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MATHEMATICAL MODELING IN PROBLEMS ABOUT DYNAMICS AND STABILITY OF ELASTIC ELEMENTS OF WING PROFILES

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Abstract

The mathematical models describing the dynamics of elastic elements of wing structures and representing the initial-boundary value problems for systems of partial differential equations are proposed. The dynamics and stability of elastic elements of wings, flown around by a gas or liquid stream in a model of an incompressible medium, are investigated. To study the dynamics of elastic elements and a gas-liquid medium, both linear and nonlinear models of the mechanics of a solid deformable body and linear models of the mechanics of liquid and gas are used. On the basis of the constructed functionals for partial differential equations, the sufficient stability conditions are obtained in analytical form. The conditions impose restrictions on the parameters of mechanical systems. The obtained stability conditions are necessary for solving the problems of controlling the parameters of the aeroelastic system. On the basis of the Galerkin method, a numerical study of the dynamics of elastic elements was carried out, the reliability of which is confirmed by the obtained analytical results.

Key words

Fluid-structure interaction; Wing profile; Elastic plate; Dynamics; Stability; Partial differential equations; Functional.

1 Introduction

In the design and operation of constructions, instruments, devices, installations for various purposes interacting with a liquid flow, an important problem is to ensure the reliability of their operation and increase their service life. The similar problems are inherent in many branches of technology. In particular, the problems of this kind arise in mechanical engineering, aircraft engineering, instrumentation, etc. The study of the stability of deformable elements is of great importance in calculating structures interacting with a liquid or gas flow, since the effect of a flow can lead to its loss.

To ensure the reliability of operation and the accuracy of the functioning of aeroelastic structures, it is necessary to control their dynamics. In this regard, it is necessary to determine the boundaries of the regions of the parameters of mechanical systems that ensure the dynamic stability of structures. When the parameters of the system go beyond the boundaries of the stable operation of the structure, it is necessary to correct the system, which involves solving the problem of controlling the parameters.

The study of the stability of elastic bodies interacting with a gas or liquid flow is devoted to many theoretical and experimental studies.

However, in recent years, most of the works have been devoted to the study of the stability of pipeline systems and cylindrical shells interacting with a liquid or gas flow. Among the latter, we note the works [Abdelbaki et al., 2019; Blinkov et al., 2018; Butt et al., 2021; Chehreghani et al., 2021; Kheiri and Paidoussis, 2015; Kontzialis et al., 2017; Moditis et al., 2016; Mogilevich et al., 2017; Mogilevich et al., 2018; Mogilevich and Ivanov, 2020; Moshkelgosha et al., 2017] and many others. Among the works of the authors of this article on the study of the dynamics, stability and controllability of pipeline systems, we note the works [Gladun and Velmisov, 2019; Velmisov and Ankilov, 2016; Velmisov and Ankilov, 2017; Velmisov and Ankilov, 2018; Velmisov and Ankilov, 2019; Velmisov and Ankilov, 2021].

It should be noted that due to the complexity of the solution, there are fewer and fewer analytical studies on the dynamics, stability and flutter of aircraft components, including wing airfoils. Among the latter, we note [Al-Mashhadani et al., 2017; Balakrishnan et al., 2014; Shubov, 2014; Yonghong Li and Ning Qin, 2021; Zachary et al., 2020]. In contrast to the direct Lyapunov method used in this work, the presented works use frequency methods suitable only for studying linear systems, various numerical methods, experimental studies.

Among the works of the authors of this article on the study of the dynamics and stability of wing structures, we note the works [Ankilov and Velmisov, 2016; Velmisov and Ankilov, 2015]. The presented work is a continuation of the research [Velmisov and Ankilov, 2015] on the study of mathematical models of wing profiles constructions. The dynamic stability of the components of these constructions – the elastic elements, which are deformable plates, is investigated. The definitions of the stability of an elastic body adopted in this work correspond to the Lyapunov concept of stability of dynamical systems. The problem can be formulated as follows: at what values of the parameters characterizing the "liquid-body" system (the main parameters are the flow velocity, strength and inertial characteristics of the body, compressive or tensile forces, friction forces), to small deformations of the bodies at the initial moment of time $t = 0$ (small initial deviations from the equilibrium position) will correspond the small deformations at any moment of time $t > 0$.

2 Dynamic stability of the elastic connecting element of the composite wing

2.1 Mathematical model

Let us consider the plane problem of aerohydroelasticity about small vibrations arising in a noncirculating flow around a wing profile, the two components of which are connected by a deformable element, a gas or liquid stream in the model of an ideal incompressible medium.

Let on the plane xOy in which the joint vibrations of the deformed element and the gas occur, the component parts correspond to the segments $[a, b]$ and $[c, d]$ on the axis Ox, and the segment $[b, c]$ to the deformed element (Figure 1). At an infinitely distant point, the gas velocity is equal V and has a direction that coincides with the direction of the axis Ox.

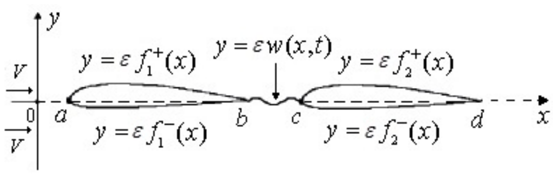

Figure 1. Cross section of the wing

Let us introduce the notation: $f_1^{\pm}(x), f_2^{\pm}(x)$ are the functions that determine the shape of the composite nondeformable parts of the profile; $u(x,t),\ w(x,t),\ x \in (b,c)$ are the functions of element deformations in the direction of the axes Ox and Oy; $P(x,t)$ is aerodynamic load on the element; $\phi(x,y,t)$ is the potential of the velocity of the disturbed flow. Then the mathematical formulation of the problem has the form:

$$\Delta\phi \equiv \phi_{xx} + \phi_{yy} = 0,\ (x,y) \in G = R^2 \backslash [a,d], \quad (1)$$

$$\phi_y^{\pm}(x,0,t) = \begin{cases} V{f_1^{\pm}}'(x),\ x \in (a,b), \\ \dot{w}(x,t) + Vw'(x,t),\ x \in (b,c), \\ V{f_2^{\pm}}'(x),\ x \in (c,d), \end{cases} \quad (2)$$

$$\lim_{x^2+y^2\to\infty} \phi_x = 0,\ \lim_{x^2+y^2\to\infty} \phi_y = 0,\ \lim_{x^2+y^2\to\infty} \phi_t = 0, \quad (3)$$

$$\begin{gathered} P(x,t) = \rho(\phi_t^+(x,0,t) - \phi_t^-(x,0,t)) + \\ + \rho V(\phi_x^+(x,0,t) - \phi_x^-(x,0,t)), \quad x \in (b,c), \end{gathered} \quad (4)$$

where the subscripts x, y, t below denote the derivatives with respect to coordinates x, y and time t; prime and dot denote derivatives with respect to x and t; $\phi_x^{\pm}(x,0,t) = \lim\limits_{y\to\pm 0} \phi_x(x,y,t)$; $\phi_y^{\pm}(x,0,t) = \lim\limits_{y\to\pm 0} \phi_y(x,y,t)$. The mathematical formulation (1)–(4) is written in a linear asymptotic approximation (corresponding to a thin profile and small deformation of an elastic element), obtained from exact equations and boundary conditions using asymptotic expansions $\phi_*(x,y,t) = Vx + \varepsilon\phi(x,y,t) + ...,\ w_*(x,t) = \varepsilon w(x,t) + ...,\ u_*(x,t) = \varepsilon^2 u(x,t) + ...,\ f_{k*}^{\pm}(x) = \varepsilon f_k^{\pm}(x) + ...$, where ε is a small parameter characterizing the thickness of the composite parts of the profile.

