PHYSCON 2011, León, Spain, 5–8 September 2011

LOCALIZATION OF THE LIMIT SET FOR A CLASS OF NONLINEAR DISTRIBUTED PARAMETER SYSTEMS

Alexander Zuyev
Institute of Applied Mathematics and Mechanics
National Academy of Sciences of Ukraine
Donetsk National University, Ukraine
al_zv@mail.ru

Abstract

In this paper, a class of bounded perturbations of a linear differential equation in a Banach space is considered. Sufficient conditions for the relative compactness of trajectories of the perturbed system is proposed. Such conditions are shown to be applicable for a nonlinear differential equation without assuming that the corresponding infinitesimal generator is dissipative.

Key words

Distributed parameter system, continuous semigroup, limit set, Lyapunov functional.

1 Introduction

Let X be a real Banach space and let $A : D(A) \to X$ be a closed linear operator defined on $D(A) \subset X$. Consider the abstract Cauchy problem on $t \in [0, +\infty)$:

$$\dot{x}(t) = Ax(t),\ x(0) = x_0 \in X. \tag{1}$$

We assume that $D(A)$ is dense in X and that A is the infinitesimal generator of a strongly continuous semigroup (C_0-semigroup) of linear operators $\{e^{tA}\}_{t\geq 0}$ on X [Pazy, 1983]. Hence, the Cauchy problem (1) is well-posed on $t \in [0, +\infty)$ and its mild solutions can be represented as follows

$$x(t) = e^{tA}x_0,\ t \geq 0. \tag{2}$$

The asymptotic behavior of solutions $x(t)$ as $t \to +\infty$ has been the subject of many publications. In particular, some important results concerning the asymptotic behavior of solutions are presented in books [Bresis, 1971; Barbu, 1993; Luo, Guo, and Morgul, 1999; Oostveen, 2000] for different classes of operators A.

A sufficient condition for the existence of the ω-limit set for solution (2) is the relative compactness (precompactness) of the trajectory $\gamma(x_0) = \{e^{tA}x_0 |\ t \geq 0\}$. For differential equations with precompact trajectories, the study of ω-limit can be carried out by means of the direct Lyapunov method and LaSalle's invariance principle [LaSalle, 1976]. Hence, this problem of compactness remains challenging for general classes of differential equations in Banach spaces. In paper [Dafermos and Slemrod, 1973], a sufficient condition for the precompactness of the trajectories $\gamma(x_0)$ has been proposed for the case of accretive operator $-A$ in (1).

Paper [Coron and d'Andrea-Novel, 1998] shows that the accretivity assumption is violated for a model of a rotating beam with precompact trajectories. In order to extend the class of systems under consideration, we introduce the following perturbed Cauchy problem on $t \geq 0$:

$$\dot{x} = Ax + f(t)R(x, t),\ x(0) = x_0 \in X, \tag{3}$$

where $f : [0, +\infty) \to \mathbb{R}$ and $R : X \times [0, +\infty) \to X$ are continuous mappings.

In the sequel, we show that the trajectories of differential equation (3) are precompact if the trajectories of (1) are precompact under certain assumptions on f and R. This result is applied for an autonomous nonlinear differential equation in a Banach space. An example shows that the technique proposed is valid for a nonlinear semigroup with non-dissipative infinitesimal generator.

2 Preliminary Results

Suppose that the Banach space X admits a basis $\{e_i\}$ $(i = 1, 2, ...)$. Denote by $\{f_j\} \subset X^*$ $(j = 1, 2, ...)$ the family of adjoint functionals, i.e. $f_j(e_i) = \delta_{ij}$, δ_{ij} is the Kronecker symbol. Thus, for each $x \in X$ and $n \in \mathbb{N}$, the following projection operators are well-defined:

$$S_n(x) = \sum_{i=1}^{n} f_i(x)e_i,\ P_n(x) = x - S_n(x).$$

As $\{e_i\}$ is a basis then then operators $S_n : X \to X$ are mutually bounded,

$$\|S_n\| \le M < \infty,\ n = 1, 2, ...$$

To describe the compact subsets of X, we formulate two preliminary lemmas which generalize the approach of [Zuyev, 2005].

