ASTEROID DYNAMICS AT THE 3:1 MEAN MOTION RESONANCE WITH JUPITER (PLANAR PROBLEM)

Vladislav Sidorenko
Keldysh Institute of Applied Mathematics
Moscow, RUSSIA
sidorenk@keldysh.ru

Abstract
The 3:1 mean-motion resonance of the planar elliptic restricted three body problem (Sun-Jupiter-asteroid) is considered. The double numeric averaging is used to obtain the evolutionary equations which describe the long-term behavior of the asteroid's argument of pericentre and eccentricity. The existence of the adiabatic chaos area in the system's phase space is shown.

Key words
Mean-motion resonance, adiabatic chaos

1 Introduction

The 3:1 mean-motion resonance in the planar elliptic restricted three-body problem (Sun–Jupiter–asteroid) has long attracted considerable attention of specialists. In order to find secular effects, the equations of motion can be averaged over fast variables, namely, over mean longitudes of the asteroid and Jupiter (see, for example, [Scholl and Froeschlé, 1974; Ferraz-Mello and Klafke, 1991]. Upon averaging, a nonintegrable system appears which describe the ”fast” and ”slow” components of secular evolution. The ”fast” evolution consists in changing resonance phase (Delaunay variable) $D = \lambda - 3\lambda'$, where λ and λ' are the mean longitudes of the asteroid and Jupiter, respectively. The ”slow” evolution reveals itself in a gradual change of perihelion longitudes of resonance asteroid orbits. In order to analyze different variants of the ”slow” evolution, one can make yet another averaging: averaging over fast processes. Previously it was done at small orbit eccentricities of the asteroid and Jupiter [Wisdom, 1985; Vashkov'yak, 1989a]. In this paper double averaging is used for studying the ”slow” evolution without restrictions on the orbit eccentricity of an asteroid.

2 Averaging over mean longitudes

We assume that the semimajor axis of the orbit of Jupiter can be taken as the unit length, while the sum of masses of the Sun and Jupiter is the unit mass. The unit time is chosen so that the period of revolution of Jupiter around the Sun is equal to 2π.

We write the equations of motion of the asteroid in the variables

$$x,\ y,\ L,\ D,$$

where x, y, and L are the elements of the second canonical Poincare system, and they are related to osculating elements by the formulas

$$x = \sqrt{2\sqrt{(1-\mu)a}[1-\sqrt{1-e^2}]}\cos\widetilde{\omega}, \qquad (2.1)$$

$$y = -\sqrt{2\sqrt{(1-\mu)a}[1-\sqrt{1-e^2}]}\sin\widetilde{\omega},$$

$$L = \sqrt{(1-\mu)a}.$$

Here, $\widetilde{\omega}, e$, and a are the longitude of perihelion, eccentricity, and semimajor axis of the asteroid orbit, and μ is the mass of Jupiter ($\mu \ll 1$).

The equations of motion have the canonical form

$$\frac{dx}{dt} = -\frac{\partial\mathcal{K}}{\partial y}, \quad \frac{dy}{dt} = \frac{\partial\mathcal{K}}{\partial x}, \qquad (2.2)$$

$$\frac{dL}{dt} = -\frac{\partial\mathcal{K}}{\partial D}, \quad \frac{dD}{dt} = \frac{\partial\mathcal{K}}{\partial L}$$

with the Hamiltonian

$$\mathcal{K} = -\frac{(1-\mu)^2}{2L^2} - 3L - \mu R. \qquad (2.3)$$

Function R in the expression for $\mathcal{K}$ is defined in the following way:

$$R = \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{(\mathbf{r}, \mathbf{r}')}{r'^3} ,$$

where $\mathbf{r} = \mathbf{r}(x, y, L, \lambda(D, \lambda'))$ and $\mathbf{r}' = \mathbf{r}'(\lambda')$ are the heliocentric radii vectors of the asteroid and Jupiter.

Formal averaging of the equations of motion consists in substitution of the function

$$W(x, y, L, D) = \frac{1}{2\pi} \int_0^{2\pi} R(x, y, L, D, \lambda')\, d\lambda'. \tag{2.4}$$

for function R in expression (2.3) for Hamiltonian. After that the equations of motion become autonomous: the mean longitude of Jupiter $\lambda' = t + \lambda'_0$ is eliminated from their right-hand sides. A detailed description of the numerical algorithm used to evaluate $W(x, y, L, D)$ is given in [Vashkov'yak, 1989a; Vashkov'yak, 1989b].

