Eqs. (2) that $U_1 > 1$ is a necessary stabilizability condition, since steady-state achievement implies that the aircraft weight be compensated.

open neighborhood $\mathcal{N} \subset \mathcal{D}$ being positively invariant with respect to Σ, and a positive definite function $T : \mathcal{N} \to \mathbb{R}_{\geq 0}$, called the settling-time function*, such that $x(t; x_0) \neq 0_n$, $\forall t \in [0, T(x_0))$, $\forall x_0 \in \mathcal{N} \setminus \{0_n\}$, and $x(t; x_0) = 0_n$, $\forall t \geq T(x_0)$, $\forall x_0 \in \mathcal{N}$. The origin is said to be a* globally finite-time stable *equilibrium of system Σ if it is finite-time stable with $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$.*

Theorem 3.1. *[Zavala-Río and Fantoni, 2013] Consider the system $\Sigma : \dot{x} = f(x)$ of Definition 3.2 with $\mathcal{D} = \mathbb{R}^n$. Suppose that f is a locally r-homogeneous vector field of degree k with domain of homogeneity $D \subset \mathbb{R}^n$. Then, the origin is a globally finite-time stable equilibrium of system Σ if and only if it is globally asymptotically stable and $k < 0$.*

The proof of Theorem 3.1 has been thoroughly developed in [Zavala-Río and Fantoni, 2013]. A partial version —namely, the sufficiency implication— of this theorem was used in [Frye, Ding, Qian and Li, 2010] to support the result presented therein.

Definition 3.3.

1. *A continuous function $\sigma : \mathbb{R} \to \mathbb{R}$ is said to be:*
   - *(a)* bounded by M *if $|\sigma(s)| \leq M$, $\forall s \in \mathbb{R}$, for some positive constant M;*
   - *(b)* strictly passive *if $s\sigma(s) > 0$, $\forall s \neq 0$;*
   - *(c)* strongly passive *if it is a strictly passive function satisfying $|\sigma(s)| \geq \kappa \big| a \operatorname{sat}(s/a) \big|^\alpha = \kappa \big(\min\{|s|, a\} \big)^\alpha$, $\forall s \in \mathbb{R}$, for some positive constants κ, α, and a.*
2. *A nondecreasing strictly passive function $\sigma : \mathbb{R} \to \mathbb{R}$ being bounded by M, locally r-homogeneous of degree $\alpha > 0$ for some $r > 0$, and locally Lipschitz-continuous on $\mathbb{R} \setminus \{0\}$, is said to be a* homogeneous saturation *(function) for (α, r, M).*
3. *A nondecreasing Lipschitz-continuous strictly passive function $\sigma : \mathbb{R} \to \mathbb{R}$ being bounded by M is said to be a* generalized saturation *(function) with bound M.*
4. *A generalized saturation function $\sigma : \mathbb{R} \to \mathbb{R}$ with bound M is said to be a* linear saturation *(function) for (L, M) if there is a positive constant $L \leq M$ such that $\sigma(s) = s$, $\forall |s| \leq L$. [Teel, 1992]*

For a generalized or homogeneous saturation function $\sigma(s)$, $M^+ \triangleq \lim_{s\to\infty} \sigma(s)$ and $M^- \triangleq -\lim_{s\to-\infty} \sigma(s)$, which are called the limit bounds *of σ, while $\bar{M} \triangleq \max\{M^+, M^-\}$ and $\underline{M} \triangleq \min\{M^+, M^-\}$.*

Observe that $\underline{M} \leq \bar{M} \leq M$, *i.e.* M^+ and M^- does not necessarily have the same value (but could be different), and M is not necessarily equal to $\bar{M}$ (but could be greater).

Remark 3.1. Note that homogeneous and generalized saturation functions are strongly passive. Indeed, let σ be any homogeneous or generalized saturation function. Notice that the strictly passive character of σ implies the existence of a sufficiently small $a > 0$ such that $|\sigma(s)| \geq \kappa |s|^\alpha$, $\forall |s| \leq a$, for some positive constants κ and α, while from its non-decreasing character we have that $|\sigma(s)| \geq |\sigma(\operatorname{sign}(s)a)| \geq \kappa a^\alpha$, $\forall |s| \geq a$, and thus $|\sigma(s)| \geq \kappa \big(\min\{|s|, a\} \big)^\alpha = \kappa \big| a \operatorname{sat}(s/a) \big|^\alpha$, $\forall s \in \mathbb{R}$. $\triangleleft$

Lemma 3.1. *Let $\sigma \in \mathcal{C}^m(\mathbb{R}; \mathbb{R})$, for some $m \in \mathbb{N}$, be a generalized saturation function with bound M. Then:*

1. *$\lim_{|s|\to\infty} s^p \sigma^{(q)}(s) = 0$, $\forall p \in \mathbb{Z}_+$, $\forall q \in \mathbb{N}_m$;*
2. *for all $p \in \mathbb{Z}_+$ and all $q \in \mathbb{N}_m$, there exist $A^{p,q} \in (0, \infty)$ such that $\big| s^p \sigma^{(q)}(s) \big| \leq A^{p,q}$, $\forall s \in \mathbb{R}$.*

Lemma 3.2. *Let $\sigma, \sigma_1, \sigma_2 : \mathbb{R} \to \mathbb{R}$ be strongly passive functions and k be a positive constant. Then:*

1. *$\int_0^s \sigma(k\nu) d\nu > 0$, $\forall s \neq 0$;*
2. *$\int_0^s \sigma(k\nu) d\nu \to \infty$ as $|s| \to \infty$;*
3. *$\sigma_1 \circ \sigma_2$ is strongly passive.*

Lemma 3.3. *Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function and k be a positive constant. Then: $s_1 \big[\sigma(ks_1 + s_2) - \sigma(s_2) \big] > 0$, $\forall s_1 \neq 0$, $\forall s_2 \in \mathbb{R}$.*

Lemma 3.4. *Consider the second-order system*

$$\dot{x}_1 = x_2 \quad , \quad \dot{x}_2 = -\sigma_1(k_1 x_1) - \sigma_2(k_2 x_2) \tag{3}$$

where $\sigma_1 : \mathbb{R} \to \mathbb{R}$ is a strongly passive function and $\sigma_2 : \mathbb{R} \to \mathbb{R}$ is strictly passive, both being locally Lipschitz on $\mathbb{R} \setminus \{0\}$, and k_1 and k_2 are (arbitrary) positive constants. For this dynamical system, $(0, 0)$ is a globally asymptotically stable equilibrium. If in addition, for every $i \in \mathbb{N}_2$, $\sigma_i(s)$ is locally r_i-homogeneous of degree α with domain of homogeneity $D_i \triangleq \big\{ s \in \mathbb{R} : |s| < \rho_i \in (0, \infty] \big\}$, for some dilation coefficients such that

$$\alpha = 2r_2 - r_1 > 0 > r_2 - r_1 \tag{4}$$

then $(0, 0)$ is globally finite-time stable.