Lemma 1. *Let $\{e_i\}$ be a basis in X. A bounded set $C \subset X$ is precompact in X if and only if*

$$\lim_{n\to\infty} \sup_{x\in C} \|P_n x\| = 0. \tag{4}$$

Lemma 2. *Let $\{e_i\}$ be a basis in X, C be a compact subset of X, and $\{e^{tA}\}_{t\ge 0}$ be a uniformly bounded C_0-semigroup of operators on X with precompact trajectories*

$$\gamma(x_0) = \{e^{tA} x_0 \,|\, t \ge 0\}$$

for all $x_0 \in C$. Then

$$\lim_{n\to\infty} \left(\sup_{t\ge 0, x\in C} \|P_n e^{tA} x\| \right) = 0. \tag{5}$$

3 Relative Compactness Under Perturbations

Let us recall that [Pazy, 1983], a continuous function $x(t) \in X$ on

$$0 \le t < T \le +\infty$$

is a mild solution of (3) if

$$x(t) = e^{tA} x_0 + \int_0^t e^{(t-s)A} f(s) R(x(s), s)\, ds. \tag{6}$$

We prove here the following result concerning the precompactness of the trajectories.

Theorem 1. *Let X be a Banach space with a basis, A be the infinitesimal generator of a uniformly bounded C_0-semigroup of linear operators $\{e^{tA}\}_{t\ge 0}$ on X, $f \in L^1[0, +\infty)$, $R(x, t) \in K$ for all $x \in X$, $t \ge 0$, and K be compact. Assume moreover that the set $\{e^{tA} y \,|\, t \ge 0\}$ is precompact for each $y \in K \cup \{x_0\}$.*

Then any mild solution $x(t)$, $t \in [0, +\infty)$ of (3) is contained in a compact subset of X.

Proof. Let $x(t)$ be a mild solution of (3) for $t \ge 0$. Integral equation (6) implies that the solution $x(t)$ is bounded as $\{e^{tA} x_0 \,|\, t \ge 0\}$ is compact and $f \in L^1[0, +\infty)$, $R \in K$.

By Lemma 1, it suffices to choose a basis $\{e_i\}$ in X and establish the limit existence

$$\lim_{n\to\infty} \sup_{t\ge 0} \|P_n x(t)\| = 0.$$

By projecting the left- and right-hand sides of (6), we get

$$\|P_n x(t)\| \le \|P_n e^{tA} x_0\| +$$

$$+ \left\| \int_0^t f(s) P_n \left(e^{(t-s)A} R(x(s), s) \right) ds \right\| \le$$

$$\le \|P_n e^{tA} x_0\| + \|f\|_{L^1} \cdot \sup_{s\in[0,t], y\in K} \|P_n e^{sA} y\|.$$

Now the assertion of Theorem 1 follows from Lemmas 1 and 2. □

A class of autonomous nonlinear differential equations with can be considered in the form (3) by computing the estimate of $f(t)$ along the trajectories by means of the direct Lyapunov method. Let us formulate the main result in this area for the abstract Cauchy problem

$$\dot{x}(t) = Ax(t) + h(x(t))B(x(t)),\ x(0) = x_0 \in X, \tag{7}$$

where $h : X \to \mathbb{R}$ and $B : X \to X$ are locally Lipschitz mappings, i.e. for any $r \ge 0$, there exists an $L(r)$ such that

$$|h(x) - h(y)| \le L(r)\|x - y\|,$$

$$\|B(x) - B(y)\| \le L(r)\|x - y\|$$

for each $\|x\| \le r$, $\|y\| \le r$. If $w : X \to \mathbb{R}$ is Fréchet differentiable then the function $w(x(t))$ is differentiable along any classical solution $x(t)$ of (7). Then, for each $x \in D(A) \subset X$, the derivative of w along the trajectories of (7) can be written as follows

$$\dot{w}(x) = [\nabla w(x), Ax + B(x)h(x)],$$

where $[\cdot,\cdot] : X^* \times X \to \mathbb{R}$ is the duality pairing of X^* and X, i.e. $[\nabla w(x), \xi]$ is the value of the linear functional $\nabla w(x) \in X^*$ at $\xi \in X$.