3 Fast-slow system describing secular effects in the motion of a resonance asteroid

In motion of the asteroid in resonance 3:1 with Jupiter the value of variable L is close to $L_0 = 1/\sqrt[3]{3}$. In the limiting case $\mu = 0$ the asteroid, moving along the orbit with the semimajor axis $a_{res} = L_0^2 \approx 0.48074$, makes exactly three revolutions around the Sun during one revolution of Jupiter.

Following the general scheme of studying resonance effects in Hamiltonian systems [Arnol'd, Kozlov and Neishtadt, 2002] we change variable L in system (2.2) averaged over λ' for variable $\Phi = (L_0 - L)/\sqrt{\mu}$ representing the normalized deviation of L from its resonance value and introduce a new independent variable $\tau = \sqrt{\mu}t$. Restricting ourselves to the leading terms in the expansion in terms of $\varepsilon = \sqrt{\mu}$ of the right-hand sides of the equations of motion in variables x, y, Φ, and D we get:

$$\frac{dx}{d\tau} = \varepsilon \frac{\partial V}{\partial y} , \; \frac{dy}{d\tau} = -\varepsilon \frac{\partial V}{\partial x} , \tag{3.1}$$

$$\frac{dD}{d\tau} = \alpha \Phi , \; \frac{d\Phi}{d\tau} = -\frac{\partial V}{\partial D} ,$$

where

$$V(x, y, D) = W(x, y, L_0, D) , \; \alpha = \frac{3}{L_0^4} = 9\sqrt[3]{3} .$$

If we take as conjugate canonical variables $x/\sqrt{\varepsilon}$ and $y/\sqrt{\varepsilon}$, D and Φ, system (3.1) is Hamiltonian with the

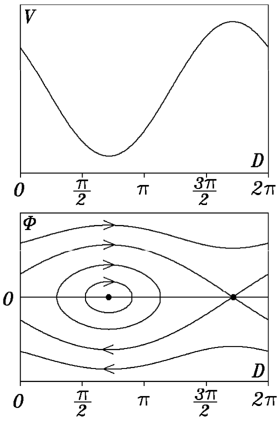

Figure 1. Phase portraits of a fast subsystem: $(x, y) \in S_*$

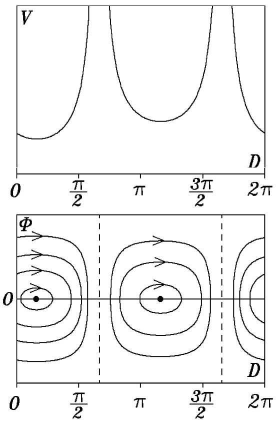

Figure 2. Phase portraits of a fast subsystem: $(x, y) \in S^*$

Hamilton function

$$\mathcal{H} = \frac{\alpha \Phi^2}{2} + V(x, y, D).$$

In the general case, variables x, y, D, and Φ have different rates of variation:

$$\frac{dD}{d\tau}, \frac{d\Phi}{d\tau} \sim 1, \; \frac{dx}{d\tau}, \frac{dy}{d\tau} \sim \varepsilon \ll 1.$$

We will refer to variables D, Φ and x, y as "fast" and "slow" variables, respectively.

System (3.1) allows one to investigate secular effects in the dynamics of resonance asteroids without constraints on the orbit eccentricity value. Evolution of osculating elements e and $\widetilde{\omega}$ with an error $O(\varepsilon)$ is described by the relations

$$e = \frac{1}{2L_0} \sqrt{(x^2 + y^2)\,[4L_0 - (x^2 + y^2)]} , \tag{3.2}$$

$$\widetilde{\omega} = \begin{cases} 2\pi - \arccos \dfrac{x}{\sqrt{x^2+y^2}},\ y \geq 0 \\ \arccos \dfrac{x}{\sqrt{x^2+y^2}}, \quad y < 0 \end{cases} \quad (x^2+y^2 \neq 0).$$

where variables x and y change in the region $S = \{(x,y), x^2+y^2 < 2L_0\}$.

Taking into account the separation of variables into "fast" and "slow" variables, we refer to system (3.1) as a fast-slow (FS) system [Arnol'd, Kozlov and Neishtadt, 2002].

