PHYSCON 2009, Catania, Italy, September, 1–September, 4 2009

ROBUSTNESS OF SYNCHRONIZATION IN HYPERCHAOTIC CIRCUITS

Arturo Buscarino
Laboratorio sui Sistemi Complessi
Scuola Superiore di Catania
Università degli Studi di Catania, Italy.

Luigi Fortuna, Mattia Frasca
Dipartimento di Ingegneria Elettrica Elettronica e dei Sistemi
Facoltà di Ingegneria
Università degli Studi di Catania, Italy.

Abstract

The onset of synchronization between two chaotic circuits has been deeply investigated in recent literature. However, the effects on the synchronous state of parameter mismatches, due to uncertainty sources, between coupled systems have not been sufficiently studied.

In this work we study numerically and experimentally the robustness of synchronization in presence of parameter mismatches in the case of hyperchaotic behavior. For robustness of synchronization we mean the capability shown by the coupling scheme of maintaining a synchronous motion even if the two circuits are not identical, due to uncertainties on circuit parameters.

Key words

Chaos, Synchronization, Robustness.

1 Introduction

Complete synchronization between two chaotic circuits can be reached when the systems are identical [Pecora & Carroll, 1990]. In presence of parameter mismatches between the systems, due to uncertainty sources, complete synchronization can be obtained only by using an open-loop–closed-loop based approach in which more than one signal is involved in the synchronization scheme [Grosu *et al.*, 2008]. Weaker forms of synchronization, however, can be observed in presence of parameter mismatches by using relatively simple coupling scheme based on the negative feedback of a unique scalar signal.

In this work we study numerically and experimentally the robustness of synchronization in presence of parameter mismatches between the coupled circuits in the case of hyperchaotic behavior. The term robustness of synchronization indicates the capability shown by the coupling scheme of maintaining the synchronization even if the two circuits are not identical, due to uncertainties on circuit parameters.

A master-slave scheme based on negative feedback is considered for achieving the synchronization and the strategy to design the slave system as an observer of the master is given following the procedure described in [Arena *et al.*, 2006]. With this approach, based on the Master Stability Function [Pecora & Carroll, 1998], the two circuits are coupled through a unique scalar signal. Experimental results obtained from two hyperchaotic circuits will be presented in order to show that synchronization widely occurs in the range of electronic component tolerances.

2 MSF-based synchronization of two hyperchaotic circuits

In this paper, a scheme, based on negative feedback [Kapitaniak, 1994], in which the slave system is designed as an observer of the master system, is proposed. Therefore, an error signal, built comparing the same linear combination of master and slave state variables (which are assumed measurable), is fed back into the slave system.

Hence, assuming that the master equations are:

$$\dot{\mathbf{X}}_{\mathbf{m}} = f(\mathbf{X}_{\mathbf{m}}), \tag{1}$$

the slave equations will be:

$$\dot{\mathbf{X}}_{\mathbf{s}} = f(\mathbf{X}_{\mathbf{s}}) + \mathrm{K}e, \tag{2}$$

where K is the gains vector, $e = \mathrm{C}\mathbf{X}_{\mathbf{m}} - \mathrm{C}\mathbf{X}_{\mathbf{s}}$ is the (scalar) error signal, and C is a vector defining the linear combination of the state variables.

In order to set values of K and C suitable for the onset of synchronization, we applied an approach based on the Master Stability Function (MSF).

The MSF was introduced in [Pecora & Carroll, 1998] and is a simple and efficient tool for the evaluation of the conditions under which N identical oscillators (coupled in an arbitrary network configuration admitting an invariant synchronization manifold) can be synchronized. The dynamics of each node can be modelled as

$$\dot{\mathbf{x}}^i = F(\mathbf{x}^i) - \sigma \sum_j G_{ij} H(\mathbf{x}^j) \tag{3}$$

where $i = 1, ..., N$, $\dot{\mathbf{x}}^i = F(\mathbf{x}^i)$ represents the dynamics of each node, σ is the coupling strength, $H : \mathbb{R}^N \to \mathbb{R}^N$ the coupling function and $G = [G_{ij}]$ is a zero-row sum matrix modelling the coupling network. The maximum conditional Lyapunov exponent Λ_{max} of the generic variational equation

$$\dot{\zeta} = [DF - (\alpha + i\beta)DH]\zeta \tag{4}$$

can be calculated as a function of α and β. In Eq. (4) DF and DH represent the Jacobian of $F(\mathbf{x}^i)$ and $H(\mathbf{x}^j)$ computed around the synchronous state. The function $\Lambda_{max} = \Lambda_{max}(\alpha + i\beta)$, which does not depend on the specific topology of the coupling network, represents the Master Stability Function (MSF). The stability of the synchronization manifold in a given network can be then evaluated by computing the eigenvalues γ_h (with $h = 2, \dots, N$) of the matrix G and studying the sign of Λ_{max} at the points $\alpha + i\beta = \sigma\gamma_h$. If all associated eigenmodes with $h = 2, \dots, N$ are stable, then the synchronous state is stable at the given coupling strength.