2.2 Solution of the aerodynamic part of the problem

In the domain G, assuming t as a parameter, we introduce the complex potential $W = f(z,t) = \phi + i\psi$, where $\psi = \psi(x,y,t)$ is a flow function, $z = x + iy$. Since $f(z,t)$ is an analytic function of the complex variable z, then $\psi_x = -\phi_y$. Using the function $\zeta = -\sqrt{\dfrac{d-z}{z-a}}$ we conformally map the domain G onto the upper half-plane $H = \{\zeta : Im\zeta > 0\}$, herewith $\sqrt{\dfrac{d-z}{z-a}} > 0$ on the upper bank of the cut $[a,d]$. Using the Schwarz integral for the half-plane, taking into account (2), (3) we find $f(z(\zeta),t)$. Passing to the limit $z \to x \pm i0,\ x \in (a,d)$, according to Sokhotskiy's formulas, we obtain the aerohydrodynamic impact (4) in form:

$$\begin{gathered} P(x,t) = -\frac{\rho}{\pi}\int\limits_b^c (\ddot{w}(\tau,t) + V\dot{w}'(\tau,t))K(\tau,x)d\tau - \\ -\frac{V\rho}{\pi}\int\limits_b^c (\dot{w}(\tau,t) + Vw'(\tau,t))K'_x(\tau,x)d\tau + \\ +\frac{V^2\rho}{2\pi}\int\limits_a^b \left({f_1^+}'(\tau) + {f_1^-}'(\tau)\right)K'_x(\tau,x)d\tau + \end{gathered} \quad (5)$$

$$+\frac{V^2\rho}{2\pi}\int\limits_c^d \left(f_2^{+\prime}(\tau)+f_2^{-\prime}(\tau)\right)K'_x(\tau,x)d\tau,\ x\in(b,c),$$

where $K(\tau,x)=$

$$=2\cdot\ln\left|\frac{\sqrt{(x-a)(d-\tau)}+\sqrt{(\tau-a)(d-x)}}{\sqrt{(x-a)(d-\tau)}-\sqrt{(\tau-a)(d-x)}}\right|, \quad (6)$$

$\tau,x\in[a,d],\ \tau\neq x$. It is not hard to see that

$$K(\tau,x)\geq 0,\quad K(\tau,x)=K(x,\tau). \quad (7)$$

The impact (5) was obtained for any methods of fixing a deformable element.

2.3 Models of deformable body

To study the dynamics of elastic elements and gas-liquid medium two models of solid mechanics are used.

I. Linear model of an elastic body:

$$\begin{aligned}&M\ddot{w}(x,t)+Dw''''(x,t)+N(t)w''(x,t)+\\&+\beta_0 w(x,t)+\beta_1\dot{w}(x,t)+\beta_2 I\dot{w}''''(x,t)=\\&=P(x,t),\ x\in(b,c).\end{aligned} \quad (8)$$

II. Nonlinear model of an elastic body:

$$\begin{cases}-\frac{EF}{2}\left(2u'(x,t)+w'^2(x,t)\right)'+\\+M\ddot{u}(x,t)=0,\\-\frac{EF}{2}\left[w'(x,t)\left(2u'(x,t)+w'^2(x,t)\right)\right]'+\\+M\ddot{w}(x,t)+Dw''''(x,t)+N(t)w''(x,t)+\\+\beta_0 w(x,t)+\beta_1\dot{w}(x,t)+\beta_2 I\dot{w}''''(x,t)=\\=P(x,t),\quad x\in(b,c),\end{cases} \quad (9)$$

where E, h, ρ_p are modulus of elasticity, thickness and density of the element; $N(t)$ is compressive or tensile force of the element; $D=EI$, $M=h\rho_p$ are flexural stiffness and linear mass of the element; $F=\frac{h}{1-\nu^2}$; $I=\frac{h^3}{12(1-\nu^2)}$; ν is Poisson's ratio; β_2, β_1 are coefficients of external and internal damping; β_0 is coefficient of stiffness of the base.

2.4 Investigation of stability for a linear model of an elastic body

Consider a linear model of an elastic body (8). Since the system of equations (5), (8) is linear, it suffices to investigate the stability of the zero solution $w(x,t)\equiv 0$ of the corresponding homogeneous equation

$$\begin{aligned}&M\ddot{w}(x,t)+Dw''''(x,t)+N(t)w''(x,t)+\beta_0 w(x,t)+\\&+\beta_1\dot{w}(x,t)+\beta_2 I\dot{w}''''(x,t)=\\&=-\frac{\rho}{\pi}\int\limits_b^c(\ddot{w}(\tau,t)+V\dot{w}'(\tau,t))K(\tau,x)d\tau-\\&-\frac{V\rho}{\pi}\int\limits_b^c(\dot{w}(\tau,t)+Vw'(\tau,t))K'_x(\tau,x)d\tau,\ x\in(b,c).\end{aligned} \quad (10)$$

Suppose that the ends of the deformable element are fixed either rigidly or hinged (in any combination), then one of the conditions is fulfilled

$$\begin{aligned}&1)\ w(\omega,t)=w'(\omega,t)=0,\\&2)\ w(\omega,t)=w''(\omega,t)=0,\end{aligned} \quad (11)$$

where $\omega=b$ or $\omega=c$.

Let us obtain the sufficient conditions for the stability of the solution to the zero solution of the integro-differential equation (10) with respect to perturbations of the initial conditions. Consider the functional

$$\begin{aligned}\Phi=&\int\limits_b^c\left\{M\dot{w}^2+Dw''^2-N(t)w'^2+\beta_0 w^2\right\}dx+\\&+\frac{\rho}{\pi}\int\limits_b^c dx\int\limits_b^c \dot{w}(x,t)\dot{w}(\tau,t)K(\tau,x)d\tau-\\&-\frac{\rho V^2}{\pi}\int\limits_b^c dx\int\limits_b^c w'(x,t)w'(\tau,t)K(\tau,x)d\tau.\end{aligned} \quad (12)$$

Integration by parts, taking into account the conditions (11), we get:

$$\begin{aligned}&\int\limits_b^c \dot{w}w''''dx=\int\limits_b^c \dot{w}''w''dx,\quad \int\limits_b^c \dot{w}w''dx=\\&=-\int\limits_b^c \dot{w}'w'dx,\quad \int\limits_b^c \dot{w}\dot{w}''''dx=\int\limits_b^c \dot{w}''^{\,2}dx.\end{aligned} \quad (13)$$

Substituting (10) into the derivative of the functional (12) taking into account the equalities (13) and symmetry of the kernel $K(\tau,x)$, we obtain

$$\dot{\Phi}=-2\int\limits_b^c\left(\frac{\dot{N}(t)}{2}w'^2+\beta_2 I\dot{w}''^2+\beta_1\dot{w}^2\right)dx. \quad (14)$$

The quadratic form under the integral sign in (14) is positively semidefinite if the conditions

$$\beta_1\geq 0,\quad \beta_2\geq 0,\quad \dot{N}(t)\geq 0. \quad (15)$$

be satisfied. Then from (14) we obtain the estimate

$$\dot{\Phi}\leq 0. \quad (16)$$

Integrating (16) from 0 to t, we obtain

$$\Phi(t) \leq \Phi(0). \tag{17}$$

It was proved that the following estimates are valid for a kernel $K(\tau, x)$ of the form (6):

$$0 \leq \int_b^c dx \int_b^c \dot{w}(x,t)\dot{w}(\tau,t)K(\tau,x)d\tau \leq \leq K_0 \int_b^c \dot{w}^2(x,t)dx, \tag{18}$$

$$0 \leq \int_b^c dx \int_b^c w'(x,t)w'(\tau,t)K(\tau,x)d\tau \leq \leq G_0 \int_b^c w'^2(x,t)dx, \tag{19}$$

where $K_0 = \sup\limits_{x\in[b,c]} \int_b^c |K(\tau,x)|\, d\tau$,

$$G_0 = \sup_{x\in[b,c]} \int_b^c |K(\tau,x) + g_1(x) + g_1(\tau)|\, d\tau,$$

$g_1(x)$ is an arbitrary function integrable over x on the segment $[b, c]$, chosen so that the value G_0 is the smallest.