4 Properties of the fast subsystem

At $\varepsilon = 0$ the equations of fast variables coincide with the equations of motion of a Hamiltonian system with a single degree of freedom which include x and y as parameters:

$$\frac{dD}{d\tau} = \alpha\Phi\,,\ \frac{d\Phi}{d\tau} = -\frac{\partial V}{\partial D}\,. \tag{4.1}$$

The qualitative behavior of trajectories on the phase portrait of system (4.1) is determined by the properties of function $V(x,y,D)$.

It turns out that there are important distinctions in fast dynamics at $e' > e'_*$ and at $e' < e'_*$, where $e'_* \approx 0.0385$ is the minimum eccentricity of the Jupiter orbit admitting its intersection with the orbit of the resonance asteroid at $\mu = 0$.

In the case $e' < e'_*$ (orbits of resonance asteroids do not intersect the Jupiter orbit at any $e < 1$) function $V(x,y,D)$ is limited. For the most (x,y) values the phase portrait of fast subsystem (4.1) is topologically equivalent to the phase portrait of a mathematical pendulum (Fig. 1). In what follows we will designate this set of (x,y) values as S_*. It can be proved that mes $S\backslash S_* \sim e'^3$.

In the case $e' > e'_*$, at appropriate choice of the initial value of variable D, the motion of the asteroid along the orbit crossing the Jupiter orbit will be accompanied by formal collisions resulting in divergence of the integrals determining functions $V(x,y,D)$. Below the set of (x,y) values corresponding to the resonance orbits with intersections of the orbit of Jupiter is designated as S^*.

Let

$$\Phi(\tau,x,y,h)\,,\ D(\tau,x,y,h) \tag{4.2}$$

be a solution to Eq. (4.1) satisfying the condition

$$\mathcal{H}(x,y,\Phi(\tau,x,y,h),D(\tau,x,y,h)) = h,$$

in which variable D changes in rotational or oscillating manner:

$$D(\tau+T,x,y,h) = D(\tau,x,y,h) \mod 2\pi$$

Here $T(x,y,h)$ is the period of the solution. We associate this solution with the following quantity

$$I(x,y,h) = \frac{\alpha}{2\pi}\int_0^{T/\sigma} \Phi^2(\tau,x,y,h)\,d\tau,$$

where the value of σ is determined by the type of solution. For rotational solutions $\sigma = 1$ and, hence, $I(x,y,h)$ is the action integral. For oscillating solutions $\sigma = 2$, and the value of $I(x,y,h)$ equals a half of the action integral.

At $\varepsilon \neq 0$ variables $x(\tau)$ and $y(\tau)$ in the right-hand sides of Eqs. (4.1) can be interpreted as slowly varying parameters. The quantity $I(x,y,h)$, coinciding to a constant factor with the action integral, will be an adiabatic invariant of system (3.1).

5 Averaging along solutions to the "fast" subsystem

Averaging the right-hand sides of the equations for x,y in system (3.1) along solutions to fast subsystem (5.2) we get evolution equations describing to the error $O(\varepsilon)$ the changes of variable x,y on the time interval with duration $\sim 1/\varepsilon$ (or $\sim 1/\mu$ in terms of original time units):

$$\frac{dx}{d\tau} = \varepsilon\left\langle\frac{\partial V}{\partial y}\right\rangle\,,\ \frac{dy}{d\tau} = -\varepsilon\left\langle\frac{\partial V}{\partial y}\right\rangle\,. \tag{5.1}$$

Here

$$\left\langle\frac{\partial V}{\partial \zeta}\right\rangle = \frac{1}{T}\int_0^T \frac{\partial V}{\partial \zeta}(x,y,D(\tau,x,y,h))\,d\tau$$

$$T = T(x,y,h),\ \zeta = x,y.$$

Construction of phase portraits of system (5.1) is the efficient method of studying the evolution of slow variables x,y. At different h the phase portraits can differ in the number of equilibrium positions and in the behavior of separatrices.