Lemma 3.4 is proven by showing that, from the properties satisfied by strictly and strongly passive functions, $V_1(x_1, x_2) = \frac{x_2^2}{2} + \int_0^{x_1} \sigma_1(k_1 s) ds$ is a radially unbounded Lyapunov function of system (3), and the application of La Salle's invariance principle [Khalil, 2002, §4.2] and Theorem 3.1.

Corollary 3.1. *For every $i \in \mathbb{N}_2$, let $\sigma_i(s) = \kappa_i \operatorname{sign}(s)|s|^{\beta_i}$, $\forall |s| < \rho_i \in (0, \infty]$, with κ_i and β_i being positive constants. Thus, for any $(r_1, r_2) \in \mathbb{R}^2_{>0}$ such that $r_1\beta_1 = r_2\beta_2 \triangleq \alpha$, $\sigma_i(s)$, $i = 1, 2$, are locally r_i-homogeneous of degree α with domain of homogeneity $D_i = \{s \in \mathbb{R} : |s| < \rho_i\}$, guaranteeing the satisfaction of (4), if and only if*

$$1 = \frac{2}{\beta_2} - \frac{1}{\beta_1} > 0 > \frac{1}{\beta_2} - \frac{1}{\beta_1} \tag{5}$$

Remark 3.2. Note from Lemma 3.4 and Corollary 3.1, that for system (3) with a strongly passive $\sigma_1(s)$ and a strictly passive $\sigma_2(s)$, both being locally Lipschitz-continuous on $\mathbb{R}\setminus\{0\}$, such that, for every $i=1,2$, $\sigma_i(s)=\kappa_i\,\mathrm{sign}(s)|s|^{\beta_i}$, $\forall|s|<\rho_i\in(0,\infty]$, with $\kappa_i>0$ and positive values of β_i satisfying (5) — equivalently expressed as: $0<\beta_1<1$ and $\beta_2=2\beta_1/(1+\beta_1)$—, $(0,0)$ is a globally finite-time stable equilibrium. In particular, for the special case generated by taking $\sigma_i(s)=k_i\,\mathrm{sign}(s)\max\{|s|^{\beta_i},|s|\}$, $\forall s\in\mathbb{R}$, $i=1,2$, with positive values of β_i satisfying (5), global finite-time stability of the origin was stated in [Frye, Ding, Qian and Li, 2010, Lemma 2.2]. ◁

Lemma 3.5. *Consider the second-order system*

$$\dot{x}_1=x_2-\sigma_1(k_1x_1)\quad,\quad\dot{x}_2=-\sigma_2(k_2x_1)\tag{6}$$

where $\sigma_1:\mathbb{R}\to\mathbb{R}$ is a strictly passive function and $\sigma_2:\mathbb{R}\to\mathbb{R}$ is strongly passive, both being locally Lipschitz on $\mathbb{R}\setminus\{0\}$, and k_1 and k_2 are (arbitrary) positive constants. For this dynamical system, $(0,0)$ is a globally asymptotically stable equilibrium. If in addition, for every $i\in\mathbb{N}_2$, $\sigma_i(s)$ is locally r_0-homogeneous of degree α_i, with domain of homogeneity $D_i=\{s\in\mathbb{R}:|s|<\rho_i\in(0,\infty]\}$, for some (common) dilation coefficient such that

$$\alpha_2=2\alpha_1-r_0>0>\alpha_1-r_0\tag{7}$$

then $(0,0)$ is globally finite-time stable.

Lemma 3.5 is proven by showing that, from the properties satisfied by strictly and strongly passive functions, $V_2(x_1,x_2)=\frac{x_2^2}{2}+\int_0^{x_1}\sigma_2(k_2s)ds$ is a radially unbounded Lyapunov function of system (6), and the application of La Salle's invariance principle and Theorem 3.1.

Corollary 3.2. *For every $i\in\mathbb{N}_2$, let $\sigma_i(s)=\kappa_i\,\mathrm{sign}(s)|s|^{\beta_i}$, $\forall|s|<\rho_i\in(0,\infty]$, with κ_i and β_i being positive constants. Thus, $\sigma_i(s)$, $i=1,2$, are locally r_0-homogeneous of degree α_i, for some (common) dilation coefficient such that* (7) *is satisfied, if and only if*

$$\beta_2=2\beta_1-1>0>\beta_1-1\tag{8}$$

Remark 3.3. Let us note, from Lemma 3.5 and Corollary 3.2, that for system (6) with strictly passive $\sigma_1(s)$ and strongly passive $\sigma_2(s)$, both being locally Lipschitz-continuous on $\mathbb{R}\setminus\{0\}$, such that, for every $i=1,2$, $\sigma_i(s)=\kappa_i\,\mathrm{sign}(s)|s|^{\beta_i}$, $\forall|s|<\rho_i\in(0,\infty]$, with $\kappa_i>0$ and positive values of β_i satisfying (8) — equivalently expressed as: $\frac{1}{2}<\beta_1<1$ and $\beta_2=2\beta_1-1$—, $(0,0)$ is a globally finite-time stable equilibrium. In particular, for the special case of system (6) generated by taking $\sigma_i(s)=k_i\,\mathrm{sign}(s)\max\{|s|^{\beta_i},|s|\}$, $\forall s\in\mathbb{R}$, $i=1,2$, with positive values of β_i satisfying (8), global finite-time stability of the origin was stated in [Frye, Ding, Qian and Li, 2010, Lemma 2.3]. ◁

4 State-feedback global stabilizer

Following a design reasoning similar to the one described in [Zavala-Río, Fantoni and Lozano, 2003], we define the following new controller

$$u_1=\sqrt{v_1^2+(1+v_2)^2}\tag{9}$$

$$v_1=-k_0\sigma_{12}\big(k_{12}\dot{x}+\sigma_{11}(k_{11}x)\big)\tag{10}$$

$$v_2=-\sigma_{22}\big(k_{22}\dot{y}+\sigma_{21}(k_{21}y)\big)\tag{11}$$

$$\begin{aligned}u_2=\sigma_{30}(\ddot{\theta}_d)&-\sigma_{31}\big(k_{31}(\theta-\theta_d)\big)\\&-\sigma_{32}\big(k_{32}(\dot{\theta}-\dot{\theta}_d)\big)\end{aligned}\tag{12}$$

$$\theta_d=\arctan(-v_1,1+v_2)\tag{13}$$

where $\arctan(a,b)$ represents the (unique) angle ϕ such that $\sin\phi=a/\sqrt{a^2+b^2}$ and $\cos\phi=b/\sqrt{a^2+b^2}$; k_{ij}, $i=1,2,3$, $j=1,2$, are (arbitrary) positive constants; k_0 is a positive constant less than unity, *i.e.*