Theorem 2. *Let X be a Banach space with a basis, A be the infinitesimal generator of a uniformly bounded C_0-semigroup of operators $\{e^{tA}\}_{t\ge 0}$ on X, $\{e^{tA} y \,|\, t \ge 0\}$ be precompact for each $y \in X$, and $B : X \to X$ be a completely continuous operator. Assume that there exists a differentiable functional $w : X \to \mathbb{R}$ satisfying the following conditions:*

1) $M_c = \{x \,|\, w(x) \le c\}$ *is bounded for each* $c \in \mathbb{R}$*;*
2) $\inf_{\|x\| \le r} w(x) > -\infty$ *for each* $r > 0$*;*
3) there exists a $k_1 > 0$ *such that*

$$\dot{w}(x) \le k_1 h(x) \le 0, \ \forall x \in D(A).$$

Then the Cauchy problem (7) has the unique solution $x(t)$ *on* $[0, +\infty)$ *for any* $x_0 \in X$*, and* $\{x(t) \,|\, t \ge 0\}$ *is precompact in* X*.*

Proof. For any $x_0 \in X$, there exists unique mild solution $x(t)$ of (7) on $t \in [0, t_{max})$ according to Theorem 1.4 of [Pazy, 1983, p. 185]. Conditions 1) and 3) ensure that $x(t)$ is bounded, hence, $t_{max} = +\infty$.

We put $R(x,t) = B(x)$ and $f(t) = h(x(t))$ in equation (3). Then conditions 2) and 3) imply that $f \in L^1[0, +\infty)$. Therefore, the trajectory $\{x(t) \,|\, t \ge 0\}$ is precompact in X by Theorem 1. □

Remark. Note that as $\dot{w}(x) \le 0$ then the sets M_c are invariant and Theorem 2 admits a local version in a subset of X which is bounded by level sets of the functional w.

4 An Example and Concluding Remarks

Consider the Hilbert space ℓ^2 whose elements are denoted by infinite columns

$$x = (u_0, v_0, u_1, v_1, u_2, v_2, ...)^T.$$

We define linear operators A and B by their infinite matrices:

$$A = \operatorname{diag}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega_1 \\ -\omega_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega_2 \\ -\omega_2 & 0 \end{pmatrix}, ... \right) +$$

$$+ \begin{pmatrix} 0 \\ 1 \\ 0 \\ -J_1 \\ 0 \\ -J_2 \\ \vdots \end{pmatrix} \cdot (-1, -1, \omega_1 J_1, 0, \omega_2 J_2, 0, ...),$$

$$B = \operatorname{diag}\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1/\omega_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1/\omega_2 & 0 \end{pmatrix}, ... \right).$$

The functional $h : \ell^2 \to \mathbb{R}$ is defined as $h(x) = -v_0^2$. We assume that the coefficients of A and B satisfy the following conditions:

$$\omega_n > 0, \ \sum_{n=1}^{\infty} J_n^2 < \infty, \ \sum_{n=1}^{\infty} \frac{1}{\omega_n^2} < \infty.$$

Differential equation (7) with such A, B, and h is a generalization of the equations introduced in paper [Zuyev, 2005] for the description of the motion of a controlled mechanical system with flexible beams.

Let us put $J = 1 + \sum_{n=1}^{\infty} J_n^2$ and define the following quadratic functional on ℓ^2:

$$w(x) = u_0^2 + J v_0^2 + \sum_{n=1}^{\infty} \left(u_n^2 + 2 J_n v_0 v_n + v_n^2 \right) \ge 0.$$

It is easy to see that $w(x)$ satisfy conditions 1)-2) of Theorem 2. By computing the time-derivative $\dot{w}(x)$ and applying the Cauchy-Schwartz inequality, we conclude that conditon 3) holds for $k_1 = 1$ if x is restricted to some ball centered at $0 \in \ell^2$. By using the Lumer-Phillips theorem [Pazy, 1983, p. 15] and Theorem 3 from paper [Dafermos and Slemrod, 1973], it is posible to show that A generates a uniformly bounded C_0-semigroup $\{e^{tA}\}_{t \ge 0}$ with precompact trajectories $\{e^{tA} y \,|\, t \ge 0\}$ in ℓ^2.

Hence, the system considered can be analyzed by means Theorem 2 in its local version, i.e. for any x_0 from some neighborhood of $0 \in \ell^2$, there exists uniue mild solution $x(t)$ of (7) on $t \in [0, +\infty)$, and the set $\{x(t) \,|\, t \ge 0\}$ is precompact. Let us denote $-F(x) = Ax + h(x)Bx$. One can check that the operator F is not monotone as the expression

$$< F(x_1) - F(x_2), x_1 - x_2 >$$

is alternating for $x_1, x_2 \in D(F)$. It means that the precompactness of the trajectories of (3) cannot be established by a straightforward application of the approach of [Dafermos and Slemrod, 1973].

The property of relative compactness, guaranteed by Theorem 2, can be used for further study of the ω-limit sets for the trajectories of (3) by means of LaSalle's invariance principle.

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