Taking into account (18), (19), we estimate $\Phi(t)$:

$$\Phi(t) \geq \int_b^c \Big\{ M\dot{w}^2 + Dw''^2 - - \left(N(t) + \frac{G_0\rho V^2}{\pi} \right) w'^2 + \beta_0 w^2 \Big\} dx. \tag{20}$$

According to the Rayleigh inequality [Kollatz, 1968], the following estimate is valid:

$$\int_b^c w''^2(x,t)dx \geq \lambda_1 \int_b^c w'^2(x,t)dx, \tag{21}$$

where λ_1 is the smallest eigenvalue of a boundary value problem $\omega^{IV}(x) = -\lambda\omega''(x),\ x \in [b, c]$ with boundary conditions corresponding to (11).

Taking into account (21), from (20) we obtain

$$\Phi(t) \geq \int_b^c \Big\{ M\dot{w}^2 + \beta_0 w^2 + + \left(\lambda_1 D - N(t) - \frac{G_0\rho V^2}{\pi} \right) w'^2 \Big\} dx. \tag{22}$$

Consider the quadratic form under the integral sign in (22). Assuming that the quadratic forms with respect to $\dot{w}(x,t),\ w'(x,t)$ will be positively definite, and the quadratic form with respect to $w(x,t)$ will be positively semidefinite, we obtain the conditions

$$M > 0, \quad \beta_0 \geq 0, \quad N(t) < \lambda_1 D - \frac{G_0\rho V^2}{\pi}. \tag{23}$$

Then we obtain the inequality $\Phi(t) \geq 0$.

From (17), (22), taking into account the Cauchy-Bunyakovsky inequality

$$w^2(x,t) \leq (c-b) \int_b^c w'^2(x,t)dx,$$

we obtain the estimate

$$\left(\lambda_1 D - N(t) - \frac{G_0\rho V^2}{\pi} \right) \frac{w^2(x,t)}{c-b} \leq \Phi(0), \tag{24}$$

from which the solution $w(x,t)$ is estimated by the initial values of this function and its derivatives.

Since, under conditions (15), (23), the functional (12) satisfies the conditions $\dot{\Phi}(t) \leq 0$, $\Phi(t) \geq 0$, then derivatives $\dot{w}(x,t),\ w'(x,t)$ are stable with respect to perturbations of the initial conditions. From inequality (24), we can conclude that the solution $w(x,t)$ is stable with respect to perturbations of the initial data. Thus, we have proved the following theorem.

Theorem 2.1. *If the function $w(x,t)$ satisfies the boundary conditions (11) and conditions (15), (23) are satisfied, then the solution $w(x,t)$ of equation (10) and the derivatives $\dot{w}(x,t), w'(x,t)$ are stable with respect to perturbations of the initial data.*

2.5 Investigation of stability for a nonlinear model of an elastic body

Consider a nonlinear model of an elastic body (9). Assuming about symmetry of wing profiles $f_1^+(x) = -f_1^-(x)$, $f_2^+(x) = -f_2^-(x)$ the system of equations (5), (9) takes the form

$$\begin{cases} -\frac{EF}{2}\left(2u'(x,t) + w'^2(x,t)\right)' + \\ +M\ddot{u}(x,t) = 0, \\ -\frac{EF}{2}\left[w'(x,t)\left(2u'(x,t) + w'^2(x,t)\right)\right]' + \\ +M\ddot{w}(x,t) + Dw''''(x,t) + Nw''(x,t) + \\ +\beta_0 w(x,t) + \beta_1\dot{w}(x,t) + \beta_2 I\dot{w}''''(x,t) = \\ = -\frac{\rho}{\pi}\int_b^c (\ddot{w}(\tau,t) + V\dot{w}'(\tau,t))K(\tau,x)d\tau - \\ -\frac{V\rho}{\pi}\int_b^c (\dot{w}(\tau,t) + Vw'(\tau,t))K'_x(\tau,x)d\tau. \end{cases} \tag{25}$$

For this system for a fixed end ($x = b$ or $x = c$), it is necessary to conditions (11) add condition $u(x,t) = 0$, and for the movable end: $u'(x,t) + 0.5w'^2(x,t) = 0$.

Based on research of the functional

$$\Phi(t) = \int_b^c \left(EF\left(u' + 0.5w'^2\right)^2 + M\dot{u}^2 + M\dot{w}^2 + \right.$$
$$\left. + Dw''^2 + \beta_0 w^2 - N(t)w'^2\right)dx +$$
$$+\frac{\rho}{\pi}\int_b^c dx \int_b^c \dot{w}(x,t)\dot{w}(\tau,t)K(\tau,x)d\tau -$$
$$-\frac{\rho V^2}{\pi}\int_b^c dx \int_b^c w'(x,t)w'(\tau,t)K(\tau,x)d\tau$$

similarly, as in section 2.4, we investigated the stability of the zero solution $u(x,t) \equiv 0$, $w(x,t) \equiv 0$ of the system of equations (25) and proved the theorem.

Theorem 2.2. *If the conditions (15), (23) are satisfied, then the solution $w(x,t)$ of the system of equations (25) and the derivatives $\dot{u}(x,t), u'(x,t), \dot{w}(x,t), w'(x,t)$ are stable with respect to perturbations of the initial data.*

3 Dynamic stability of the elastic aileron of wing

3.1 Mathematical model

The plane problem of aeroelasticity about small oscillations of a deformable aileron of the wing, in the trace of which another wing is located (the wings are located sequentially one after another along one line) with a subsonic flow of an ideal incompressible gas around the wings is also considered. Let on the plane Oxy, in which the joint oscillations of the deformable aileron and the subsonic flow of an ideal gas (liquid) occur, to the first wing corresponds a segment $[a_1, c_1]$ on the axis Ox, to the aileron – a segment $[b_1, c_1]$, to the second wing – a segment $[a_2, b_2]$, where $a_2 > c_1$ (Figure 2). At an infinitely distant point, the gas velocity is equal V and has a direction that coincides with the direction of the axis Ox.