$$0<k_0<1\tag{14a}$$

σ_{11}, σ_{12}, σ_{21}, and σ_{22} are strictly increasing twice continuously differentiable generalized saturations with bounds M_{11}, M_{12}, M_{21}, and limit bounds M_{22}^+ and M_{22}^- such that

$$M_{12}^2+(1+M_{22}^-)^2\le U_1^2\quad,\quad M_{22}^+<1\tag{14b}$$

σ_{30} is a linear saturation, σ_{31} a homogeneous saturation, and σ_{32} a strictly increasing homogeneous saturation for (L_{30},M_{30}), $(\alpha_{31},r_{31},M_{31})$, $(\alpha_{32},r_{32},M_{32})$, and limit bounds such that

$$M_{30}+M_{31}+M_{32}\le U_2\tag{15a}$$

$$\bar{M}_{30}+\bar{M}_{31}<\underline{M}_{32}\tag{15b}$$

$$\alpha_{31}=\alpha_{32}=2r_{32}-r_{31}>0>r_{32}-r_{31}\tag{15c}$$

and $\dot{\theta}_d\triangleq\frac{d}{dt}\theta_d$ and $\ddot{\theta}_d\triangleq\frac{d^2}{dt^2}\theta_d$ are given by

$$\begin{aligned}\dot{\theta}_d&=k_0\dot{\bar{\theta}}_d\\\dot{\bar{\theta}}_d&=\frac{\bar{v}_1\dot{v}_2-(1+v_2)\dot{\bar{v}}_1}{u_1^2}\end{aligned}\tag{16}$$

$$\ddot{\theta}_d = k_0\ddot{\bar{\theta}}_d$$
$$\dddot{\bar{\theta}}_d = \frac{\bar{v}_1\ddot{v}_2 - (1+v_2)\ddot{\bar{v}}_1}{u_1^2} - \frac{2\dot{u}_1\dot{\bar{\theta}}_d}{u_1} \tag{17}$$

where $\bar{v}_1 \triangleq v_1/k_0$, $\dot{\bar{v}}_1 \triangleq \frac{d}{dt}\bar{v}_1$, $\ddot{\bar{v}}_1 \triangleq \frac{d^2}{dt^2}\bar{v}_1$, $\dot{v}_2 \triangleq \frac{d}{dt}v_2$, $\ddot{v}_2 \triangleq \frac{d^2}{dt^2}v_2$, and $\dot{u}_1 \triangleq \frac{d}{dt}u_1$ are given by $\bar{v}_1 = -\sigma_{12}(s_{12})$,

$$\dot{\bar{v}}_1 = -\sigma_{12}'(s_{12})\dot{s}_{12} \tag{18}$$
$$\ddot{\bar{v}}_1 = -\sigma_{12}''(s_{12})\dot{s}_{12}^2 - \sigma_{12}'(s_{12})\ddot{s}_{12} \tag{19}$$
$$\dot{v}_2 = -\sigma_{22}'(s_{22})\dot{s}_{22} \tag{20}$$
$$\ddot{v}_2 = -\sigma_{22}''(s_{22})\dot{s}_{22}^2 - \sigma_{22}'(s_{22})\ddot{s}_{22} \tag{21}$$
$$\dot{u}_1 = \frac{v_1\dot{v}_1 + (1+v_2)\dot{v}_2}{u_1} \tag{22}$$

with $\dot{v}_1 \triangleq \frac{d}{dt}v_1 = k_0\dot{\bar{v}}_1$, and s_{i2}, $\dot{s}_{i2} \triangleq \frac{d}{dt}s_{i2}$, and $\ddot{s}_{i2} \triangleq \frac{d^2}{dt^2}s_{i2}$, $i = 1,2$, given by

$$\begin{aligned}
s_{12} &= k_{12}\dot{x} + \sigma_{11}(k_{11}x)\\
\dot{s}_{12} &= k_{12}a_x + \sigma_{11}'(k_{11}x)k_{11}\dot{x}\\
\ddot{s}_{12} &= k_{12}\dot{a}_x + \sigma_{11}''(k_{11}x)(k_{11}\dot{x})^2 + \sigma_{11}'(k_{11}x)k_{11}a_x\\
s_{22} &= k_{22}\dot{y} + \sigma_{21}(k_{21}y)\\
\dot{s}_{22} &= k_{22}a_y + \sigma_{21}'(k_{21}y)k_{21}\dot{y}\\
\ddot{s}_{22} &= k_{22}\dot{a}_y + \sigma_{21}''(k_{21}y)(k_{21}\dot{y})^2 + \sigma_{21}'(k_{21}y)k_{21}a_y
\end{aligned}$$

and a_x (taken from the horizontal dynamics in Eqs. (2)), $\dot{a}_x \triangleq \frac{d}{dt}a_x$, a_y (taken from the vertical dynamics in Eqs. (2)), and $\dot{a}_y \triangleq \frac{d}{dt}a_y$ given by

$$\begin{aligned}
a_x &= -u_1\sin\theta \quad , \quad \dot{a}_x = -\dot{u}_1\sin\theta - u_1\dot{\theta}\cos\theta\\
a_y &= u_1\cos\theta - 1 \quad , \quad \dot{a}_y = \dot{u}_1\cos\theta - u_1\dot{\theta}\sin\theta
\end{aligned}$$

Proposition 4.1. *Consider the PVTOL aircraft dynamics* (2) *with input saturation bounds $U_1 > 1$ and $U_2 > 0$. Let the input thrust u_1 be defined as in* (9)*, with constant k_0, bounds M_{11}, M_{12}, M_{21}, and limit bounds M_{22}^+ and M_{22}^- of the strictly increasing twice continuously differentiable generalized saturation functions σ_{ij}, $i,j = 1,2$, in* (10) *and* (11) *satisfying inequalities* (14)*, and (arbitrary) positive gains k_{ij}, $i,j = 1,2$, and the input rolling moment u_2 as in* (12)*, with parameters (L_{30}, M_{30}), $(\alpha_{31}, r_{31}, M_{31})$, $(\alpha_{32}, r_{32}, M_{32})$, and limit bounds of the linear, homogeneous, and strictly increasing homogeneous saturations σ_{3l}, $l = 1,2,3$, in* (12) *satisfying conditions* (15)*, and (arbitrary) positive gains k_{3l}, $l = 1,2,3$. Then, for any $(x,y,\theta,\dot{x},\dot{y},\dot{\theta})(0) \in \mathbb{R}^6$:*

