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MATHEMATICAL MODELING OF FRINGE FIELDS IN BEAM LINE CONTROL SYSTEMS

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Abstract
Usually beam dynamic modeling in beam lines is implemented using a piecewise approximation for of steering magnet fields. Real magnet fields have a bell-shaped form, which describes a field distribution along the electrical axis of a steering element. Influence of fringe fields in beam lines is taken into account only on tuning stage, practically on final steps before usage. This paper deals with methodology, which allows including fringe field effects during an initial stage of a modeling process. Also wide spectrum of modeling functions for fringe field distributions is presented. It is known, that fringe field effects are intrinsic and unremovable effects and could heavily impact on the beam dynamic and corresponding beam characteristics. Mathematical and computer models for fringe field help to estimate their influence and include information of real magnet field into the designing model. Research is based on the matrix formalism for Lie algebraic tools. This approach gives large flexibility, because it could admit the usage of computer algebra methods and technologies. Computer algebra methods can be easily paralleling, which gives line benefit with increasing number of processors (cores). Besides, this approach can be extended for nonlinear aberrations without loss of generality.

Key words
Fringe fields, beam optics, matrix formalism, Lie methods.

1 Introduction

In the last years Focusing Probe Systems (FPS) like microprobe or nanoprobe became very popular. Such systems allow forming particle beams with specified sizes, forms and even particle distributions. Also recently there was attracted huge interest in design of nanosize beam systems. Nanoprobes are very high-precision systems and sensitive to linear and nonlinear aberrations in accordance with experimental (see e. g. [Andrianov, 2004; Andrianov, Dymnikov and Osetinsky, 1978]) data and theoretical investigations (see, e. g. [Tereshonkov, Andrianov, 2008]). Moreover, the latest researches show that selection problem of *optimal* FPS variant is not limited with one or two alternatives (see e. g. [Tereshonkov, Andrianov, 2008]). To solve this problem it is necessary to estimate thoroughly a large number of possible variants and ensure many contradictory requirements (beam quality, misalignments, errors and so on). An approximate definition of FPS could be formulated as the following

Definition 1. *Focusing Probe System (FPS) is a transport and forming beam line, which satisfied the following conditions:*
1) *beam line is intended for "transfer" a beam from one part of space to another part ("transportation");*
2) *the main purpose is crosscut beam focusing, where the demagnification value of beam size are in the range from 50 to 100 or even more ("focusing");*
3) *approximation models of beam evolution description admit hierarchical structure of particle motion equations (meanwhile so-called linear model is considered as elementary model);*
4) *a linear model admits optical analogs of performance criterion;*
5) *nonlinear models is constructed as disturbed with respect to linear models and described with the aids of different nature aberration concept.*

High sensitivity of FPS and similar beam lines leads to necessity of thorough preliminary analysis and synthesis of beam line. Moreover, is is required to include a possible configuration influence of control systems and unremovable aberrations in their technological characteristics. This analysis must be made on basis of intensional *mathematical models* with performing series of calculating experiments (with obligatory stage of verification and comparison with known experimental data).

In the present paper, one of the most significant factor, which affects on beam characteristics, is so-called *fringe field* of control elements (magnet lenses) (see e. g. [Berz, Erdélyi and Makino, 2000; Venturini, Abell, and Dragt, 1998]):

Definition 2. *Fringe field is a changeable part of electromagnetic field, which is generated by control FPS.*

According to the features of control elements (magnet lenses) it is required to differ the "in" and "out" fringe fields.

Information about fringe fields could be obtained using experimental data from various sources or field maps, or as the result of numerical solution of Laplace's equation. All applied papers are attached to specified magnet lenses or some types of lenses. There are no mass measuring of real fringe fields due to high cost of similar measuring. So it is impossible to create some new system structures. One can create systems, which based on previous systems. In order to construct absolutely new system structure it is necessary to consider a set of approximation modeling functions for fringe fields. These functions help investigating of fringe field influence on beam characteristics. In spite of sufficiently great list of publication according to fringe fields (see e. g. [Berz, Erdélyi and Makino, 2000; Venturini, Abell, and Dragt, 1998]), there is not enough of investigations of fringe fields features and their impact on beam characteristics. In this paper a methodology of constructing modeling functions for fringe fields are described, many examples and basic principles are presented. This methodology gives necessary toolbox for system designers and helps selecting appropriate modeling functions which are optimal for approximating real steering field. The modeling functions have to be sectionally continuous and satisfy smoothness conditions.