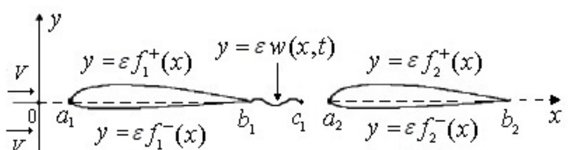

Figure 2. Cross-section of system of two wings of "tandem" type

The mathematical formulation of the problem has the form:

$$\phi_{xx} + \phi_{yy} = 0, \quad (x,y) \in G = R^2 \backslash \left([a_1, c_1] \cup [a_2, b_2]\right), \tag{26}$$

$$\phi_y^{\pm} = \begin{cases} V f_k^{\pm\prime}(x),\ x \in (a_k, b_k),\ k = 1,2, \\ \dot{w}(x,t) + Vw'(x,t),\ x \in (b_1, c_1), \end{cases} \tag{27}$$

$$\left(\phi_x^2 + \phi_y^2 + \phi_t^2\right)_\infty = 0, \tag{28}$$

$$P(x,t) = \rho(\phi_t^+ - \phi_t^-) + \rho V(\phi_x^+ - \phi_x^-), \quad x \in (b_1, c_1). \tag{29}$$

3.2 Solution of the aerodynamic part of the problem

In the domain G, assuming t as a parameter, we introduce the complex potential $W = f(z,t) = \phi + i\psi$, where $\psi = \psi(x,y,t)$ is a flow function, $z = x + iy$. For the velocity function $f_z(z,t) = \phi_x - i\phi_y$ according to equation (26) and conditions (27) we have the following integral representation

$$f_z(z,t) = \frac{1}{\pi\sqrt{h(z)}}\left(-\int_{b_1}^{c_1} \frac{\dot{w}(\tau,t) + Vw'(\tau,t)}{\tau - z}\sqrt{h(\tau)}d\tau + \Gamma(t) + \frac{V}{2}\int_{a_2}^{b_2} \frac{f_2^{+\prime}(\tau) + f_2^{-\prime}(\tau)}{\tau - z}\sqrt{h(\tau)}d\tau - \frac{V}{2}\int_{a_1}^{b_1} \frac{f_1^{+\prime}(\tau) + f_1^{-\prime}(\tau)}{\tau - z}\sqrt{h(\tau)}d\tau\right) - \frac{V}{2\pi}\int_{a_1}^{b_1} \frac{f_1^{+\prime}(\tau) - f_1^{-\prime}(\tau)}{\tau - z}d\tau - \frac{V}{2\pi}\int_{a_2}^{b_2} \frac{f_2^{+\prime}(\tau) - f_2^{-\prime}(\tau)}{\tau - z}d\tau, \tag{30}$$

where $h(z) = (z - a_1)(z - c_1)(z - a_2)(b_2 - z)$; $\Gamma(t)$ is a function determining the circulation of the gas velocity around each plate. Branch of the root in the formula (30) is fixed by the condition

$$\sqrt{h(z)} = i\sqrt{(x - a_1)(x - c_1)(x - a_2)(x - b_2)}, \quad z = x > b_2.$$

We select the function $\Gamma(t)$ so that the circulation around each wing equaled zero. Then, integrating (30), we find the complex potential. Passing to the limit $z \to x \pm i0$, $x \in (a_1, c_1)$, according to the formulas Sokhotskiy, we obtain the aerohydrodynamic action (29) in the form:

$$P(x,t) = -\frac{\rho}{\pi}\int_{b_1}^{c_1} (\ddot{w}(\tau,t) + V\dot{w}'(\tau,t))K(\tau,x)d\tau -$$

$$
\begin{aligned}
&-\frac{\rho V}{\pi}\int\limits_{b_1}^{c_1}\left(\dot w(\tau,t)+Vw'(\tau,t)\right)K'_x(\tau,x)d\tau-\\
&-\frac{\rho V^2}{\pi\sqrt{h(x)}}\int\limits_{a_1}^{b_1}\frac{{f_1^+}'(\tau)+{f_1^-}'(\tau)}{\tau-x}\sqrt{h(\tau)}d\tau+\\
&+\frac{\rho V^2}{\pi\sqrt{h(x)}}\int\limits_{a_2}^{b_2}\frac{{f_2^+}'(\tau)+{f_2^-}'(\tau)}{\tau-x}\sqrt{h(\tau)}d\tau,
\end{aligned}
\tag{31}
$$

where

$$
\begin{aligned}
K(\tau,x)=\int\limits_{a_1}^{x}\frac{\sqrt{h(\tau)}}{\sqrt{h(x)}(\tau-x)}dx-\\
-\left(\int\limits_{a_1}^{c_1}\frac{dx}{\sqrt{h(x)}}\right)^{-1}\cdot\int\limits_{a_1}^{x}\frac{dx}{\sqrt{h(x)}}\cdot\int\limits_{a_1}^{c_1}\frac{\sqrt{h(\tau)}}{\sqrt{h(x)}(\tau-x)}dx.
\end{aligned}
\tag{32}
$$

Note that kernel (32), in contrast to kernel (6), is asymmetric and indefinite in sign. We decompose the kernel into symmetric and skew-symmetric parts

$$
\begin{aligned}
K(\tau,x)&=G_1(\tau,x)+G_2(\tau,x),\\
G_1(\tau,x)&=\frac{K(\tau,x)+K(x,\tau)}{2},\\
G_2(\tau,x)&=\frac{K(\tau,x)-K(x,\tau)}{2}.
\end{aligned}
\tag{33}
$$

3.3 Investigation of stability for a linear model of an elastic body

Since the right end is free, then $N(t)\equiv 0$. Consider a linear model of an elastic body (8). Since the system of equations (8), (31) is linear, it is sufficient to investigate the stability of the zero solution $w(x,t)\equiv 0$ of the corresponding homogeneous equation:

$$
\begin{aligned}
&M\ddot w+Dw''''+\beta_0 w+\beta_1\dot w+\beta_2 I\dot w''''=\\
&=-\frac{\rho}{\pi}\int\limits_{b_1}^{c_1}(\ddot w(\tau,t)+V\dot w'(\tau,t))K(\tau,x)d\tau-\\
&-\frac{\rho V}{\pi}\int\limits_{b_1}^{c_1}\left(\dot w(\tau,t)+Vw'(\tau,t)\right)K'_x(\tau,x)d\tau.
\end{aligned}
\tag{34}
$$

Consider the case of elastic fastening of the left end of the aileron with the wing and the free right end, then the boundary conditions at the ends of the element are:

$$
\begin{aligned}
w(b_1,t)=0,\ w''(b_1,t)=\alpha w'(b_1,t),\\
w''(c_1,t)=0,\ w'''(c_1,t)=0,
\end{aligned}
\tag{35}
$$

where number α is the coefficient of stiffness of the elastic connection between the aileron and the wing.