1. $0 < 1 - M_{22}^+ \leq u_1(t) \leq \sqrt{(k_0M_{12})^2 + \left(1+M_{22}^-\right)^2} < U_1$ *and* $|u_2(t)| < M_{30} + M_{31} + M_{32} \leq U_2$, $\forall t \geq 0$*;*
2. *$x(t)$, $\dot{x}(t)$, $y(t)$, $\dot{y}(t)$, $\theta(t)$, and $\dot{\theta}(t)$ are bounded on $[0,\tau]$ for any $\tau \in (0,\infty)$;*
3. *there exist initial-condition-independent positive constants $B_{\dot{\bar{v}}_1}$, $B_{\dot{v}_2}$, $B_{\dot{u}_1}$, and $B_{\dot{\bar{\theta}}_d}$ such that (along the closed-loop system trajectories) $|\dot{v}_1(t)| < |\dot{\bar{v}}_1(t)| \leq B_{\dot{\bar{v}}_1}$, $|\dot{v}_2(t)| \leq B_{\dot{v}_2}$, $|\dot{u}_1(t)| \leq B_{\dot{u}_1}$, and $|\dot{\theta}_d(t)| < |\dot{\bar{\theta}}_d(t)| \leq B_{\dot{\bar{\theta}}_d}$, $\forall t \geq 0$;*
4. *there exists a finite time $t_1 \geq 0$ such that:*
   (a) $|\dot{\theta}(t)| \leq B_{\dot{\theta}}$, $\forall t \geq t_1$,
   (b) *and* $|\ddot{\bar{v}}_1(t)| \leq B_{\ddot{\bar{v}}_1}$, $|\ddot{v}_2(t)| \leq B_{\ddot{v}_2}$, $|\ddot{\theta}_d(t)| \leq k_0B_{\ddot{\bar{\theta}}_d}$, $\forall t \geq t_1$,

   for some initial-condition-independent positive constants $B_{\dot{\theta}}$, $B_{\ddot{\bar{v}}_1}$, $B_{\ddot{v}_2}$, and $B_{\ddot{\bar{\theta}}_d}$;
5. *provided that k_0 is sufficiently small, from t_1 on, $\theta_d(t)$ is globally finite-time stabilized in the rotational coordinate space, i.e. $\theta_d(t)$ becomes a stable solution of the rotational motion closed-loop dynamics and, for any $(\theta,\dot{\theta})(t_1) \in \mathbb{R}^2$, there exists a finite time $t_2 \geq t_1$ such that $\theta(t) = \theta_d(t)$, $\forall t \geq t_2$;*
6. *from t_2 on, $(x,y)(t) \equiv (0,0)$ becomes a stable solution of the translational motion closed-loop dynamics and, for any $(x,y,\dot{x},\dot{y})(t_2) \in \mathbb{R}^4$, $(x,y,\theta)(t) \to (0,0,0)$ as $t \to \infty$.*

Proof.

1. Item 1 of the statement follows directly from the definition of u_1, u_2, v_1, and v_2 in Eqs. (9)–(12), the consideration of inequalities (14) and (15a), and the strictly increasing character of σ_{32}. Its proof is consequently straightforward.
2. Observe from the system dynamics in Eqs. (2), item 1 of the statement, and inequality (14a) that

$$\begin{aligned}
|\ddot{x}(t)| &< \sqrt{M_{12}^2 + \left(1+M_{22}^-\right)^2} \triangleq B_{u_1}\\
|\ddot{y}(t)| &< B_{u_1} + 1\\
|\ddot{\theta}(t)| &< M_{30} + M_{31} + M_{32} \triangleq B_{u_2}
\end{aligned}$$

Hence, for any $\tau \in (0,\infty)$:

$$\begin{aligned}
|\dot{x}(t)| &< |\dot{x}(0)| + B_{u_1}\tau\\
|\dot{y}(t)| &< |\dot{y}(0)| + (B_{u_1}+1)\tau\\
|\dot{\theta}(t)| &< |\dot{\theta}(0)| + B_{u_2}\tau
\end{aligned}$$

and

$$\begin{aligned}
|x(t)| &< |x(0)| + |\dot{x}(0)|\tau + B_{u_1}\tau^2/2\\
|y(t)| &< |y(0)| + |\dot{y}(0)|\tau + (B_{u_1}+1)\tau^2/2\\
|\theta(t)| &< |\theta(0)| + |\dot{\theta}(0)|\tau + B_{u_2}\tau^2/2
\end{aligned}$$

$\forall t \in [0,\tau]$.

3. Note that Eq. (18) may be rewritten as

$$\dot{\bar{v}}_1 = -\sigma_{12}'(s_{12})\Big[k_{12}a_x + \frac{k_{11}}{k_{12}}\sigma_{11}'(k_{11}x)\big(s_{12} - \sigma_{11}(k_{11}x)\big)\Big]$$

Then, by applying Lemma 3.1, we have that (along the closed-loop system trajectories)

$$|\dot{\bar{v}}_1(t)| \leq k_{12}A_{12}^{0,1}B_{u_1} + \frac{k_{11}}{k_{12}}A_{11}^{0,1}\Big(A_{12}^{1,1} + A_{12}^{0,1}M_{11}\Big) \triangleq B_{\dot{\bar{v}}_1}$$

$\forall t \geq 0$; recall further that $v_1 = k_0\bar{v}_1$ and consequently, under the consideration of (14a), we have that $|\dot{v}_1(t)| < |\dot{\bar{v}}_1(t)| \leq B_{\dot{\bar{v}}_1}$, $\forall t \geq 0$. Following a similar procedure for Eq. (20), we get

$$|\dot{v}_2(t)| \leq k_{22}A_{22}^{0,1}(B_{u_1}+1) + \frac{k_{21}}{k_{22}}A_{21}^{0,1}\Big(A_{22}^{1,1} + A_{22}^{0,1}M_{21}\Big) \triangleq B_{\dot{v}_2}$$

$\forall t \geq 0$. Observe now that Eq. (22) may be rewritten as

$$\begin{aligned}\dot{u}_1 &= \dot{v}_2\cos\theta_d - \dot{v}_1\sin\theta_d \\ &= \sqrt{\dot{v}_1^2 + \dot{v}_2^2}\,\cos\big(\theta_d + \arctan(\dot{v}_1, \dot{v}_2)\big)\end{aligned}$$

whence we have that $|\dot{u}_1(t)| \leq \sqrt{B_{\dot{\bar{v}}_1}^2 + B_{\dot{v}_2}^2} \triangleq B_{\dot{u}_1}$, $\forall t \geq 0$. Furthermore, notice that the right-hand-side equation in (16) may be rewritten as

$$\dot{\bar{\theta}}_d = \frac{\bar{v}_1\dot{v}_2}{u_1^2} - \frac{\dot{\bar{v}}_1\cos\theta_d}{u_1}$$

wherefrom we get that

$$|\dot{\bar{\theta}}_d(t)| \leq \frac{M_{12}B_{\dot{v}_2}}{(1-M_{22}^+)^2} + \frac{B_{\dot{\bar{v}}_1}}{1-M_{22}^+} \triangleq B_{\dot{\bar{\theta}}_d}$$

$\forall t \geq 0$. Observe further from left-hand-side equation in (16) that, under the consideration of (14a), we have that $|\dot{\theta}_d(t)| < |\dot{\bar{\theta}}_d(t)| \leq B_{\dot{\bar{\theta}}_d}$, $\forall t \geq 0$.