The aim of the approximation process is to prepare a set of suitable functions in some appropriate class, in which the search of optimal configuration of control FPS will be performed. In other words, it is required to solve an "inverse problem": using the required final beam characteristics it is necessary to find a number of possible variants of steering fields.

2 Simulation scheme of FPS

A structure of control FPS could be defined in terms of linear model with particle beam description in terms of beam matrix envelope [Andrianov, 2004]. Meanwhile, on the *first stage* this structure is described in terms of so-called "rectangular model" (piecewise constant approximation) of steering field. On this stage it is required to select collection of optimal (in some sense) structures. The detailed investigation of selected collection implements on the next stages of modeling process.

On the *second stage* it is required to realize investigation of fringe fields influence on basic beam characteristics in linear model. In terms of this investigation it is considered some set of modeling approximating functions for fringe fields. An ability of simulation using some set of modeling functions allows selecting the most appropriate functions from the set. Thereby it helps to give a certain recommendations for "manufacturer" of control elements (magnet lenses).

Finally, on the *third stage* it is discussed issues of fringe fields impact on basic beam characteristics including third-order aberrations. Similar aberrations are intrinsic for a control FPS, which intends for transport and focusing particle beams with linear axis.

3 Mathematical model of beam dynamics

In the present paper the mathematical model of FPS is constructed with the aids of Lie algebraic tools [Dragt, 1982], which are realized in terms of matrix formalism [Andrianov, 2004]. It helps using advantages of matrix algebra, group theory methods and Lie algebras for building the matrix beam propagator with the aids of investigation of different effects impact on basic beam characteristics.

In this paper, evolution operator (matrix propagator) is constructed as infinite dimensional up-triangular matrix, which consists of block matrices according to representation in Poincare-Witta basis(see e. g. [Andrianov, 2004]). With a glance of above mentioned assumptions matrix propagator is not depend on initial beam status if a space charge is neglected.

Particle motion equation in the neighborhood of optical axis in common case can be written as following:

$$\frac{d\mathbf{X}(s)}{ds} = \mathbf{F}(\mathbf{X}, s), \quad \mathbf{F}(\mathbf{0}, s) \equiv 0. \tag{1}$$

Using quite suitable assumption one can solve the initial value problem (1) in Poincare-Witta basis and obtain an infinite dimensional Taylor series as a solution. The corresponding equation and its solution in terms of matrix formalism can be written as following

$$\frac{d\mathbf{X}(s)}{ds} = \sum_{k=1}^{\infty} \mathbb{P}^{1k}(s)\mathbf{X}^{[k]}(s), \; \mathbf{X}(s) = \sum_{k=1}^{\infty} \mathbb{R}^{1k}(s|s_0)\mathbf{X}_0^{[k]}, \tag{2}$$

where $\mathbf{X}^{[k]}(s)$ is the Kronecker k-th power for the phase vector $\mathbf{X}(s)$, $\mathbf{X}_0 = \mathbf{X}(s_0)$ is an initial phase vector, s_0 is an initial point. Here $\mathbb{P}^{1k}(s)$ are matrices with the entries equal to k-th derivative of the components of vector function $\mathbf{F}(\mathbf{X}(s), s)$. The matrices $\mathbb{R}^{1k}(s|s_0)$, $k \geq 2$ are called k-th order aberrations matrices, and they store the influence of all nonlinear effects up to k-th order.