Consider the functional:

$$
\begin{aligned}
\Phi=&\int\limits_{b_1}^{c_1}\left((M+\beta_1\theta_2)\,\dot w^2+2\left(\theta_1 M+\beta_0\theta_2\right)w\dot w+\right.\\
&+(D+\beta_2 I\theta_1)\,w''^2+(\beta_0+\beta_1\theta_1)\,w^2+\beta_2 I\theta_2\dot w''^2+\\
&\left.+\,2D\theta_2\dot w''w''\right)dx+2\alpha D\theta_2 w'(b,t)\dot w'(b,t)+\\
&+\alpha\beta_2 I\theta_2\dot{w'}^2(b,t)+\alpha\left(D+\beta_2 I\theta_1\right)w'^2(b,t)+\\
&+\frac{\rho}{\pi}\int\limits_{b_1}^{c_1}dx\int\limits_{b_1}^{c_1}\dot w(x,t)\dot w(\tau,t)G_1(\tau,x)d\tau-\\
&-\frac{\rho V^2}{\pi}\int\limits_{b_1}^{c_1}dx\int\limits_{b_1}^{c_1}w'(x,t)w'(\tau,t)G_1(\tau,x)d\tau-\\
&-\frac{4\rho V}{\pi}\int\limits_{b_1}^{c_1}dx\int\limits_{b_1}^{c_1}w'(x,t)\dot w(\tau,t)G_2(\tau,x)d\tau-\\
&-\frac{2\rho V\theta_2}{\pi}\int\limits_{b_1}^{c_1}dx\int\limits_{b_1}^{c_1}\dot w'(x,t)\dot w(\tau,t)K(\tau,x)d\tau-\\
&-\frac{2\rho V^2\theta_2}{\pi}\int\limits_{b_1}^{c_1}dx\int\limits_{b_1}^{c_1}\dot w'(x,t)w'(\tau,t)K(\tau,x)d\tau-\\
&-\frac{2\rho V\theta_1}{\pi}\int\limits_{b_1}^{c_1}dx\int\limits_{b_1}^{c_1}w'(x,t)w(\tau,t)G_1(\tau,x)d\tau-\\
&-\frac{2\rho V\theta_1}{\pi}\int\limits_{b_1}^{c_1}dx\int\limits_{b_1}^{c_1}w'(x,t)w(\tau,t)G_2(\tau,x)d\tau+\\
&+\frac{2\rho\theta_1}{\pi}\int\limits_{b_1}^{c_1}dx\int\limits_{b_1}^{c_1}w(x,t)\dot w(\tau,t)G_2(\tau,x)d\tau,
\end{aligned}
\tag{36}
$$

where θ_1,θ_2 are some positive parameters.

Let us introduce the notation

$$
\begin{aligned}
G_{10}&=\sup_{x\in[b_1,c_1]}\int\limits_{b_1}^{c_1}\left|G_1(\tau,x)\right|d\tau,\\
G_{20}&=\sup_{x\in[b_1,c_1]}\int\limits_{b_1}^{c_1}\left|G_2(\tau,x)\right|d\tau,\\
K_{10}&=\sup_{x\in[b_1,c_1]}\int\limits_{b_1}^{c_1}\left|K(\tau,x)\right|d\tau,\\
K_{20}&=\sup_{x\in[b_1,c_1]}\int\limits_{b_1}^{c_1}\left|K(x,\tau)\right|d\tau.
\end{aligned}
\tag{37}
$$

Taking into account (37), we obtain the following estimate for the derivative of the functional (36):

$$\dot{\Phi} \le -2\int\limits_{b_1}^{c_1} \Big\{ (\beta_2 I - D\theta_2)\,\dot{w''}^2 - -\frac{\rho V^2}{\pi}\left(G_{20} + 4\theta_2 G_{10} + \theta_1 G_{10}\right)\dot{w}'^2(x,t) + +\left(\beta_1 - \beta_0\theta_2 - M\theta_1 - \frac{\rho G_{20}}{\pi}\right)\dot{w}^2 + \Big(M\theta_2 - -\frac{\rho}{\pi}\left[3G_{20} + 4\theta_2 G_{10} + \theta_1 G_{10} + \theta_1 G_{20}\right]\Big)\ddot{w}^2 - -\frac{\rho V^2}{\pi}\left(3G_{20} + 2\theta_1 G_{10}\right)w'^2(x,t) + + D\theta_1 w''^2 + \theta_1\left(\beta_0 - \frac{\rho}{\pi}\left[2G_{10} + G_{20}\right]\right)w^2 \Big\}\,dx - -2\alpha\left(\beta_2 I - D\theta_2\right)\dot{w}'^2(b_1,t) - 2\alpha D\theta_1 w'^2(b_1,t). \quad (38)$$

To estimate the integrals in (38), we use the Cauchy-Bunyakovsky and Rayleigh inequalities:

$$\int\limits_{b_1}^{c_1} w''^2(x,t)dx \ge \ge \frac{2}{(c_1-b_1)^2}\int\limits_{b_1}^{c_1}\left(w'(x,t) - w'(b_1,t)\right)^2 dx,$$
$$\int\limits_{b_1}^{c_1} \dot{w}''^2(x,t)dx \ge \ge \frac{2}{(c_1-b_1)^2}\int\limits_{b_1}^{c_1}\left(\dot{w}'(x,t) - \dot{w}'(b_1,t)\right)^2 dx, \quad (39)$$

$$\int\limits_{b_1}^{c_1} \dot{w}''^{\,2}(x,t)dx + \alpha\,\dot{w}'^{\,2}(b_1,t) \ge \ge \mu_1\int\limits_{b_1}^{c_1}\dot{w}^2(x,t)dx,$$
$$\int\limits_{b_1}^{c_1} w''^2(x,t)dx + \alpha w'^2(b_1,t) \ge \ge \mu_1\int\limits_{b_1}^{c_1} w^2(x,t)dx, \quad (40)$$

where μ_1 is the smallest eigenvalue of the boundary value problem for an equation $\psi^{IV}(x) = \mu\psi(x)$, $x \in [b_1, c_1]$ with boundary conditions (35). This problem is self-adjoint and fully defined under a certain condition

$$\alpha \ge 0. \quad (41)$$

Let the conditions

$$\beta_2 I - D\theta_2 > 0, \quad \chi_1 \in (0,1], \quad \chi_2 \in (0,1], \quad (42)$$

$$M \ge \frac{\rho}{\theta_2\pi}\left[3G_{20} + 4\theta_2 G_{10} + \theta_1 G_{10} + \theta_1 G_{20}\right], \quad (43)$$

be satisfied, where

$$\chi_1 = \begin{cases} 1, & \delta_1 \ge 0, \\ 1 + \dfrac{\delta_1}{(\beta_2 I - D\theta_2)\,\mu_1}, & \delta_1 < 0, \end{cases}$$
$$\delta_1 = \beta_1 - \beta_0\theta_1 - M\theta_1 - \frac{\rho G_{20}}{\pi},$$
$$\chi_2 = \begin{cases} 1, & \delta_2 \ge 0, \\ 1 + \dfrac{\delta_2}{D\pi\mu_1}, & \delta_2 < 0, \end{cases}$$
$$\delta_2 = \beta_0 - \frac{\rho}{\pi}\left[2G_{10} + G_{20}\right],$$

then from (38) we obtain

$$\dot{\Phi} \le -2\int\limits_{b_1}^{c_1}\Big\{ -\frac{4\left(\beta_2 I - D\theta_2\right)\chi_1}{(c_1-b_1)^2}\dot{w}'(x,t)\dot{w}'(b_1,t) + +\left[\frac{2\left(\beta_2 I - D\theta_2\right)\chi_1}{(c_1-b_1)^2} - \frac{\rho V^2}{\pi}\left(G_{20} + G_{10}(4\theta_2 + \theta_1)\right)\right]\times \times\dot{w}'^2(x,t) + \frac{\left(\beta_2 I - D\theta_2\right)\chi_1}{(c_1-b_1)^2}\left[2 + \alpha(c_1-b_1)\right]\times \times\dot{w}'^2(b_1,t) + \left[\frac{2D\chi_2\theta_1}{(c_1-b_1)^2} - \frac{\rho V^2}{\pi}\left(3G_{20} + 2\theta_1 G_{10}\right)\right]\times \times w'^2(x,t) - \frac{4D\chi_2\theta_1}{(c_1-b_1)^2}w'(x,t)w'(b_1,t) + + \frac{D\chi_2\theta_1}{(c_1-b_1)^2}\left[2 + \alpha(c_1-b_1)\right]w'^2(b_1,t)\Big\}\,dx. \quad (44)$$