4a. Let σ_{33} be a generalized saturation function with bound M_{33} such that[2]

$$M_{33} < \underline{M}_{32} - \bar{M}_{31} - \bar{M}_{30} \tag{23}$$

Let us further define the positive function $V_1 = \dot{\theta}^2/2$. Its derivative along the trajectories of the closed-loop rotational motion dynamics is given by $\dot{V}_1 = \dot{\theta}\ddot{\theta}$, *i.e.*

$$\dot{V}_1 = \dot{\theta}\Big[\sigma_{30}(\ddot{\theta}_d) - \sigma_{31}\big(k_{31}(\theta-\theta_d)\big) - \sigma_{32}\big(k_{32}(\dot{\theta}-\dot{\theta}_d)\big)\Big]$$

which may be rewritten as

$$\dot{V}_1 = -\dot{\theta}\sigma_{33}(\dot{\theta}) + \dot{\theta}\Big[\sigma_{30}(\ddot{\theta}_d) - \sigma_{31}\big(k_{31}(\theta-\theta_d)\big) - \sigma_{32}\big(k_{32}(\dot{\theta}-\dot{\theta}_d)\big) + \sigma_{33}(\dot{\theta})\Big] \tag{24}$$

Observe, from item 3 of the statement and the strictly increasing character of σ_{32}, that[3]

$$\begin{aligned}\dot{\theta} \geq B_{\dot{\theta}}^+ &\triangleq \frac{\sigma_{32}^{-1}(\bar{M}_{30}+\bar{M}_{31}+M_{33})}{k_{32}} + B_{\dot{\bar{\theta}}_d} > 0 \\ \Longrightarrow\ \dot{\theta} - \dot{\theta}_d &\geq \frac{\sigma_{32}^{-1}(\bar{M}_{30}+\bar{M}_{31}+M_{33})}{k_{32}} + B_{\dot{\bar{\theta}}_d} - \dot{\theta}_d \\ &\geq \frac{\sigma_{32}^{-1}(\bar{M}_{30}+\bar{M}_{31}+M_{33})}{k_{32}} \\ \Longrightarrow\ &\sigma_{32}\big(k_{32}(\dot{\theta}-\dot{\theta}_d)\big) \geq \bar{M}_{30} + \bar{M}_{31} + M_{33} \\ \Longrightarrow\ &\sigma_{32}(\cdot) - \sigma_{30}(\cdot) + \sigma_{31}(\cdot) - \sigma_{33}(\cdot) \\ &\geq \bar{M}_{30} - \sigma_{30}(\cdot) + \bar{M}_{31} + \sigma_{31}(\cdot) + M_{33} - \sigma_{33}(\cdot) \geq 0 \\ \Longrightarrow\ &\sigma_{30}(\ddot{\theta}_d) - \sigma_{31}\big(k_{31}(\theta-\theta_d)\big) - \sigma_{32}\big(k_{32}(\dot{\theta}-\dot{\theta}_d)\big) + \sigma_{33}(\dot{\theta}) \leq 0\end{aligned}$$

while analogous developments show that

$$\begin{aligned}\dot{\theta} \leq B_{\dot{\theta}}^- &\triangleq \frac{\sigma_{32}^{-1}(-\bar{M}_{30}-\bar{M}_{31}-M_{33})}{k_{32}} - B_{\dot{\bar{\theta}}_d} < 0 \\ \Longrightarrow\ &\sigma_{30}(\ddot{\theta}_d) - \sigma_{31}\big(k_{31}(\theta-\theta_d)\big) - \sigma_{32}\big(k_{32}(\dot{\theta}-\dot{\theta}_d)\big) + \sigma_{33}(\dot{\theta}) \geq 0\end{aligned}$$

From these expressions we see that

$$\begin{aligned}|\dot{\theta}| &\geq \max\Big\{B_{\dot{\theta}}^+, -B_{\dot{\theta}}^-\Big\} \\ \Longrightarrow\ &\dot{\theta}\Big[\sigma_{30}(\ddot{\theta}_d) - \sigma_{31}\big(k_{31}(\theta-\theta_d)\big) - \sigma_{32}\big(k_{32}(\dot{\theta}-\dot{\theta}_d)\big) + \sigma_{33}(\dot{\theta})\Big] \leq 0\end{aligned}$$

[2]Note that the satisfaction of (15b) ensures positivity of the right-hand side of inequality (23).

[3]Let us note that its strictly increasing character renders σ_{32} an invertible function mapping $\mathbb{R}$ onto $(-M_{32}^-, M_{32}^+)$ —and consequently σ_{32}^{-1} is a well-defined function mapping $(-M_{32}^-, M_{32}^+)$ onto $\mathbb{R}$— and observe that, by (23), we have that $\bar{M}_{30}+\bar{M}_{31}+M_{33} \in (0, \underline{M}_{32}) \subset (-M_{32}^-, M_{32}^+)$.

whence, in view of (24), we conclude that

$$\dot{V}_1 \le -\dot{\theta}\sigma_{33}(\dot{\theta}) \qquad \forall|\dot{\theta}| \ge \max\left\{B_{\dot{\theta}}^+ , -B_{\dot{\theta}}^-\right\}$$

with $\dot{\theta}\sigma_{33}(\dot{\theta})$ being a positive definite function of $\dot{\theta}$ in view of the strictly passive character of σ_{33}. Then, according to [Khalil, 2002, Theorem 4.18],[4] there exists a finite time $t_1 \ge 0$ such that $|\dot{\theta}(t)| \le \max\left\{B_{\dot{\theta}}^+ , -B_{\dot{\theta}}^-\right\} \triangleq B_{\dot{\theta}}, \forall t \ge t_1$.

4b. Notice that Eq. (19) may be rewritten as

$$\begin{aligned}\dddot{v}_1(t) &= -\sigma_{12}''(s_{12})\Big[-k_{12}u_1\sin\theta+\sigma_{11}'(k_{11}x)\Delta\Big]^2\\ &- \sigma_{12}'(s_{12})\Big[-\big(k_{12}\dot{u}_1 + k_{11}u_1\sigma_{11}'(k_{11}x)\big)\sin\theta\\ &\qquad - k_{12}u_1\dot{\theta}\cos\theta + \sigma_{11}''(k_{11}x)\Delta^2\Big]\end{aligned}$$

with $\Delta = \frac{k_{11}}{k_{12}}\big(s_{12} - \sigma_{11}(k_{11}x)\big)$. Thus, by applying Lemma 3.1 and considering items 3 and 4a of the statement, we have that (along the closed-loop system trajectories)

$$\begin{aligned}|\dddot{v}_1(t)| &\le k_{12}^2 A_{12}^{0,2} B_{u_1}^2\\ &+ 2k_{11}B_{u_1}A_{11}^{0,1}\big(A_{12}^{1,2} + A_{12}^{0,2}M_{11}\big)\\ &+ \left(\frac{k_{11}A_{11}^{0,1}}{k_{12}}\right)^2\big(A_{12}^{2,2} + 2A_{12}^{1,2}M_{11} + A_{12}^{0,2}M_{11}^2\big)\\ &+ A_{12}^{0,1}C_1 + \left(\frac{k_{11}}{k_{12}}\right)^2 A_{11}^{0,2}\big(A_{12}^{2,1} + 2A_{12}^{1,1}M_{11}\\ &\qquad + A_{12}^{0,1}M_{11}^2\big) \triangleq B_{\dddot{v}_1}\end{aligned}$$