In the coupling scheme considered in this paper, the coupling matrix is $G = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$, whose eigenvalues, i.e. $\gamma_1 = 0$ and $\gamma_2 = -1$, are real. In this case the MSF can be computed as function of α only.

The choice of vectors C and K can be then performed on the basis of the sign of the corresponding MSF.

3 Bifurcation analysis of the designed hyperchaotic circuit

The robustness of the MSF approach with respect to parameter mismatches will be evaluated considering the case study of two hyperchaotic circuits. The considered four-dimensional dynamical system is described by the following dimensionless equations [Wang *et al.*, 2009]:

$$\begin{aligned} \dot{x} &= a(y - x) + yz \\ \dot{y} &= cx - xz - y - \tfrac{1}{2}w \\ \dot{z} &= xy - 3z \\ \dot{w} &= \tfrac{1}{2}xz - bw \end{aligned} \tag{5}$$

These equations, each characterized by a cross-product term, represent an hyperchaotic extension of the Lorenz system. Choosing $a = 40$, $b = -1.5$, and $c = 88$ the system exhibit an hyperchaotic behavior.

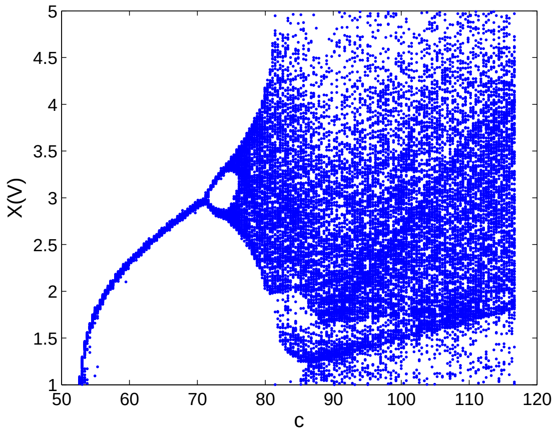

Figure 1. Experimental bifurcation diagram for the proposed circuit with respect to parameter c. The other parameters are fixed as indicated in the text.

An electronic circuit reproducing the dynamics of Eqs. (5) has been designed and implemented following the approach in [Manganaro *et al.*, 1999]. The sensitivity of the circuit with respect to parameter c has been investigated and the corresponding experimental bifurcation diagram is reported in Figure 1. All the data collected in the experiments have been acquired by using a data acquisition board (National Instruments USB-6009) with sampling frequency $f_s = 40kHz$ for $T = 2s$ (i.e. 80000 samples for each time series).

Changing the parameter c the circuit shows a wide range of dynamical behaviors according to the numerical analysis carried out by [Wang *et al.*, 2009].

4 Robustness of the synchronization of two hyperchaotic systems

Two circuits reproducing the dynamics of the Eqs. (5) have been coupled in a negative feedback scheme obtaining a synchronous behavior by choosing the coupling gains on the basis of a Master Stability Function analysis.

Choosing $C = \bar{C} = [1\ 1\ 1\ 1]$, and $K = \bar{K} = [k_1\ k_2\ k_3\ k_4] = [1\ 1\ 0\ 0]$ means that the master and slave systems are coupled through the sum of their state variables, defined by C, and the error signal $e = x_m + y_m + z_m + w_m - x_s - y_s - z_s - w_s$ is fed back only to the first and the second dynamical equations of the slave, as defined by K. Thus, the equations of the master-slave system will read as follows:

$$\begin{aligned} \dot{x_m} &= a(y_m - x_m) + y_m z_m \\ \dot{y_m} &= cx_m - x_m z_m - y_m - \tfrac{1}{2}w_m \\ \dot{z_m} &= x_m y_m - 3z_m \\ \dot{w_m} &= \tfrac{1}{2}x_m z_m - bw_m \dot{x_s} = a(y_s - x_s) + y_s z_s + k_1 e \\ \dot{y_s} &= cx_s - x_s z_s - y_s - \tfrac{1}{2}w_s + k_2 e \\ \dot{z_s} &= x_s y_s - 3z_s \\ \dot{w_s} &= \tfrac{1}{2}x_s z_s - bw_s \end{aligned} \tag{6}$$