Thus, we have obtained two quadratic forms with respect to $\dot{w}'(x,t),\ \dot{w}'(b_1,t)$ and $w'(x,t),\ w'(b_1,t)$. Let us write down the conditions for their positive semi-definiteness:

$$\left[2\left(\beta_2 I - D\theta_2\right)\pi\chi_1 - \rho V^2(c_1-b_1)^2\left(G_{20} + +4\theta_2 G_{10} + \theta_1 G_{10}\right)\right]\left[2 + \alpha(c_1-b_1)\right] \ge \ge 4\pi\left(\beta_2 I - D\theta_2\right)\chi_1, \quad (45)$$
$$\left[2D\pi\chi_2\theta_1 - \rho V^2(c_1-b_1)^2\left(3G_{20} + +2\theta_1 G_{10}\right)\right]\left[2 + \alpha(c_1-b_1)\right] \ge 4D\pi\chi_2\theta_1.$$

Taking into account (45), the inequality (44) takes the form:

$$\dot{\Phi} \le 0 \quad \Rightarrow \quad \Phi(t) \le \Phi(0). \quad (46)$$

We obtain an estimate for functional (36) using inequalities (39), (40):

$$\Phi \geq \int_{b_1}^{c_1} \{((1-\chi_5-\chi_6)\beta_2 I\theta_2\mu_1 + M + \beta_1\theta_2 - \frac{\rho}{\pi}[G_{10}+2G_{20}+\theta_2 K_{20}+\theta_1 G_{20}]\Big)\dot{w}^2(x,t) + 2(\beta_0\theta_2 + M\theta_1)\dot{w}(x,t)w(x,t) + ((1-\chi_3-\chi_4)\times (D+\beta_2 I\theta_1)\mu_1 - \frac{\rho\theta_1}{\pi}[K_{20}+G_{20}] + \beta_0 + \beta_1\theta_1)w^2(x,t)\Big)dx + \int_{b_1}^{c_1}\{(D+\beta_2 I\theta_1)\times \chi_4 w''^2(x,t) + 2D\theta_2\dot{w}''(x,t)w''(x,t) + \beta_2 I\theta_2\chi_6\dot{w}''^2(x,t)\}dx + \int_{b_1}^{c_1}\Big\{\Big(\frac{2\chi_3(D+\beta_2 I\theta_1)}{(c_1-b_1)^2} - \frac{\rho V^2}{\pi}[G_{10}+2G_{20}+\theta_2 K_{20}+\theta_1 K_{10}]\Big)w'^2(x,t) - \frac{4\chi_3(D+\beta_2 I\theta_1)}{(c_1-b_1)^2}w'(x,t)w'(b_1,t) + \frac{D+\beta_2 I\theta_1}{(c_1-b_1)^2}[2\chi_3+\alpha(\chi_3+\chi_4)(c_1-b_1)]w'^2(b_1,t) + \Big(\frac{2\beta_2 I\theta_2\chi_5}{(c_1-b_1)^2} - \frac{2\rho V^2\theta_2 K_{10}}{\pi}\Big)\dot{w}'^2(x,t) - \frac{4\beta_2 I\theta_2\chi_5}{(c_1-b_1)^2}\dot{w}'(x,t)\dot{w}'(b_1,t) + \frac{\beta_2 I\theta_2}{(c_1-b_1)^2}[2\chi_5+\alpha(\chi_5+\chi_6)(c_1-b_1)]\dot{w}'^2(b_1,t) + \frac{2\alpha D\theta_2}{c_1-b_1}w'(b_1,t)\dot{w}'(b_1,t)\Big\}dx, \quad (47)$$

where additional parameters

$$\begin{gathered}\chi_3\in(0,1],\ \chi_4\in(0,1],\ \chi_5\in(0,1],\\ \chi_6\in(0,1],\ \chi_3+\chi_4\in(0,1],\ \chi_5+\chi_6\in(0,1]\end{gathered} \quad (48)$$

are introduced.

The quadratic form with respect to $\dot{w}(x,t), w(x,t)$, in (47) will be positive definite if the following conditions

$$\begin{gathered}(1-\chi_5-\chi_6)\beta_2 I\theta_2\mu_1 + M + \beta_1\theta_2 - \frac{\rho}{\pi}[G_{10}+2G_{20}+\theta_2 K_{20}+\theta_1 G_{20}] > 0,\\ ((1-\chi_5-\chi_6)\beta_2 I\theta_2\mu_1 + M + \beta_1\theta_2 - \frac{\rho}{\pi}[G_{10}+2G_{20}+\theta_2 K_{20}+\theta_1 G_{20}])\times\\ \times((1-\chi_3-\chi_4)(D+\beta_2 I\theta_1)\mu_1 + \beta_0 + \beta_1\theta_1 - \frac{\rho\theta_1}{\pi}[K_{20}+G_{20}]) - (\beta_0\theta_2+M\theta_1)^2 \geq 0,\\ (D+\beta_2 I\theta_1)\beta_2 I\chi_4\chi_6 - D^2\theta_2 \geq 0,\\ a_{11}>0,\ a_{33}>0,\ a_{11}a_{22}-a_{12}^2>0,\\ (a_{11}a_{22}-a_{12}^2)(a_{33}a_{44}-a_{34}^2) - a_{11}a_{33}a_{24}^2 > 0\end{gathered} \quad (49)$$

are satisfied, where

$$a_{11} = \frac{2\chi_3(D+\beta_2 I\theta_1)}{(c_1-b_1)^2} - \frac{\rho V^2}{\pi}[G_{10}+2G_{20}+\theta_2 K_{20}+\theta_1 K_{10}],\quad a_{12} = -\frac{2\chi_3(D+\beta_2 I\theta_1)}{(c_1-b_1)^2},$$

$$a_{22} = \frac{D+\beta_2 I\theta_1}{(c_1-b_1)^2}[2\chi_3+\alpha(\chi_3+\chi_4)(c_1-b_1)],$$

$$a_{33} = \frac{2\beta_2 I\theta_2\chi_5}{(c_1-b_1)^2} - \frac{2\rho V^2\theta_2 K_{10}}{\pi},$$

$$a_{44} = \frac{\beta_2 I\theta_2}{(c_1-b_1)^2}[2\chi_5+\alpha(\chi_5+\chi_6)(c_1-b_1)],$$

$$a_{34} = -\frac{2\beta_2 I\theta_2\chi_5}{(c_1-b_1)^2},\quad a_{24} = \frac{\alpha D\theta_2}{c_1-b_1}.$$

Let conditions (48), (49) be satisfied, then, taking into account the Cauchy-Bunyakovsky inequality

$$w^2(x,t) \leq (c_1-b_1)\int_{b_1}^{c_1} w'^2(x,t)dx$$

from (46), (47) we obtain the estimate

$$\frac{a_{11}a_{33}a_{24}^2 + a_{12}^2 a_{33}a_{44} - a_{12}^2 a_{34}^2}{a_{22}(a_{33}a_{44}-a_{34}^2)(c_1-b_1)} w^2(x,t) \leq \Phi(0). \quad (50)$$

Thus, we have proved the following theorem.