6 Forbidden area and uncertainty curve on phase portraits

Following [Neishtadt and Sidorenko, 2004], let us consider the auxiliary functions

$$H_*(x,y) = \min_D V(x,y,D), \tag{6.1}$$

$$H^*(x,y) = \max_D V(x,y,D).$$

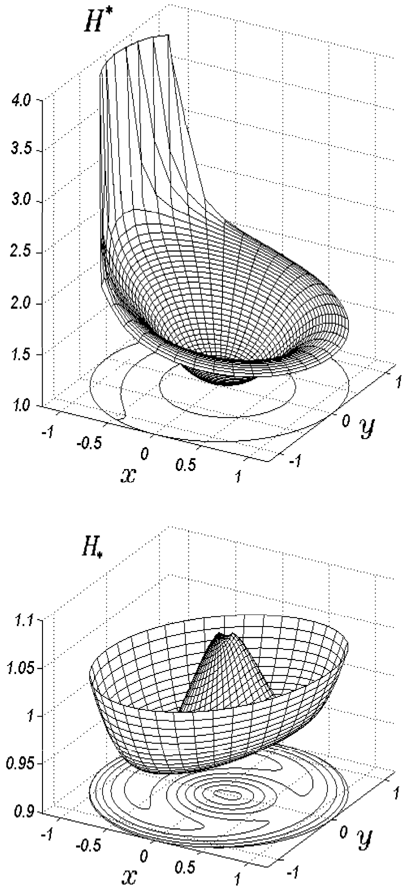

Figure 3. 3D graphs of $H_*(x, y)$ and $H^*(x, y)$ at $e' = 0.048$

The behavior of these functions is determined by the eccentricity e' of the orbit of Jupiter. In particular, it worth while to mention that $H^*(x, y) = \infty$ for $(x, y) \in S^*$ (Fig. 3).

The region

$$M(h) = \{(x, y) \in S, H_*(x, y) > h\} \qquad (6.2)$$

is forbidden for phase trajectories of system (5.1). At a given h the slow variables cannot assume values from $M(h)$.

The curve

$$\Gamma(h) = \{(x, y) \in S, H^*(x, y) = h\}$$

is called the uncertainty curve. In the case $H^*(x, y) = h$ the trajectory of the fast system is a separatrix and, hence, one cannot use averaging. If $\Gamma(h)$ is present on the phase portrait of system (5.1), it consists of several fragments undergoing a series of bifurcations when h is varied.

When a projection of the phase trajectory of system (5.1) onto the plane x, y crosses the curve $\Gamma(h)$, a quasi-random jump of adiabatic invariant $I(x, y, h)$ occurs [Neishtadt, 1987]. When studying the evolution of slow variables on a time interval of order of $1/\varepsilon$, this violation of adiabatic invariance is usually neglected, and solutions of the averaged system on curve $\Gamma(h)$ are glued in accordance with the following rule

$$I_{before} = I_{after},$$

where I_{before} is the value of $I(x, y, h)$ along the part of the phase trajectory of system (5.1) approaching $\Gamma(h)$, and I_{before} is the value of $I(x, y, h)$ on the trajectory part going away from curve $\Gamma(h)$. For most initial conditions, the accuracy of such an approximation is $O(\varepsilon)$ on the specified time interval.

The phenomena taking place at multiple intersections of the uncertainty curve will be discussed in Sec.**8**.

7 Investigation of slow evolution based on averaged equations

As an example Fig. 4 presents the phase portraits of system (5.1) constructed for the case $e' = 0.048 > e_*$. To choose such value of eccentricity of Jupiter is traditional for numerical investigations of the dynamics of asteroids in the context of the restricted elliptical three-body problem [Wisdom, 1982]. The discussion of the slow evolution for the case $e' < e_*$ can be found in [Sidorenko, 2006]. For better visualization the phase portraits present the behavior of quantities $\widehat{x}$ and $\widehat{y}$ which are related to variables x, y and osculating orbital elements $e, \widetilde{\omega}$ as

$$\widehat{x} = \frac{x}{2L_0}\sqrt{4L_0 - (x^2 + y^2)} = e\cos\widetilde{\omega},$$

$$\widehat{y} = \frac{y}{2L_0}\sqrt{4L_0 - (x^2 + y^2)} = -e\cos\widetilde{\omega}.$$

Fig 4 demonstrates the reconnection of separatrices at $h \approx 1.0955$. The location of the uncertainty curve on the phase portrait determines the size of the region of adiabatic chaos in the phase space of a nonaveraged system

Note. If $e' > e'_*$, one needs to take into account the existence of resonance orbits crossing the orbit of Jupiter (the orbits with parameters from region S^*). Region S^* in variables $\widehat{x}, \widehat{y}$ looks like a narrow strip on the periphery of region S. If one chooses $(x, y) \in S^*$ in system (3.1), there are, in the general case, two "fast" processes over which averaging is possible. Therefore, the right-hand sides of Eqs. (5.1) are ambiguously determined in S^*. At numerical integration, in the situation when evolving orbit of a resonance asteroid begins to intersect the orbit of Jupiter, one can choose an appropriate solution to fast subsystem (4.1) as a closest to the solution along which averaging was made on the preceding step.

