Control of dynamical behaviour in networks

S. Vakulenko[1], I. Kotlyarov[2]

[1] Institute of Mechanical Engineering Problems, Saint-Petersburg, Russia

[2] Saint Petersburg State University of Technology and Design, Saint-Petersburg, Russia

1. Introduction. In last decades, a large attention has been given to problems of global organization, stability and evolution of complex networks such as neural and gene networks, economical circuits, Internet etc. (see [1]). In this paper we consider dynamical network models having the form

$$u_i(t+1) = \sigma(\sum_{j=1}^{N} K_{ij}\bar{u}_j(t) + h_i - \xi_i(t)), \qquad (1.1)$$

where $t = 0, 1, 2, ...,$, ξ_i^t are random real valued functions of discrete time t, the $\bar{u}_i$ denote time averages with the weight ϕ: $\bar{u}_i = \sum_{\tau=0,1,...} u_i(t-\tau)\phi(\tau)$, where ϕ is a decreasing function (a natural choice is $\phi = \exp(-\gamma t)$). Initial conditions are $u_i(0) \equiv s_i$, The function σ is an increasing function satisfying $\lim_{z\to-\infty} \sigma(z) = 0, \quad \lim_{z\to\infty} \sigma(z) = 1$ (for example, $\sigma(z) = \sigma_m(z) = \frac{z^m}{1+z^m}$) for $z > 0$ and 0 for $z \le 0$, $m > 0$). This model and the corresponding continuous time analogoes are paradigmactical for neural and gene networks [7, 6, 9, 8].

The goal of this paper is to describe a new class of networks where we can perform an effective control of network dynamics. This dynamics can be very complex, even chaotic. Also we describe effects connecting with long memory in these networks which lead to a global network cruch.

Let us consider (1.1) with special K_{ij}. Consider a directed weighted graph (V, E) associated with (1.1), with N vertices, where the edge $(i, j) \in E$ only and only if $K_{ij} \neq 0$. Denote by V_i the connectivity of the i-th node. Suppose that there are $n << N$ special "key" nodes (leading centers) (they are indexed by $i = 1, 2, ..., n$) and $m = N - n$ "usual" nodes. We assume that i-th usual node can be connected only with the key nodes and the key nodes can be connected only with usual nodes. For $n = 1$ one has a model of a supercentralized system (an Empire, with a capital and many regions, a central bank and many small banks, or the world financial center like the USA and other countries that depend on this center's financial activity, or the financial market and the banks working on it). Besides such a network topology, we also consider more general situations where usual modes can interact.

Main results can be outlined as follows. When noises are absent ($\xi_i = 0$), one has

A Circuits (1.1) can simulate any dynamical systems. Namely, if $q(t+1) = F(q), q \in \mathbf{D}$ a discrete time dynamical system and $\epsilon > 0$ then there is a choice of $N > n$ and K_{ij} such that the dynamics of the key nodes is defined by the map $q(t+1) = F(q) + \epsilon G(q), q \in D$, where $|G| < 1$. Here D is a open ball in $\mathbf{R}^n$;

The results on dynamical complexity were known [11], however, in contrast to [11], for our network class the new control method admits a simple physical interpretation: we perform a control by adjusting connections between the central node and usual ones.

B Using the result **A** and results of [4], one can show that (1.1) simulate any Turing machines. An interesting effect is possible when the network with a larger noise could be more effective in computations than the corresponding network with smaller noise level.

Noised networks can be reduced, for large N, to networks without noise, but with a new sigmoidal function.

For networks with a memory and a more nontrivial interaction between usual nodes one has the following. Stability depends on some main parameters, in particular, the averaged connectivity W of interactions between usual nodes and γ, the memory fading rate. We show that such a network without noise can have a trivial dynamic for large γ when all trajectories converge to a trivial equilibria and a nontrivial bistable dynamics if γ is small enough. If a noise exists, in the limit $\gamma \to 0$ dynamically stable steady states can become unstable as a result of fluctuation onset.

2. Complicated behavior. Suppose the memory and the noises are absent, i.e., $\xi_i = 0$, and $\phi(\tau) = 1$ for $\tau = 0$ and $\phi = 0$ for $\tau > 0$. Then $\bar{u}_i = u_i$. First we show that dynamics of (1.1) is completely captured by states $q_l(t) = u_l(t), l = 1, 2, ..., n$ of leading centers. Suppose $\xi_i = 0$. Under our assumptions on the structure of (1.1) one has

$$q_l(t+1) = \sigma(\sum_{j=n+1}^{N} K_{lj} u_j(t)) \tag{2.1}$$

(we set $h_1 = h_2 = ... = h_n = 0$) and

$$u_j(t+1) = \sigma(\sum_{l=1}^{n} K_{jl} q_l(t) + h_j). \tag{2.2}$$

and thus $q_l(t+2) = G(q)$, where $G_l = \sigma(\sum_{j=n+1}^{N} K_{lj} \sigma(\sum_{l=1}^{n} K_{jl} q_l(t) + h_j)$.