$\forall t \ge t_1$, with

$$C_1 \triangleq \sqrt{\Big(k_{12}B_{\dot{u}_1} + k_{11}B_{u_1}A_{11}^{0,1}\Big)^2 + \big(k_{12}B_{u_1}B_{\dot{\theta}}\big)^2}$$

Following a similar procedure for Eq. (21), we get

$$\begin{aligned}|\dddot{v}_2(t)| &\le k_{22}^2 A_{22}^{0,2}(B_{u_1}+1)^2\\ &+ 2k_{21}(B_{u_1}+1)A_{21}^{0,1}\big(A_{22}^{1,2} + A_{22}^{0,2}M_{21}\big)\\ &+ \left(\frac{k_{21}A_{21}^{0,1}}{k_{22}}\right)^2\big(A_{22}^{2,2} + 2A_{22}^{1,2}M_{21} + A_{22}^{0,2}M_{21}^2\big)\\ &+ A_{22}^{0,1}\big(C_2 + k_{21}A_{21}^{0,1}\big)\\ &+ \left(\frac{k_{21}}{k_{22}}\right)^2 A_{21}^{0,2}\big(A_{22}^{2,1} + 2A_{22}^{1,1}M_{21}\\ &\qquad + A_{22}^{0,1}M_{21}^2\big) \triangleq B_{\dddot{v}_2}\end{aligned}$$

[4]Theorem 4.18 of [Khalil, 2002] is being applied by considering the closed-loop rotational motion dynamics a first order subsystem with respect to $\dot{\theta}$, *i.e.* $\frac{d}{dt}\dot{\theta} = u_2(t,\dot{\theta})$ where (along the closed loop trajectories) the rest of the system variables, involved in u_2, are considered time-varying functions.

$\forall t \ge t_1$, with

$$C_2 \triangleq \sqrt{\Big(k_{22}B_{\dot{u}_1} + k_{21}B_{u_1}A_{21}^{0,1}\Big)^2 + \big(k_{22}B_{u_1}B_{\dot{\theta}}\big)^2}$$

Furthermore, note that the right-hand-side equation in (17) may be rewritten as

$$\ddot{\bar{\theta}}_d = \frac{\bar{v}_1\ddot{v}_2}{u_1^2} - \frac{\ddot{\bar{v}}_1\cos\theta_d + 2\dot{u}_1\dot{\bar{\theta}}_d}{u_1}$$

whence we get that

$$|\ddot{\bar{\theta}}_d(t)| \le \frac{M_{12}B_{\ddot{v}_2}}{(1-M_{22}^+)^2} + \frac{B_{\ddot{v}_1} + 2B_{\dot{u}_1}B_{\dot{\bar{\theta}}_d}}{1-M_{22}^+} \triangleq B_{\ddot{\bar{\theta}}_d}$$

$\forall t \ge t_1$. Hence, from the left-hand-side equation in (17), we conclude that $|\ddot{\theta}_d(t)| = k_0|\ddot{\bar{\theta}}_d(t)| \le k_0 B_{\ddot{\bar{\theta}}_d}, \forall t \ge t_1$.

5. From items 4b of the statement and 4 of Definition 3.3, one sees that by choosing a sufficiently small value of k_0 —such that $k_0 B_{\ddot{\bar{\theta}}_d} \le L_{30}$—, we have (along the system trajectories) that $\sigma_{30}(\ddot{\theta}_d(t)) = \ddot{\theta}_d(t)$, $\forall t \ge t_1$. Then, from t_1 on, the rotational motion dynamics becomes

$$\ddot{\theta} = \ddot{\theta}_d - \sigma_{31}\big(k_{31}(\theta-\theta_d)\big) - \sigma_{32}\big(k_{32}(\dot{\theta}-\dot{\theta}_d)\big)$$

By defining $e_1 = \theta - \theta_d$ and $e_2 = \dot{\theta} - \dot{\theta}_d$, this subsystem adopts a state-space representation of the form

$$\dot{e}_1 = e_2 \quad , \quad \dot{e}_2 = -\sigma_{31}(k_{31}e_1) - \sigma_{32}(k_{32}e_2)$$

Thus, by Lemma 3.4 and Remark 3.1, we conclude that $(e_1,e_2) = (0,0)$ is a globally finite-time stable equilibrium of this subsystem. Hence, $\theta_d(t)$ becomes a globally finite-time stable solution of the rotational motion closed-loop dynamics, or equivalently, it becomes a stable solution of this subsystem and, for any $(\theta,\dot{\theta})(t_1) \in \mathbb{R}^2$, there exists a finite time $t_2 \ge t_1$ such that $\theta(t) = \theta_d(t)$, $\forall t \ge t_2$.

6. Observe from item 2 of the statement that up to t_2 (and actually for any arbitrarily long finite time), the closed-loop system solutions exist and are bounded. Further, from item 5 of the statement, the definitions of θ_d in (13) and u_1 in (9), and Eqs. (2), one sees that, from t_2 on, we have that $\ddot{x}_1 = -u_1\sin\theta_d = v_1$ and $\ddot{y}_1 = u_1\cos\theta_d - 1 = v_2$ with v_1 and v_2 as defined in Eqs. (10)-(11), *i.e.* the translational motion closed-loop dynamics in the transformed coordinates becomes

$$\begin{aligned}\ddot{x} &= -k_0\sigma_{12}\big(k_{12}\dot{x} + \sigma_{11}(k_{11}x)\big)\\ \ddot{y} &= -\sigma_{22}\big(k_{22}\dot{y} + \sigma_{21}(k_{21}y)\big)\end{aligned} \qquad (25)$$

By defining $z \triangleq \left(x, \dot{x}, y, \dot{y}\right)^T$, this subsystem adopts a consequent state-space representation $\dot{z} = f(z)$ with $f(0_4) = 0_4$. More precisely,

$$\dot{z}_1 = z_2 \tag{26a}$$
$$\dot{z}_2 = -k_0\sigma_{12}\big(k_{12}z_2 + \sigma_{11}(k_{11}z_1)\big) \tag{26b}$$
$$\dot{z}_3 = z_4 \tag{26c}$$
$$\dot{z}_4 = -\sigma_{22}\big(k_{22}z_4 + \sigma_{21}(k_{21}z_3)\big) \tag{26d}$$