Theorem 3.1. *Let the function $w(x,t)$ satisfy the boundary conditions (35) and for any moment of time will be found the parameters $\theta_1 > 0, \theta_2 > 0, \chi_i,\ i=\overline{1,6}$ such that conditions (41), (42), (43), (45), (48), (49) are satisfied. Then the solution $w(x,t)$ of equation (34) and the derivatives $\dot{w}(x,t), w'(x,t)$ are stable with respect to perturbations of the initial data.*

4 Galerkin's method

4.1 Linear model

The solution of equations (10), (34) is found by the Galerkin's method in the form

$$w(x,t) = \sum_{k=1}^{m} a_k(t)g_k(x), \quad (51)$$

where $g_k(x)$ are basis functions, selected so that the specified boundary conditions (11), (35) are fulfilled, and the functions $a_k(t)$ are determined from the condition of orthogonality of the residual of the equation to the system of the basis functions.

As a basis we take the functions

$$\begin{gathered}g_k(x) = A_k\cos\gamma_k(x-b) + B_k\sin\gamma_k(x-b) + C_k ch\gamma_k(x-b) + D_k sh\gamma_k(x-b), \quad (52)\\ k=1,2,3,\ldots\end{gathered}$$

We choose the coefficients A_k, B_k, C_k, D_k and the parameter γ_k so that at each endpoint of the segment $[b, c]$ the boundary conditions are satisfied. Note that γ_k and $g_k(x)$ are the eigenvalues and eigenfunctions of boundary value problems for the equation $g_k^{IV}(x) = \gamma^4 g_k(x)$. These problems are self-adjoint and fully defined, therefore, the system of functions $\{g_k(x)\}_{k=1}^{\infty}$ is orthogonal on $[b, c]$. In this case, according to the decomposition theorem, any function $U(x)$, four times continuously differentiable in (b, c) and satisfying the corresponding boundary conditions can be expanded in a series $U(x) = \sum\limits_{k=1}^{\infty} a_k g_k(x)$, absolutely and converging uniformly in (b, c).

Taking into account (51), the conditions for orthogonality of the residual of equations (10), (34) to basis functions $\{g_j(x)\}_{j=1}^{m}$ of the form (52) allow us to write the system of equations

$$\begin{gathered}\left[M\ddot{a}_j(t)+(\beta_2 I\gamma_j^4+\beta_1)\dot{a}_j(t)+(D\gamma_j^4+\beta_0)a_j(t)\right]\delta_j+\\ +N(t)\sum_{k=1}^{m} a_k(t)\int_b^c g_k''(x)g_j(x)dx = \\ = \int_b^c P(x,t)g_j(x)dx,\ \delta_j = \int_b^c g_j^2(x)dx,\ j=\overline{1,m}.\end{gathered} \tag{53}$$

The conditions for orthogonality of the residuals of the initial conditions $w(x, 0) = f_1(x)$, $\dot{w}(x, 0) = f_2(x)$ to the basis functions make it possible to find the initial values:

$$\begin{aligned} a_j(0) &= \frac{1}{\delta_j}\int_b^c f_1(x)g_j(x)dx,\\ \dot{a}_j(0) &= \frac{1}{\delta_j}\int_b^c f_2(x)g_j(x)dx. \end{aligned} \tag{54}$$

Thus, we have obtained the Cauchy problems for systems of ordinary differential equations (53) with initial conditions (54).

4.2 Nonlinear model

In the case of a motionless fixation of the ends of the element, according to the Galerkin's method, the solution of the system of equations (25) is sought in the form

$$\begin{aligned} u(x,t) &= \sum_{k=1}^{m} a_k(t)g_k^{(1)}(x),\\ w(x,t) &= \sum_{k=1}^{m} b_k(t)g_k^{(2)}(x). \end{aligned} \tag{55}$$

In (55) we select the basis functions $g_k^{(1)}(x),\ g_k^{(2)}(x)$ so that so that the given boundary conditions are satisfied, and the functions $a_k(t),\ \ b_k(t)$ are determined from the condition of orthogonality of the residual of the first equation of the system to all basis functions $g_k^{(1)}(x)$, and the residuals of the second equations – to $g_k^{(2)}(x),\ k = 1 \div m$. As basic functions $g_k^{(2)}(x)$ we take functions of the form (52), and as basis functions $g_k^{(1)}(x)$ take the functions

$$\begin{gathered} g_k^{(1)}(x) = A_k\cos\gamma_k^{(1)}x + B_k\sin\gamma_k^{(1)}x,\\ k=1,2,3,\ldots \end{gathered} \tag{56}$$

We choose the coefficients A_k, B_k and the parameter $\gamma_k^{(1)}$ so that at each endpoint of the segment $[b, c]$ one of the following conditions

$$1)\ g_k^{(1)}(x) = 0,\ 2)\ g_k^{(1)\prime}(x) = 0,\ k = 1,2,3,\ldots \tag{57}$$

is fulfilled. Note that $\gamma_k^{(1)}$ and $g_k^{(1)}(x)$ are eigenvalues and eigenfunctions of the boundary value problem $g''(x) = -\gamma^2 g(x)$ with boundary conditions (57). These problems are self-adjoint and completely definite, therefore, the system of functions $\{g_k(x)\}_{k=1}^{\infty}$ is orthogonal on $[b, c]$.

Substituting (55) into the system of equations (25), from the condition of the orthogonality of the residuals of the first equation (25) to the basis functions $\{g_j^{(1)}(x)\}_{j=1}^{m}$, the second to $\{g_j^{(2)}(x)\}_{j=1}^{m}$ we obtain the system of ordinary differential equations for $a_j(t), b_j(t)$:

$$\begin{cases} M\delta_j^{(1)}\ddot{a}_j(t) + EF{\gamma_j^{(1)}}^2\delta_j^{(1)}a_j(t)-\\ -EF\sum\limits_{i=1}^{m}\sum\limits_{s=1}^{m}A_{isj}b_i(t)b_s(t) = 0,\\ -EF\sum\limits_{i=1}^{m}\sum\limits_{s=1}^{m}B_{isj}b_i(t)a_s(t)-\\ -\frac{3EF}{2}\sum\limits_{i=1}^{m}\sum\limits_{s=1}^{m}\sum\limits_{k=1}^{m}C_{iskj}b_i(t)b_s(t)b_k(t)+\\ +N(t)\sum\limits_{k=1}^{m}b_k(t)\int\limits_b^c {g_k^{(2)}}''(x)g_j^{(2)}(x)dx+\\ +\left[D{\gamma_j^{(2)}}^4b_j(t) + M\ddot{b}_j(t) + \beta_2 I{\gamma_j^{(2)}}^4\dot{b}_j(t)+\right.\\ \left.+\beta_1\dot{b}_j(t)+\beta_0 b_j(t)\right]\delta_j^{(2)}=\int\limits_b^c P(x,t)g_j^{(2)}(x)dx, \end{cases} \tag{58}$$

where

$$\delta_j^{(1)} = \int_b^c {g_j^{(1)}}^2(x)dx,\quad \delta_j^{(2)} = \int_b^c {g_j^{(2)}}^2(x)dx,$$

$$A_{isj} = \int_b^c {g_i^{(2)}}'(x){g_s^{(2)}}''(x)g_j^{(1)}(x)dx,\quad B_{isj} =$$

$$= \int_b^c \left({g_i^{(2)}}'(x){g_s^{(1)}}''(x) + {g_i^{(2)}}''(x){g_s^{(1)}}'(x)\right)g_j^{(2)}(x)dx,$$

$$C_{iskj} = \int_b^c {g_i^{(2)}}'(x){g_s^{(2)}}'(x){g_k^{(2)}}''(x)g_j^{(2)}(x)dx.$$

The conditions of orthogonality of the residuals of the initial conditions $w(x,0) = f_1(x)$, $\dot{w}(x,0) = f_2(x)$, $u(x,0) = f_3(x)$, $\dot{u}(x,0) = f_4(x)$ to the basis functions allows to find the initial values:

$$\begin{aligned} a_j(0) &= \frac{1}{\delta_j^{(1)}} \int_b^c f_3(x) g_j^{(1)}(x) dx, \\ \dot{a}_j(0) &= \frac{1}{\delta_j^{(1)}} \int_b^c f_4(x) g_j^{(1)}(x) dx, \\ b_j(0) &= \frac{1}{\delta_j^{(2)}} \int_b^c f_1(x) g_j^{(2)}(x) dx, \\ \dot{b}_j(0) &= \frac{1}{\delta_j^{(2)}} \int_b^c f_2(x) g_j^{(2)}(x) dx. \end{aligned} \tag{59}$$

Thus, we have obtained the Cauchy problem for a system of ordinary differential equations (58) with initial conditions (59).