FPS can be considered as some sequence of parts of control influence. The whole matrix propagator can be presented *exactly* as production of partial matrix propagators using group property. For propagator block matrices one can write the following recurrent equality:

$$\mathbb{R}^{ik}(s_2|s_0) = \sum_{j=i+1}^{k} \mathbb{R}^{ij}(s_2|s_1)\mathbb{R}^{jk}(s_1|s_0), \tag{3}$$

where $j \geq 1, k \geq 2$, $\mathbb{R}^{ij}(s|s_0)$ are an auxiliary matrices for constructing $\mathbb{R}^{1k}(s|s_0)$, which are the main. Eq. (3) is the exact representation of matrix propagator for any number of segmentation intervals, any distribution function $k(s)$ of steering field along the optical axis of a beam line. Moreover, the propagator is not depend on segmentation algorithm.

The segmentation is generated using the structure of FPS, and helps marking the intervals with different steering field behavior. It is a pity, but there is only narrow class of functions, for which there are known the analytical representations of linear matrix propagator. One of the purpose of the present paper is the investigation of different fringe field forms and length influence on beam characteristics. Special attention is given to representation of fringe fields in some classes of functions (function approximation). In this case, the basis set of control parameters are: locations of magnet lenses, length and field gradients of magnet lenses, and locations of drifts. On the intervals with nonconstant field (intervals with fringe fields) the character of field variation can be quite arbitrary. Since on stand one can check only specific control element, so similar information is not enough for the whole system simulation. During the modeling and synthesis of a new system designer must formulate some recommendations about fringe fields, which should be realized in order to obtain the necessary characteristics. The selection of splitting intervals and their number are determined by a set of control elements and form of representation of steering fields along the optical axis. The schematic

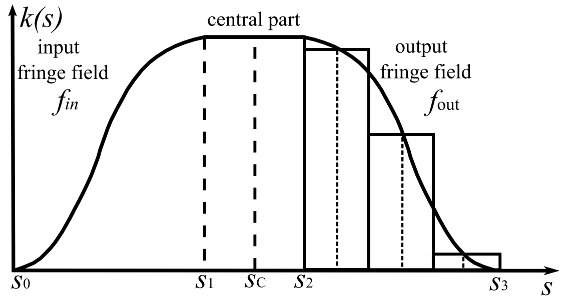

Figure 1. Fringe field of control element.

representation of fringe field of some control element is given on Fig. 1. The influence of similar element is determined by combination of input and output fringe fields, and a central part. Is is known several paper, which deals with different forms of fringe fields representation and investigations of fringe fields influence on beam characteristics with a glance of experimental data (see, e. g. [Berz, Erdélyi and Makino, 2000; Venturini, Abell, and Dragt, 1998]. In the present paper the fringe field modeling performs in some classes of functions in order to find an influence of fringe fields in arbitrary FPS. It allows giving some recommendations to designers of similar facilities in order to make *purposeful selection* of control elements characteristics, which is also depends on decided tasks.

In order to determine the matrix propagator on one of the intervals with steering field one can approximate the generated field in class of suitable functions (see, e. g. [Andrianov, 2004]). In common case it is possible to approximate steering field with piecewise constant (see Fig. 1), piecewise linear or more smoothness functions, for which the analytical matrix propagators are known or can be determined. Sometimes for some classes of steering fields one can find modeling functions (including composite), for which it is known the analytical solution (matrix propagator), and these functions are very closely to steering fields in norm. Usage of such functions is reducing a calculation time and allows increasing the accuracy of numerical modeling.

Moreover, analytical propagators for necessary classes of functions can be stored in specific database, which can be used for any further calculations (both analytical and numerical). This is exactly the approach, which is proposed in present paper. With the aids of proposed approach one can use native methods of parallel and distributed calculus, due to using matrix algebraic tools.

Finally, on conclusive stages of FPS modeling with a glance of fringe fields it is efficient to use *numerical calculation schemas* in order to check and correct the results (according to known experimental data), which is obtained analytically. In the present paper, analytical calculations are privileged due to its flexibility, lightness of parameters varying and multiple applicability in similar classes of tasks. It helps to make a decision about fundamental construction possibility of alike FPS with the glance of technological features of manufacturing and alignment for a quite wide class of facilities.

