COMPUTATION OF THE FIRST LYAPUNOV QUANTITY FOR SECOND-ORDER DYNAMICAL SYSTEM [1]

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Abstract: The new method for computation of Lyapunov quantities for second-order dynamical system, permitting us to narrow the requirements on a smoothness of system, is obtained.

Keywords: Lyapunov quantities, polynomial system, small amplitude limit cycle

1. INTRODUCTION

The classical method for computation of Lyapunov quantities involves the introduction of the polar coordinates and the reducing of original system to normal form [Lyapunov, 1892; Bautin, 1962; Lloyd & Pearson, 1990; Yu, 1998; Lynch, 2005]. Here it is suggested the substantially different method, not requiring the direct reduction to normal form. The quality of the method suggested is ideological simplicity and visualization. We require a less smoothness of the right-hand sides of differential equations in comparison with the classical consideration. In the present work we follow ideas, proposed in [Leonov 2006, 2007].

2. COMPUTATION OF LYAPUNOV QUANTITY

Consider the system

$$\begin{aligned} \dot{x} &= -y + u_f(t), \\ \dot{y} &= x + u_g(t). \end{aligned} \tag{1}$$

Then for a solution with initial data $x(0) = 0, y(0) = 0$ we have

$$\begin{aligned} x &= u_g(0)\cos(t) + \\ &\quad + \cos(t)\int_0^t \cos(\tau)(u_g'(\tau) + u_f(\tau))\mathrm{d}\tau + \\ &\quad + \sin(t)\int_0^t \sin(\tau)(u_g'(\tau) + u_f(\tau))\mathrm{d}\tau - u_g(t) \\ y &= u_g(0)\sin(t) + \\ &\quad + \sin(t)\int_0^t \cos(\tau)(u_g'(\tau) + u_f(\tau))\mathrm{d}\tau - \\ &\quad - \cos(t)\int_0^t \sin(\tau)(u_g'(\tau) + u_f(\tau))\mathrm{d}\tau \end{aligned} \tag{2}$$

Consider the equation

$$\begin{aligned} \dot{x} &= -y + f(x,y) \\ \dot{y} &= x + g(x,y) \end{aligned} \tag{3}$$

Here $f(0,0) = g(0,0) = 0$ and in a certain neighborhood of the point $(x,y) = (0,0)$ the functions $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ have partial derivative up to the order 2, and $f_x'(0,0) = f_y'(0,0) = g_x'(0,0) = g_y'(0,0) = 0$.

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We shall use a smoothness of the functions f and g and shall follow the first Lyapunov method on finite time interval [Lefschetz, 1957; Cesari, 1959].

and using the smoothness of the functions f and g, we can write

$$\begin{aligned} f(x,y) &= f_{20}x^2 + f_{11}xy + f_{02}y^2 + o((|x|+|y|)^2) = \\ &= f_2(x,y) + o((|x|+|y|)^2), \\ g(x,y) &= g_{20}x^2 + g_{11}xy + g_{02}y^2 + o((|x|+|y|)^2) = \\ &= g_2(x,y) + o((|x|+|y|)^2). \end{aligned} \tag{4}$$

Consider the solution

$$x(t,h) = x(t,x(0),y(0)), y(t,h) = y(t,x(0),y(0))$$

of system (3) with the initial data

$$\begin{aligned} x(0,x(0),y(0)) &= 0, \\ y(0,x(0),y(0)) &= h, \end{aligned} \tag{5}$$

Then for the first approximation $x_1(t,h), y_1(t,h)$ of the solution $x(t,x(0),y(0)), y(t,x(0),y(0))$, from the equation

$$\begin{aligned} \dot{x}_1 &= -y_1, \quad x_1(0,h) = 0, \\ \dot{y}_1 &= x_1, \quad y_1(0,h) = h, \end{aligned} \tag{6}$$

we obtain

$$x_1(t,h) = -h\sin(t), y_1(t,h) = h\cos(t).$$

By the assumption on the smoothness of f, g we obtain that the right-hand side of system (3) has 2 continuous partial derivatives with respect to x and y. Then [Hartman, 1964] the solution of system (3), i.e. $x(t,h), y(t,h)$ have partial derivative up to the order 2 with respect to the initial data h.

We shall seek sequential approximations for $x(t,h), y(t,h)$ in the form of the sum

$$\begin{aligned} x_2(t,h) &= x_1(t)h + x_2(t)h^2, \quad x_2(0) = 0, \\ y_2(t,h) &= y_1(t)h + y_2(t)h^2, \quad y_2(0) = 0, \end{aligned} \tag{7}$$

where, according to the local Taylor formula, at fixed moment $t = t^*$ the following representation holds

$$\begin{aligned} x(t^*,h) &= x_2(t^*,h) + o(h^2), \\ y(t^*,h) &= y_2(t^*,h) + o(h^2). \end{aligned} \tag{8}$$

Substituting (7) in (4) and then in (3) and determining the coefficients $u_2^x(t)$ and $u_2^y(t)$ of h^2 in $f(x_1(t,h), y_1(t,h))$ and $g(x_1(t,h), y_1(t,h))$ correspondingly we obtain the following approximations

$$\begin{aligned} u_2^x(t,h) &= u_2^x(t)h^2, \\ u_2^y(t,h) &= u_2^y(t)h^2, \end{aligned} \tag{9}$$