5 Program complex

For solving of the obtained Cauchy problems a complex of programs "Aerohydroelasticity" has been developed [Ankilov and Velmisov, 2021].

To start the calculations, enter:
– type of construction;
– model of a deformable solid;
– the type of fastening of the ends of the element;
– initial conditions;
– parameters of the mechanical system;
– order of approximation m;
– estimated time T.

Then the program checks:
– correspondence between the type of structure and the type of fastening of the elastic element;
– correspondence of the initial and boundary conditions

and produces:
– calculation of the coefficients D, M, I, F;
– calculation of eigenvalues and eigenfunctions;
– calculation of integral members of systems;
– solution of systems of ordinary differential equations;
– construction of three-dimensional graphs of element deformations;
– construction of animation graphs of element deformations;
– construction of flat graphs of deformations and strain rate in a given point or at a given moment in time.

6 Numerical experiment

Let's consider an example of calculations using the programs complex. We introduce:
1) type of construction – wing with a connecting element;
2) model of a deformable rigid body (9);
3) parameters of the mechanical system:

$$V = 20,\ \rho = 1,\ E = 20.6 \cdot 10^{10},\ \rho_p = 7850,\ a = 0,$$
$$b = 1,\ c = 1.3,\ d = 2,\ h = 0.01,\ \nu = 0.25,\ \beta_0 = 400,$$
$$\beta_1 = 40,\ \beta_2 = 20,\ N(t) = 1000;$$

4) type of fastening "rigid-hinged";
5) profile forms

$$\begin{aligned} f_1^+(x) &= 0.05x(b-x)^2, \\ f_1^-(x) &= -0.05x(b-x)^2, \\ f_2^+(x) &= 0.0125(x-c)(d-x)^2, \\ f_2^-(x) &= -0.0125(x-c)(d-x)^2; \end{aligned}$$

6) initial conditions

$$f_1(x) = 0.001 g_1^{(2)}(x),\ f_2(x) = -0.0005 g_2^{(2)}(x),$$
$$f_3(x) = 0.0001 g_1^{(1)}(x),\ f_4(x) = 0.00005 g_2^{(1)}(x);$$

7) order of approximation $m = 4$ and estimated time $T = 5$.

For the proposed parameters of the mechanical system, conditions (15) and the first two conditions (23) are satisfied. For the type of fastening "rigid-hinged" we count $\lambda_1 = \left(\frac{4.4934}{c-b}\right)^2 = 224.34$. Choosing a function $g_1(x) = 3.7\sqrt{(x-b)(c-b)} - 3.5$, we find $G_0 = 0.54$. Consequently, the third condition (23) is satisfied. According to Theorem 2.2, the solution $w(x,t)$ of the system of equations (25) and the derivatives $\dot{u}(x,t), u'(x,t), \dot{w}(x,t), w'(x,t)$ are stable with respect to perturbations of the initial data.

Figures 3, 4 show examples of calculations of the transverse $w(x,t)$ and longitudinal $u(x,t)$ deformation at the point $x_0 = 1.15$.

Figures 5 and 6 show examples of calculations of the transverse $w(x,t)$ and longitudinal $u(x,t)$ deformation of the element at time $t_0 = 1$.

Indeed, according to Figure 3 and to the continuation of the graph on a larger time interval, the stability of vibrations of the elastic element is observed.

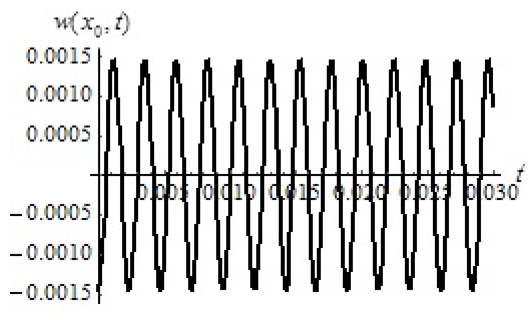

Figure 3. Transverse deformation of element at point x_0

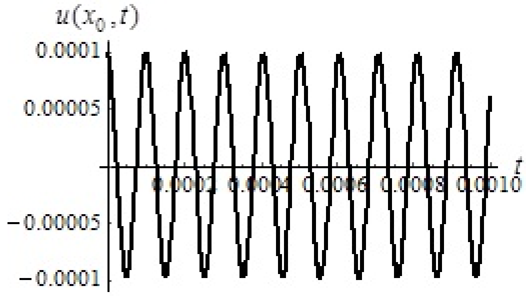

Figure 4. Longitudinal deformation of element at point x_0

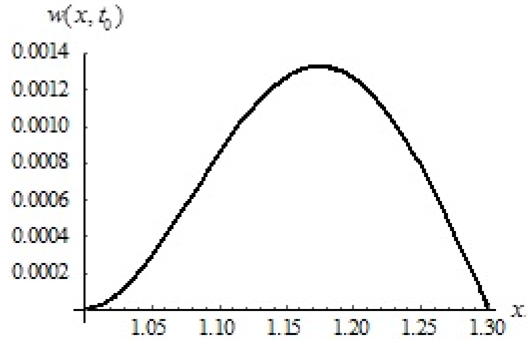

Figure 5. Transverse deformation of element at moment of time t_0

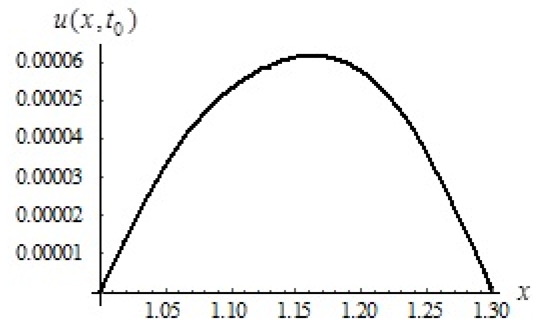

Figure 6. Longitudinal deformation of element at moment of time t_0

7 Conclusion

On the basis of the proposed mathematical models of wing constructions, streamlined by a subsonic flow of an ideal liquid or gas, a study of the dynamics and stability of elastic deformable elements, which are components of these constructions, has been carried out. The obtained stability conditions impose restrictions on the linear mass and bending stiffness of the elements, compressive (tensile) forces, the incident flow velocity and other parameters of the mechanical system. These conditions obviously contain the main parameters of the mechanical system, and in this form they are most suitable for solving problems of optimization, automatic control, computer-aided design.

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