Let us now define the continuously differentiable scalar function

$$V_2 = \frac{z_2^2}{2k_0} + \int_0^{z_1} \sigma_{12}\big(\sigma_{11}(k_{11}s)\big)ds + \frac{z_4^2}{2} + \int_0^{z_3} \sigma_{22}\big(\sigma_{21}(k_{21}s)\big)ds$$

Note, under the consideration of Lemma 3.2 and Remark 3.1, that $V_2(z)$ is radially unbounded and positive definite. Its derivative along the system trajectories is given by

$$\dot{V}_2 = \frac{z_2\dot{z}_2}{k_0} + z_2\sigma_{12}\big(\sigma_{11}(k_{11}z_1)\big) + \frac{z_4\dot{z}_4}{k_0} + z_4\sigma_{22}\big(\sigma_{21}(k_{21}z_3)\big)$$

From Lemma 3.3 —in view of the strictly increasing character of σ_{i2}, $i = 1, 2$— one sees that $\dot{V}_2(z) \leq 0$, $\forall z \in \mathbb{R}^4$, with $\dot{V}_2(z) = 0 \iff z_2 = z_4 = 0$, whence 0_4 is concluded to be a stable equilibrium of the state equations (26), or equivalently $(x, y)(t) \equiv (0, 0)$ is concluded to be a stable solution of subsystem (25). Further, from Eqs. (26) and the strictly passive character of the involved generalized saturation functions, one sees that $\big(z_2(t) \equiv 0\big) \wedge \big(z_4(t) \equiv 0\big) \implies \big(\dot{z}_2(t) \equiv 0\big) \wedge \big(\dot{z}_4(t) \equiv 0\big) \implies \big(z_1(t) \equiv 0\big) \wedge \big(z_3(t) \equiv 0\big)$. Then, from La Salle's invariance principle, one concludes that, for any $z(t_2) \in \mathbb{R}^4$, $z(t) \to 0_4$ as $t \to \infty$. Finally, observe from this asymptotic convergence that, since (along the closed-loop system trajectories) $(\theta, \dot{\theta})(t) = (\theta_d, \dot{\theta}_d)(t)$, $\forall t \geq t_2$, and, as functions of the system variables, $\theta_d(z)\big|_{z=0_4} = \dot{\theta}_d(z)\big|_{z=0_4} = 0$, then $(\theta, \dot{\theta})(t) = (\theta_d, \dot{\theta}_d)(t) \to 0_2$ as $t \to \infty$.

5 Output-feedback global stabilizer

With $u \triangleq (u_1, u_2)^T$, let $u(x, \dot{x}, y, \dot{y}, \theta, \dot{\theta})$ represent the (state) feedback controller presented in the precedent section. Suppose now that position measurements are available while the velocity signals are not. In this case we show that the globally stabilizing objective is achievable through the precedent algorithm with the velocities replaced by estimation variables coming from a finite-time observer defined through a generalized dynamics that includes that used in [Frye, Ding, Qian and Li, 2010] as a particular case. More specifically, we consider the closed loop generated by taking $u = u(x, \hat{z}_2, y, \hat{z}_4, \theta, \hat{z}_6)$ under the additional consideration of the auxiliary dynamics

$$\dot{\hat{z}}_1 = \hat{z}_2 + \sigma_{41}\big(k_{41}(x - \hat{z}_1)\big) \tag{27a}$$
$$\dot{\hat{z}}_2 = -u_1 \sin\theta + \sigma_{42}\big(k_{42}(x - \hat{z}_1)\big) \tag{27b}$$
$$\dot{\hat{z}}_3 = \hat{z}_4 + \sigma_{51}\big(k_{51}(y - \hat{z}_3)\big) \tag{27c}$$
$$\dot{\hat{z}}_4 = u_1 \cos\theta - 1 + \sigma_{52}\big(k_{52}(y - \hat{z}_3)\big) \tag{27d}$$
$$\dot{\hat{z}}_5 = \hat{z}_6 + \sigma_{61}\big(k_{61}(\theta - \hat{z}_5)\big) \tag{27e}$$
$$\dot{\hat{z}}_6 = u_2 + \sigma_{62}\big(k_{62}(\theta - \hat{z}_5)\big) \tag{27f}$$

where, for every $i \in \{4, 5, 6\}$, k_{i1} and k_{i2} are (arbitrary) positive constants, $\sigma_{i1}(s)$ is a strictly passive function and $\sigma_{i2}(s)$ is strongly passive, both being locally Lipschitz-continuous on $\mathbb{R} \setminus \{0\}$ and locally r_i-homogeneous of degree α_{i1} and α_{i2}, respectively, for some r_i such that

$$\alpha_{i2} = 2\alpha_{i1} - r_i > 0 > \alpha_{i1} - r_i \tag{28}$$

As in the previous section, we denote $z = (x, \dot{x}, y, \dot{y}, \theta, \dot{\theta})^T$, while we define $\hat{z} \triangleq (\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4, \hat{z}_5, \hat{z}_6)^T$.

Proposition 5.1. *Assuming input saturation bounds $U_1 > 1$ and $U_2 > 0$, consider the PVTOL aircraft dynamics* (2) *with $u = u(x, \hat{z}_2, y, \hat{z}_4, \theta, \hat{z}_6)$, i.e. in closed loop with the output feedback scheme generated from the control algorithm considered in Proposition 4.1, with the horizontal, vertical, and rotational velocity variables in the control law expressions* (9) *and* (12) *respectively replaced by estimation variables $\hat{z}_2$, $\hat{z}_4$, and $\hat{z}_6$ dynamically computed through the auxiliary subsystem represented in Eqs.* (27)*, under the satisfaction of the parametric conditions* (28) *(concerning the previously described functions σ_{i1} and σ_{i2}, $i = 4, 5, 6$) and the consideration of (arbitrary) positive constants k_{ij}, $i = 4, 5, 6$, $j = 1, 2$. Then, for any $(z^T, \hat{z}^T)^T(0) \in \mathbb{R}^{12}$:*

1. *items 1 and 2 of Proposition 4.1 hold, i.e. along the closed loop trajectories, input saturation is avoided and the position and velocity variables exist and are bounded at any finite time;*
2. *there exists a finite time $t_0 \geq 0$ such that $\hat{z}(t) = z(t)$, $\forall t \geq t_0$;*
3. *from t_0 on, items 3–6 of Proposition 4.1 are retrieved with $t_1 \geq t_0$, i.e. in particular, there exist a finite time $t_1 \geq t_0$ such that $|\ddot{\theta}_d(t)| \leq k_0 B_{\ddot{\theta}_d}$, $\forall t \geq t_1$, and a finite time $t_2 \geq t_1$ such that, provided that k_0 is sufficiently small, $\theta(t) = \theta_d(t)$, $\forall t \geq t_2$, and such that, from t_2 on, $(x, y)(t) \equiv (0, 0)$ becomes a stable solution of the translational motion closed-loop dynamics and, for any*

$(x, y, \dot{x}, \dot{y})(t_2) \in \mathbb{R}^4$, $(x, y, \theta)(t) \to (0, 0, 0)$ *as* $t \to \infty$.