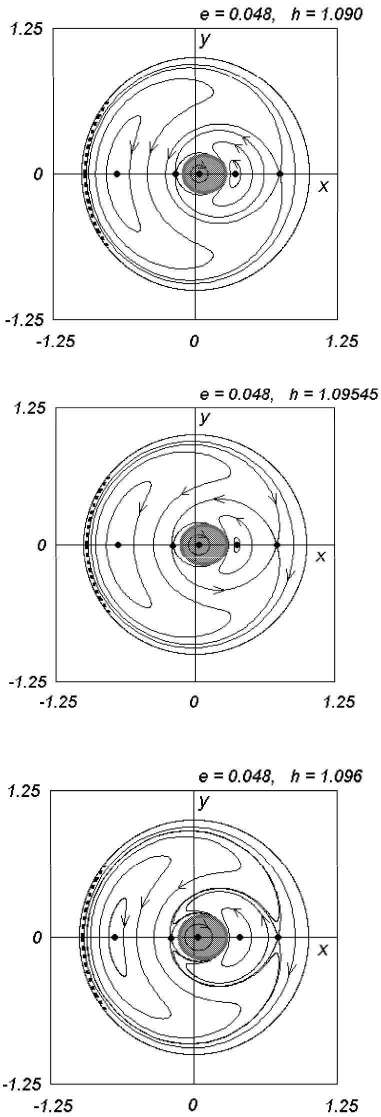

Figure 4. Example of the separatrices reconnection. Dashed line shows the position of the boundary between regions S_* and S^*

8 Adiabatic chaos

As it was mentioned in Sec. **6** in the neighborhood of uncertainty curve $\Gamma(h)$ the projection of a phase point of system (3.1) onto the plane x, y jumps from one trajectory of averaged system (5.1) to another in a quasi-random way: $|I_{after} - I_{before}| \approx \varepsilon$. As a result of a series of such jumps, the phase trajectories of system (3.1) with ε-close initial data can go away to a distance of ~ 1. Their projections onto the plane x, y will fill the region $\Sigma(h)$ which is a set of all trajectories of evolution equations (6.1) intersecting $\Gamma(h)$ (Fig. 5). In the phase space of FS-system (3.1) diverging trajectories will be located in the region

$$\Sigma^*(h) =$$

$$\{x, y, D, \Phi : \mathcal{H}(x, y, D, \Phi) = h,\ (x, y) \in \Sigma(h)\}.$$

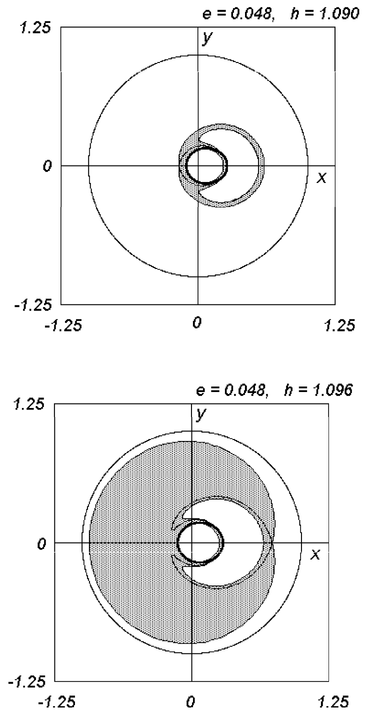

Figure 5. The region of adiabatic chaos before and after reconnection of separatrices.

We call $\Sigma^*(h)$ the region of adiabatic chaos: the complex behavior of trajectories in $\Sigma^*(h)$ is associated with violation of adiabatic invariance in the neighborhood of $\Gamma(h)$.