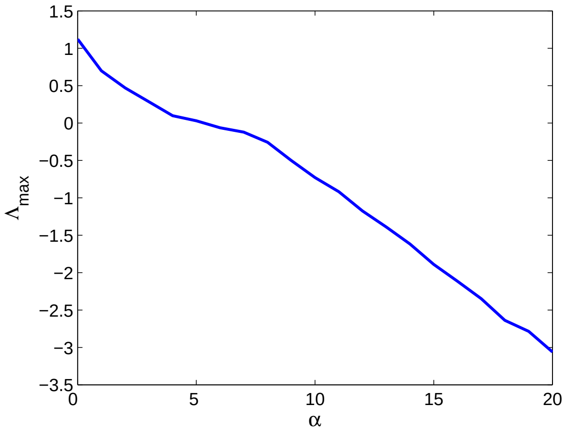

Figure 2. Master stability function for the system in Eqs. (5) in the hyperchaotic region (i.e., $a = 40$, $b = -1.5$, and $c = 88$, with $K = \bar{K}$ and $C = \bar{C}$.

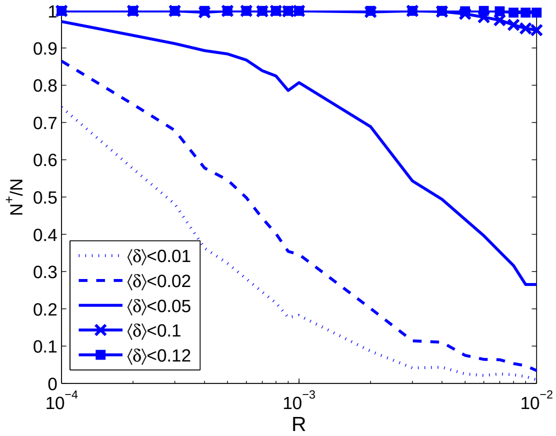

Figure 3. Percentage of parameter sets chosen inside the uncertainty ellipsoid for which the error index is lower than a threshold.

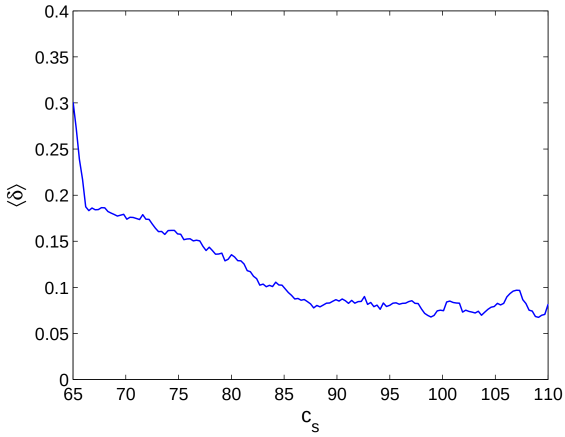

Figure 4. Mean error $\langle\delta\rangle$ vs. c_s.

The MSF computed for the hyperchaotic circuit considered in this paper with $C = \bar{C}$ and $K = \bar{K}$ is reported in Fig. 2, and has negative values for $\alpha < \alpha_c \approx -6$.

In our experimental setup, we set the coupling coefficient $\sigma = \bar{\sigma} = 8$ in order to have $\bar{\sigma}\gamma_2 < \alpha_c$ ensuring the stability of the synchronous manifold.

However, dealing with real circuital components, the two coupled circuits cannot be exactly identical, due to tolerances in electrical components. Our aim is to measure the robustness of synchronization in presence of parametric uncertainty. In this case, in fact, complete synchronization can be achieved only transmitting to the slave system three different signals [Grosu *et al.*, 2008]. In any case, using the proposed scheme based on a reduced order observer, general synchronization can be reached.