Then for determining $x_2(t), y_2(t)$ we have the equation

$$\begin{aligned} \dot{x}_2 &= -y_2 + u_2^x(t) \\ \dot{y}_2 &= x_2 + u_2^y(t). \end{aligned} \tag{10}$$

Let find the solution of (10) by (2).

$$\begin{aligned} x_2(t) = \frac{1}{3}(\\ & 2\cos(t)f_{02}\sin(t) - g_{11}\sin(t)\cos(t) \\ & -2\sin(t)f_{20}\cos(t) + \cos(t)g_{02} \\ & -g_{20} - g_{20}\cos(t)^2 + g_{02}\cos(t)^2 \\ & +g_{11}\sin(t) + f_{02}\sin(t) + 2\cos(t)g_{20} \\ & -\cos(t)f_{11} - 2g_{02} - f_{11} + 2f_{11}\cos(t)^2 + 2\sin(t)f_{20}) \end{aligned}$$

$$\begin{aligned} y_2(t) = \frac{1}{3}(\\ & -f_{02}\cos(t)^2 + f_{20} + 2f_{02} - g_{11} \\ & -g_{11}\cos(t) - 2f_{20}\cos(t) + \sin(t)g_{02} \\ & +2\sin(t)g_{20} - \sin(t)f_{11} + 2g_{11}\cos(t)^2 \\ & +f_{20}\cos(t)^2 - \cos(t)f_{02} + 2g_{02}\cos(t)\sin(t) \\ & -2g_{20}\sin(t)\cos(t) + f_{11}\sin(t)\cos(t)) \end{aligned}$$

Here $x_2(0) = y_2(0) = x_2(2\pi) = y_2(2\pi) = 0$.

Lemma. *Let be*

$$\begin{aligned} x_1(2\pi) &= 0, \ y_1(2\pi) = 1, \\ x_2(2\pi) &= y_2(2\pi) = 0. \end{aligned} \tag{11}$$

Then for sufficiently small h the solution $x(t,h), y(t,h)$ on a phase plane crosses the half-line $(x = 0, y > 0)$ at time

$$T = 2\pi + o(h). \tag{12}$$

Proof.

Since $x_2(2\pi,h) = 0$ and $y_2(2\pi,h) = h$, we obtain that for $t = 2\pi$ the trajectory $(x(t,h), y(t,h))$ on phase plane (8) is in the neighborhood of radius $o(h^2)$ of the point $(x = 0, y = h)$.

For fixed $t = t^*$, according [Hartman, 1964] and (8) we have

$$\dot{x}(t^*,h) = -h\cos t^* + o(h).$$

Since $\dot{x}(t,h)$ bounded with respect to t and h in a certain neighbourhood of $(x = 0, y = h)$ and $t = 2\pi$, we obtain the relation

$$\dot{x}(t,h) \le -ch$$

for sufficiently small h and for t from certain neighborhood 2π for certain number $c > 0$. Then

$$T = 2\pi + o(h).$$

■

Consider a function

$$V(x,y) = x^2 + y^2. \tag{13}$$

For the derivative of the function V along the solutions of system (3) the relation

$$\dot{V}(x,y) = 2xf(x,y) + 2yg(x,y) \tag{14}$$

is valid.

The following notation are needed for the sequel

$$L = V\big(x(T,h), y(T,h)\big) - V\big(x(0,h), y(0,h)\big). \tag{15}$$

Integrating (14) from 0 to $T = 2\pi + o(h)$ we obtain

$$L = \int_0^T \dot{V}(x(t,h), y(t,h))\mathrm{d}t = \\ = \int_0^{2\pi} \dot{V}(x(t,h), y(t,h))\mathrm{d}t + o(h^4).$$

Substituting (14), we finally have

$$L = \int_0^{2\pi} 2x_2(t,h) f_2(x_2(t,h), y_2(t,h)) + \\ +2y_2(t,h) g_2(x_2(t,h), y_2(t,h))\mathrm{d}t + o(h^4). \tag{16}$$

Substituting $x_2(t,h), y_2(t,h)$ in $f_2(x,y)$, and $g_2(x,y)$ and then in (16) and using terms grouping up to h^4, since we obtain

$$L = L_1 h^3 + o(h^4). \tag{17}$$

where $L_1/2$ is the k-th Lyapunov quantity $\mathbf{L_1}$.

$$\mathbf{L_1} = \frac{\pi}{4}(\\ f_{11}f_{02} + 2f_{02}g_{02} - 2f_{20}g_{20} - g_{11}g_{20} - g_{11}g_{02} + f_{11}f_{20})$$

Here the sign $\mathbf{L_1}$ characterizes an unwinding or a twisting of trajectory of system $(x(t,h), y(t,h))$ on a phase plane.

We remark that for computing L_1 it is sufficient that in the neighborhood of considered stationary point the relation $f, g \in \mathbb{C}^2$ is satisfied, what is one less than conventional assumptions on a smoothness [Marsden & McCracken, 1976].

3. CONCLUSION

Note also that a wide class of polynomial systems for which a given technique permits us to construct small cycles (see, for example, [Bautin, 1952; Leonov, 1998; Lloyd & Pearson, 1997; Lynch, 2005; Yu & Han, 2005] and others).

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