Proposition 2.1. *Let us consider a dynamical system with discrete time defined on a closed ball B_R^n of the radius $R > 0$ in $\mathbf{R}^n$: $q(t+2) = F(q(t)), \quad q = (q_1, q_2, ..., q_n) \in B_R^n$, where F is a*

C^1-map from B_R^n to an open subset of B_R^n. Suppose that $\sigma \in C^\infty$ is a stictly increasing. Then for each $\epsilon > 0$ there are such coefficients a_{lj}, b_{jl} and thresholds h_j that $|G - F|_{C^1(B_R^n)} < \epsilon$.

Sketch of proof: to prove this assertion, we can use approximation theorems of the multilayered network theory [2]. Since σ is a strictly increasing function, one can define a C^1 -smooth vector field Q by $Q_l(q) = \sigma^{-1}(F_l(q))$. On B_R^n, this field can be approximated in C^1 -norm, within arbitrary precision, by a field $S_l(q, a, b, h) = \sum_{j=n+1}^N a_{lj}\sigma(\sum_{l=1}^n b_{jl}q_l(t) - h_j)$, for appropriate a, b, h. The proof is completed.

This result shows that any (up to topological equivalency) structurally stable dynamics can be obtained by centralized networks. In particular, we conclude that centralized networks can generate complicated chaotic attractors.

Proposition 2.1 entails such a corollary.

Proposition 2.2. *Centralized networks (1.1) can simulate any Turing machines.*

Recall that the Turing machine is an abstract model of computer. The Church thesis asserts that all possible computations can be done by the Turing machines. To prove this assertion, one can use results of [4]. In this paper it is established, in particular, that if a class of networks is capable to simulate all piecewise linear maps $q \to F = L(q)$, then, inside this class, we can simulate any prescribed Turing machine. It is obvious (due to Prop.2.1) that our class enjoys this property. Moreover, even it suffices to approximate all piecewise linear maps in dimension 2, $q \in \mathbf{R^2}$ [4].

So, circuits (1.1) can have complicated attractors (for $n \geq 1$) and also produce any program of development (for $n \geq 2$). To prescribe a complex behavioural program to a centralized system, we need at least two "controlling" centers, $n = 2$.

To create a chaos, it is sufficient to have only a single center, $n = 1$. In fact, it well known that maps from $[0, 1]$ to $[0, 1]$ can have chaotic attractors. Similar results can be obtained in time continuous case. Then, to create a chaos, our system should contain at least three centers and two centers to induce periodocal oscillations.

Let us consider the noisy case. Suppose ξ_i are mutually independent identically distributed noises without time correlation: ξ_i are random numbers drawn by a fixed distrubution, $Prob(\xi < x) = F(x)$, where $F(x)$ is a smooth function. We can apply the Central Limit Theorem that gives us (2.2) with a new $\sigma(z) = \bar{\sigma}(z)$, where $\bar{\sigma}$ is an averaged sigmoidal function: $\bar{\sigma} = \int_{-\infty}^{\infty} \sigma(z + \xi)\rho(\xi)d\xi$, $\rho(x) = F'(x)$ is distribution density of ξ_i.

3. Networks with memory and more complex topology.

We consider the simplest case of a single center, $n = 1$, but, on the other hand, let us assume that $\phi = \exp(-\gamma\tau)$. To make the problem more analytically tractable, we suppose $\sigma(z) = \sigma_m$ and $h_i = 0$. Let us consider now such a system

$$q(t+1) = q(t) - \beta q(t) + a\sum_{i=1}^{N} u_i, \tag{3.1}$$

$$u_i(t+1) = \sigma_m(\sum_j K_{ij}\bar{u}_j(t) + N^{-1}\beta q(t)). \tag{3.2}$$

This system admits, for example, the following economical interpretation. A center (state) concentrates resources q. A part βq of these resources are distributed for many enterprises u_i, on the other hand, these enterprises return to the center the quantity $a\sum_{i=1}^{N} u_i$ (as a tax, for exmaple). Here $\beta \in (0, 1)$, $a = O(1)$. We seek for equilibrium states of this network for small γ (a long memory dynamics). Let us denote $V_i = \sum_j K_{ij}$ (the connectivity of the i-th node).