Being δ_a, δ_b, and δ_c, the uncertainties on parameters a, b, and c, the uncertain parameters of the slave system can be written as follows: $\bar{a} = a(1+\delta_a)$, $\bar{b} = b(1+\delta_b)$ and $\bar{c} = c(1+\delta_c)$, where $a = 40$, $b = -1.5$ and $c = 88$ are the nominal values of the parameters. In this case, the equations of the slave system become:

$$\begin{array}{l} \dot{x_s} = (a + a\delta_a)(y_s - x_s) + y_s z_s + k_1 e \\ \dot{y_s} = (c + c\delta_c)x_s - x_s z_s - y_s - \frac{1}{2}w_s + k_2 e \\ \dot{z_s} = x_s y_s - 3z_s \\ \dot{w_s} = \frac{1}{2}x_s z_s - (b + b\delta_b)w_s \end{array} \qquad (7)$$

with $a = 40$, $b = -1.5$, and $c = 88$.

It can be numerically observed that mismatches on parameter c lead to smaller effects in the error index compared to the effects introduced by mismatches on a and b. Therefore, we can assume that the uncertainties on a and b should be lower than uncertainty on c, i.e. $R = 10\delta_a = 10\delta_b = \delta_c$.

The considered uncertainties identify an ellipsoid in the parameter space. Inside the ellipsoid, whose axes are defined by the uncertainties on the three parameters, we numerically verified that the system behaves hyperchaotically.

Let us define a synchronization error as $\langle\delta\rangle = \frac{1}{4}\left(\sqrt{\langle|x_m - x_s|^2\rangle} + \sqrt{\langle|y_m - y_s|^2\rangle} + \sqrt{\langle|z_m - z_s|^2\rangle} + \sqrt{\langle|z_m - z_s|^2\rangle}\right)$. A Monte-Carlo like approach allows to evaluate the error index for $N = 10000$ parameter sets chosen inside the uncertainty range. Fixing an error threshold $\bar{\delta}$, the number of points N^+ corresponding to error indexes lower than $\bar{\delta}$ is taken into account. In Fig. 3 the fraction $\frac{N^+}{N}$ of parameter sets for which $\langle\delta\rangle < \bar{\delta}$ is reported for increasing values of the uncertainty and for different values of $\bar{\delta}$. Given a value for the uncertainty R, Fig. 3 allows to derive the maximum value of the synchronization error corresponding to that uncertainty.

The results observed through numerical simulations have been experimentally validated. Our experiments were restricted to the case of one parameter mismatches. In particular, the parameter c_s of the slave system has been varied in the range $65 < c < 110$, keeping constant $c_m = 88$ in the master (*i.e.*, ensuring that the master is in the hyperchaotic regime). In Fig. 4 the error index $\langle\delta\rangle$ is plotted for each value of c_s. It should be noticed that the range experimentally investigated is larger than that numerically examined. In fact, there are values of c_s ($c_s < 75$) inducing a periodic behavior in the slave. For such values the synchronization error is greater. When the slave is in the hyperchaotic regime, the synchronization error is in the range predicted by the numerical analysis.

These results show that a suitable level of synchronization can be reached also in presence of parameter mismatches.

5 Conclusions

In this paper the robustness of synchronization in presence of parametric uncertainties is investigated, both numerically and experimentally. The experiment was performed through the electrical analogous of a recently introduced Lorenz-like system able to show hyperchaotic behavior. Dealing with real circuital components, the two coupled circuits cannot be identical, due to tolerances in electrical components. However, the results presented in this work allow to remark that the MSF approach is a robust observer design tool for critical systems like the hyperchaotic ones.

References

Arena P., Buscarino A., Fortuna L., Frasca M. (2006) *Separation and synchronization of piecewise linear chaotic systems.* Phys. Rev. E 74:026212.

Grosu I., Padmanaban E., Prodyot K. Roy, Dana S. K. (2008) *Designing coupling for synchronization and amplification of chaos.* Phys. Rev. Lett. 100:234102.

Kapitaniak T. (1994) *Synchronization of chaos using continuous control.* Phys. Rev. E, 50:1642–1644.

Manganaro G., Arena P., Fortuna L. (1999) *Cellular Neural Networks: Chaos, Complexity and VLSI processing.* New York: Springer-Verlag.

Pecora L. M., Carroll T. L. (1990) *Synchronization in chaotic systems.* Phys. Rev. Lett. 61:821–824.

Pecora L. M., Carroll T. L. (1998) *Master Stability Functions for Synchronized Coupled Systems.* Phys. Rev. Lett. 80:2109–2112.

Wang J., Chen Z., Chen G., Yuan Z. (2009) *A novel hyperchaotic system and its complex dynamics.* Int. J. Bif. Chaos, 18:3309.