The properties of adiabatic chaos in Hamiltonian systems were studied in [Neishtadt, Treschev and Sidorenko, 1997; Neishtadt and Sidorenko, 2004]. The existence in this region of numerous ($\sim 1/\varepsilon$) stable periodic solutions was proved in [Neishtadt, Treschev and Sidorenko, 1997]. In [Neishtadt and Sidorenko, 2004], using numerical methods, such solutions were sought for an autonomous FS-system with two degrees of freedom.

The diverging trajectories of original (unaveraged) system (2.2) correspond to the trajectories of system (3.1) diverging in $\Sigma^*(h)$. Thus, the regions of chaotic dynamics originating due to violations of adiabatic invariance will also exist in the phase space of system (2.2). The stable periodic solutions to FS-system (3.1) turn into stable invariant tori in the extended phase space x, y, L, D, λ'.

9 Conclusions

Studies of 3:1 mean-motions resonance are of great importance for understanding the evolution of orbits of many celestial bodies. Asteroids of the Hestia family move in resonance 3:1 with Jupiter. In [Tittemore and Wisdom, 1990] the possibility of such a resonance was considered for Uranus' moons Miranda and Umbriel.

A hypothesis of planet motion in resonance 3 : 1 in the system 55 Cancri was discussed in [Ji, Kinoshita, Liu, and Li, 2003].

The approach used in this paper allows us to get a sufficiently detailed description of secular effects in motion of resonance objects in the context of a planar restricted elliptical three-body problem.

10 Acknowledgments

This investigation was supported by RFFI (grant NSh–2448.2006.1)

References

Arnol'd,V.I., Kozlov,V.V., and Neishtadt,A.I. (2002) Matematicheskie aspekty klassicheskoi i nebesnoi mekhaniki (Mathematical Aspects of Classical and Celestial Mechanics). Editorial URSS, Moscow (in Russian)

Ferraz-Mello, S., and Klafke, J.C. (1991). A model for the study of very-high-eccentricity asteroidal motion. The 3:1 resonance. *Predictability, Stability and Chaos in N-body Dynamical Systems.* Ed. Roy A.E. New York: Plenum Press (1991), 177–184.

Ji,J., Kinoshita,H., Liu,L., and Li,G. (2003) Could the 55 Cancri planetary system really be in the 3:1 mean motion resonance. *Astrophys. J.*, **585**, L139–L142.

Neishtadt,A.I. (1987). Sudden Changes of Adiabatic Invariant at Crossing the Separatrix and Origin of the 3:1 Kirkwood Gap. *Dokl. Akad. Nauk SSSR*, **295**, 47–50.

Neishtadt, A.I., and Sidorenko, V.V. (2004). Wisdom system: dynamics in the adiabatic approximation. *Celest. Mech. Dyn. Astron.*, **90**, 307–330.

Neishtadt,A.I, Treschev,D.V., and Sidorenko,V.V. (1997) Stable periodic motions in the problem of passage through a separatrix. *CHAOS*, **7**, 2–11.

Scholl,H., and Froeschlé,C. (1974). Asteroidal motion at the 3/1 commensurability. *Astron. Astrophys.*, **33**, 455–458.

Sidorenko,V.V. (2006) Evolution of asteroid orbits at the 3 : 1 their mean motion resonance with Jupiter (planar problem). arXiv:astro-ph/0610025.

Tittemore,W.C., and Wisdom,J. (1990) Tidal evolution of the uranian satellites III. Evolution through the Miranda-Umbriel 3:1, Miranda-Ariel 5:3, and Ariel-Umbriel 2:1 mean-motion commensurabilities. *ICARUS*, **85**, 394–443.

Vashkov'yak, M.A. (1989a). Method of Numerical Averaging in the Problem of Evolution of Orbits of Resonance Asteroids. *Kosm. Issled.*, **27**, 9–14.

Vashkov'yak,M.A. (1989b).Investigation of Evolution of Almost Circular Orbits of Distant Satellites in Resonance 3:1 with the Moon Using a Numerical-Analytical Method. *Kosm. Issled.*, **27**, 817–826.

Wisdom,J. (1982). The origin of the Kirkwood gaps: a mapping for the asteroidal motion near the 3/1 commensurability. *Astr. J.* (1982), **87**, 577–593.

Wisdom,J. (1985). A perturbative treatment of motion near the 3/1 commensurability. *ICARUS*, **63**, 272–286.