4 Linear Model

A character of considered tasks impose some constraints on steering fields: stationarity, absence of currents and charges, which generate fields in the region of particle beam motion, representation of steering fields as expansion in a series by multipole components. The latest point means that, we do not use Maxwell equations for obtaining a magnet fields. In the present paper only magneto-static steering fields are considered.

4.1 Particle motion equations.

In linear approximation particle motion equations can be represented as following for quadruples and linear optical axis:

$$\begin{cases} x'' + k(s)x = 0, & x' = dx/ds, \\ y'' - k(s)y = 0, & y' = dy/ds, \end{cases} \tag{4}$$

where $k(s) = qG/(m_0 c\beta\gamma)$, c is a light speed, $\beta = |v|/c$, $\gamma = 1/\sqrt{1-\beta^2}$, $G = \partial B_x/\partial y|_{x=y=0} = \partial B_y/\partial x|_{x=y=0}$ is a gradient of magnet field, s is a length, which is measured along some reference orbit. Scalar equations (4) could be written in vector form $\dfrac{d\mathbf{X}(s)}{ds} = \mathbb{P}(s)\mathbf{X}(s)$, where $\mathbf{X}_0 = \mathbf{X}(s_0)$ is a initial vector. The aim of linear approximation (4) is to construct linear propagator $\mathbb{R}^{11}(s|s_0)$ (matrizant) for the whole system, with a glance of steering fields along the optical axis.

In order to work with rigorous mathematical models is is required to make dimentionless the equation (4). It can be done by choosing attached length of the beam line. For example, one can select length of magnet lens, the whole system length, period length for cyclic accelerator, one meter or any other unit of length.

4.2 Control functions and parameters

In the present paper FPS structure allows representing control function $k(s)$ as piecewise smooth functions

$$k(s) = \begin{cases} 0, s \in [s_0, s_1), \Delta s_1 = s_1 - s_0, \\ k_2(s), s \in [s_1, s_2), \Delta s_2 = s_2 - s_1, \\ 0, s \in [s_2, s_3), \Delta s_3 = s_3 - s_2, \\ \dots \\ k_n(s), s \in [s_{n-1}, s_n), \Delta s_n = s_n - s_{n-1}, \\ 0, s \in [s_n, s_{n+1}), \Delta s_{n+1} = s_n - s_{n+1}. \end{cases} \quad (5)$$

where $k_i(s)$ is a field of i-th control element, which can be also split on intervals. In mathematical model input and output fringe fields are control functions. Let us introduce the additional segmentation, which allows interpreting fringe fields as virtual control parameters, during modeling process for optimal solutions retrieval.

After similar segmentation the task of optimization of FPS can be formulated in terms of control functions and control parameters. The right part in (1) can be rewritten as $\mathbf{F}(\mathbf{X}, \mathbf{U}, \mathbf{B}, s)$, where $\mathbf{U}(s) = (u_1(s), \dots, u_n(s)) = (0, k_2, \dots, k_n)$ is a vector of control functions and $\mathbf{B} = (\Delta s_1, \dots, \Delta s_{n+1})$ is a vector of control parameters. Segmentation performs according to control elements location. Presentation of $k_i(s)$ as input and output parts of fringe field, and central part leads to increasing a set of control functions and parameters. In other words, $\mathbf{U}(s) = (k_2^{\text{in}}, k_2^{\text{out}}, \dots, k_n^{\text{in}}, k_n^{\text{out}})$, where $k_i^{\text{in}}, k_i^{\text{out}}$ are modeling functions for input and output fringe field parts of i-th control element correspondingly. The vector of control parameters in this case $\mathbf{B} = (\Delta s_1, k_{\max}^2, L_0^2, \dots, k_{\max}^n, L_0^n, \Delta s_{n+1})$, where $k_{\max}^i, L_0^i$ is a maximum value of field gradient and length of central part of i-th control element. After introducing virtual modeling functions one can convert the control of functions and parameters to the whole set of control parameters, where parameters: $\Delta s_i, k_{\max}^{i+1}, L_0^{i+1}$ and $A_{i+1}^{\text{in}}, A_{i+1}^{\text{out}}$ are vectors of parameters describing $i+1$-th input and output fringe field modeling functions.