Below we consider two main cases.

ER) Erdos -Renyi topology [5], where almost all V_i have the same order. Here we set, for simplicity, that $V_i = W$.

SF) Scale - free case [1], where there are a few strongly connected nodes and many weakly connected nodes.

We obtain the following results as $N \to \infty$ in the case **ER**. I) If W, β are small enough, $\gamma = O(1)$ there exists a unique steady state $q = 0$ attracting globally all the dynamics;

II) For small γ a there are possible three steady states: two stable, in particular, a trivial state $q = 0$, a nontrivial state $q_{eq} \neq 0$ (that corresponds to global network activity) and a unstable state.

Stability of the nontrivial steady state decreases as $\gamma \to 0$: the corresponding multiplicators λ have the form $\lambda(\gamma) \approx \exp(-\gamma) + g(W, \gamma, a)$, where $g(W, \gamma, a) = O(\gamma^m)$. If $\lambda > 1$ one has instability; if $\lambda < 1$ our steady state is stable. If $m < 1$ (the sigmoid is not sharp) this steady state losts stability dynamically. If $m > 1$, one can see that $\lambda \to 1$ as $\gamma \to 0$ and $\lambda < 1$ if γ is small enough, in this case we should take into account fluctuations. In this case glaoblly active state of the network can be stable longtime but finally it falls. There exists an optimal value of γ which gives a maximal stability for the globally active steady state of the network. One can show that the stability of this steady state is an increasing function of the connectivity W.

For the case **SF** the steady state existence can be shown in asymptotical limit $\gamma \to 0$. To find an exact estimate of the stability is now a more difficult task, however, we can apply standard varitional methods to estimate multiplicators. Rough estimates by test functions show that, the instability onset starts with strongly connected nodes (therefore, to support system at the critical moment we should support system forming enterprises). Notice that physical network approaches to economics have been developed last decades (see [1, 3, 10]), here we concentrate on memory effects.

Memory may have the meaning of credits, therefore, we come to the following conclusions: too short money does not allow to activate weakly connected economical networks; too long money can destroy these network; to increase stability we should increase connectivity; at the crash moment it is better to support billioners than simple men.

This phenomenon has clear economical meaning: 1. Absence of credits does not allow the economy to develop - when money is too short, companies have to repay credits before they earn profits. So companies which decided to take loans can go bankrupt, while companies that do not use borrowed money (in order to avoid the risk of insolvency) cannot develop. The same is true for situations with high interest rate; 2. When credits are given at attractive terms (long repayment period and low interest rate), it helps to speed up the economical development, but more credits were taken, heavier the financial burden is. Companies have the tendency to take as many credits as possible, and this burden crushes them even before the date of payment comes. Therefore, credit terms should not be too attractive in order to prevent a long financial memory. It is important to mention that the same is true for banks working on financial market (in this case memory means credits given and investments made). Accumulating this memory leads to the problem of non-sufficient assets. Banks try to hidden this memory by making financial operations beyond the balance sheet, but it helps only in short-run. In long-run it makes the situation worse: as these operations are beyond the balance, they are also beyond audit and beyond the legal control. Banks have the impression that they may perform as many operations as they wish. But accumulating bad memory leads to a crash. For example, a huge volume of such operations destroyed the Lehman Brothers bank.

References

[1] R. Albert and A. L. Barabási, Rev. Modern Physics, 2002, **74**, 47-97

[2] A. Barron, IEEE Trans. on Inf. theory, 1993 **39**, 930-945.

[3] G. Bianconi, A-L. Barabasi, (2001). Phys. Rev. Lett. 2001, **86** 5632-35.

[4] Blondel V., Bournez O., Koiran P. and Tsitsiklis J. Journal of Computer and System Sciences, 2001, **62**, 442-462

[5] Erdos P., Rényi A., Publ. Math. Inst. Hungarian Academy of Sciences, 1960, **5**, 17-61

[6] L. Glass and S. Kauffman, J. Theor. Biology, 1973 **34**, 103-129.

[7] J. J. Hopfield, Proc. of Natl. Acad. USA,1982, **79** 2554-2558.

[8] A. Lesne. Letters in Math. Phys., 2006, **78**, 235-262

[9] J. Reinitz and D. H. Sharp, Mechanisms of Development, 1995, **49**, 133-158.

[10] D. Sornette Why Stock Markets Crash?. Princeton University Press 2003.

[11] S. A. Vakulenko, Advances in Diff. Equations, **5**, (2000), 1139-1178

[12] Vakulenko S., Grigoriev D., J. Math. Sci., 2006, **138**, p. 5644-5662