Proof.

1. By reproducing the proof of items 1 and 2 of Proposition 4.1 under the consideration of estimation auxiliary states replacing the velocity variables in the control law expressions, one observes that both items hold, whence item 1 of the statement is concluded.
2. Let us define the observation error variables $\bar{z}_i = z_i - \hat{z}_i$, $i = 1, \ldots, 6$. From the closed-loop system equations, the observation error variable dynamics is obtained as

$$\dot{\bar{z}}_i = \bar{z}_j - \sigma_{i1}(k_{i1}\bar{z}_i) \quad , \quad \dot{\bar{z}}_j = -\sigma_{j2}(k_{j2}\bar{z}_i)$$

   for all $i \in \{1, 3, 5\}$, with $j = i + 1$. Hence, from Lemma 3.5, item 2 of the statement is concluded.
3. Let us first note that in view of item 1 of the statement and the stability properties of the observation error dynamics, up to t_0 (and actually for any arbitrarily long finite time), all the closed-loop system variables, and consequently all the expressions involved in the definition of the control algorithm, exist and are bounded. On the other hand, in view of item 2 of the statement, from t_0 on, the state-feedback closed-loop dynamics considered in Proposition 4.1 is retrieved, and it is further mirrored by the auxiliary subsystem in Eqs. (27). Hence, from Proposition 4.1, item 3 of the statement is concluded.

6 Simulation tests

Numerical tests were implemented under the consideration of input bound values $U_1 = 10$ and $U_2 = 10$ (for the sake of simplicity, units will be omitted). Defining the following generalized/homogeneous saturation functions:

$$\sigma_{ij}(s) = M_{ij}\tanh(s/M_{ij}) \qquad \forall (i,j) \in \mathbb{N}_2 \times \mathbb{N}_2 \setminus \{(2,2)\}$$

$$\sigma_{22}(s) = \begin{cases} M_{22}^- \tanh(s/M_{22}^-) & \forall s < 0 \\ M_{22}^+ \tanh(s/M_{22}^+) & \forall s \geq 0 \end{cases}$$

$$\sigma_{30}(s) = M_{30}\,\mathrm{sat}(s/M_{30})$$

$$\sigma_{31}(s) = \mathrm{sign}(s)\min\{|s|^{\beta_{31}}, M_{31}\}$$

$$\sigma_{32}(s) = \begin{cases} \mathrm{sign}(s)\frac{L_{32}^{1-\beta_{32}}}{\beta_{32}}|s|^{\beta_{32}} & \forall |s| < L_{32} \\ \mathrm{sign}(s)\frac{L_{32}}{\beta_{32}} + \varrho(s) & \forall |s| \geq L_{32} \end{cases}$$

with $\varrho(s) = \left(M_{32} - \frac{L_{32}}{\beta_{32}}\right)\tanh\left(\frac{s - \mathrm{sign}(s)L_{32}}{M_{32} - L_{32}/\beta_{32}}\right)$, and

$$\sigma_{mn}(s) = \mathrm{sign}(s)|s|^{\beta_{mn}} \quad \forall (m,n) \in \{4,5,6\} \times \{1,2\}$$

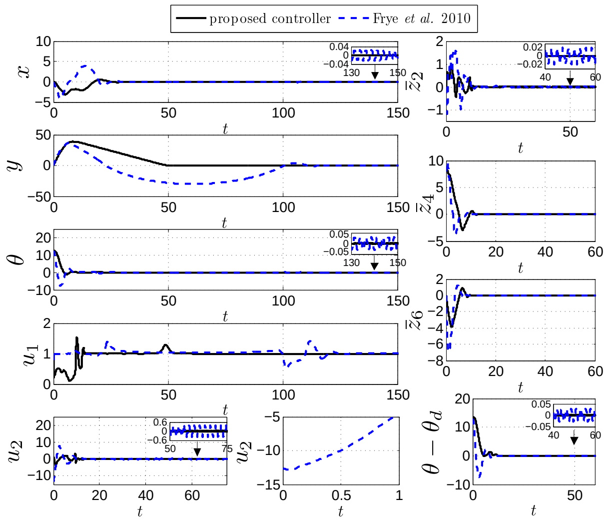

Figure 1. Simulation results

with $M_{11} = M_{21} = 3$, $M_{12} = 7$, $M_{22}^- = 6$, $M_{22}^+ = 0.9$, $M_{30} = M_{31} = 2$, $M_{32} = 6$, $L_{32} = 2$, $\beta_{31} = 1/3$, $\beta_{32} = 1/2$, $\beta_{41} = \beta_{51} = \beta_{61} = 2/3$, $\beta_{42} = \beta_{52} = \beta_{62} = 1/3$, the simulations were run fixing $k_0 = 0.1$, $k_{11} = k_{21} = 2$, $k_{12} = k_{22} = 3$, $k_{ij} = 1$, $i = 3, \ldots, 6$, $j = 1, 2$, and taking $z(0) = (x, \dot{x}, y, \dot{y}, \theta, \dot{\theta})(0) = (0, 0, 0, 8, 4\pi, 0)$ and $\hat{z}(0) = (3, 0, 3, 0, 0, 0)$. For comparison purposes, the output-feedback algorithm of [Frye, Ding, Qian and Li, 2010] was implemented too taking $\beta_1 = \alpha_2 = 1/3$, $\beta_2 = 1/2$, $\alpha_1 = 2/3$, and $k_1 = k_2 = 1$; input saturation bounds were not included for this controller. Several simulations were run involving model (1) with diverse nonnegative values of ε. The proposed controller satisfactorily achieved the control objective, avoiding input saturation, at every implemented test. This is observed, for instance, in Fig. 1 where the results obtained with $\varepsilon = 0.5$ are shown. Note from the graphs that the rolling moment u_2 produced by the algorithm of [Frye, Ding, Qian and Li, 2010] takes absolute values greater than $U_2 = 10$ during the transient; in a bounded input context, this controller would have undergone input saturation. Notice further that with the algorithm of [Frye, Ding, Qian and Li, 2010] non-negligible ripple effects are observed on some of the closed loop variables.

7 Conclusions

An output feedback scheme for the global stabilization of uncoupled PVTOL aircraft with bounded inputs has been proposed. To deal with the lack of velocity measurements, the proposed algorithm involves a finite-time observer. The generalized versions of the involved finite-time stabilizers have not only permitted to solve the output-feedback stabilization problem in a bounded input context, but also give unlimited possibilities in the control design which may be used in aid of performance improvements. Successful simulation results corroborated the efficiency of the proposed scheme. Future work will focus on the extension of the developed study to the consideration of lateral force coupling, *i.e.* $\varepsilon \geq 0$, in the PVTOL aircraft dynamics.

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