4.3 Fringe field forming problem

Some special case of fringe field are discussed, for example, in [Berz, Erdélyi and Makino, 2000; Tereshonkov, Andrianov, 2008; Venturini, Abell, and Dragt, 1998]. However it is not enough for thorough analysis and detailed modeling. Mathematical and computer models of fringe fields are not work out in detail. Therefore, the present paper deals with thorough and consistent developmental of mathematical tools and computer models of fringe fields. Moreover, the proposed approaches allow calculating matrix propagators for quite wide class of modeling functions and binding a fringe field form to specific experimental data.

Topology and geometry of fringe fields can be differ even for one-type control elements. Therefore, the investigation of influence of a fringe field form is very significant.

All standard control elements generate magnet field, which is symmetric relative to the center of control element. Thereby, in the present paper we consider only symmetric fringe fields (see e. g. [Berz, Erdélyi and Makino, 2000]) relative to the center point $s_c = (s_2 - s_1)/2$ for each element (see Fig. 1). Possible deviation from serial parameters of control elements can be investigated with the aids of perturbation theory methods or using new fringe fields models. In the present paper is supposed, that fringe fields of nearby control elements (doublets, triplets) do not interact and the result steering field is determined as linear fields superposition.

Due to laws of electrodynamics and experimental data, control magnet field is a smoothness function, which can be presented in following form:

$$f(s) = f_0 \begin{cases} f_{\text{in}}(s), & s \in [s_0, s_1), \\ k_{\max}, & s \in [s_1, s_2), \\ f_{\text{out}}(s), & s \in [s_2, s_3]. \end{cases} \quad (6)$$

Functions $f_{\text{in}}(s)$ and $f_{\text{out}}(s)$ describe input and output fringe fields correspondingly. In order to make $f(s)$ smooth in positions of joint it is required to demand the additional conditions:

$$\begin{aligned} f_{\text{in}}(s_0) = f_{\text{out}}(s_3) = f'_{\text{in}}(s_0) = f'_{\text{out}}(s_1) = f'_{\text{in}}(s_2) \\ = f'_{\text{out}}(s_3) = 0,\ f_{\text{in}}(s_1) = f_{\text{out}}(s_2) = 1, \end{aligned} \quad (7)$$

where $f'(s) = df(s)/ds$. Then $f_{\text{in}}(s)$ is determined with the aids of different functions approximation. The output fringe field $f_{\text{out}}(s)$ can be automatically found using the symmetric reflection relative to s_c (see Fig. 1). With a glance of (7), we will consider fringe field approximation for its modeling by quite simple functions in terms of their construction and analysis.

Instead of Eq. (7) it is possible to introduce the asymptotic analogues, which allow considering more extensive class of modeling functions:

$$\begin{aligned} &\lim_{s \to +s_0} f_{\text{in}}(s) = \lim_{s \to +s_0} f'_{\text{in}}(s) = \lim_{s \to -s_1} f'_{\text{in}}(s) = 0, \\ &\lim_{s \to -s_1} f_{\text{in}}(s) = \lim_{s \to +s_2} f_{\text{out}}(s) = 1, \\ &\lim_{s \to -s_3} f_{\text{out}}(s) = \lim_{s \to +s_2} f'_{\text{out}}(s) = \lim_{s \to -s_3} f'_{\text{out}}(s) = 0. \end{aligned} \quad (8)$$

In terms of functions $f_{\text{in}}(s), f_{\text{out}}(s)$ topology it is possible to discuss two extreme cases. The first one is the case of sharp boundary field (rectangular representation or piecewise constant form. The second case is a bell-shaped form. All the rest cases are intermediate in a manner.

Using the approach which is described in [Antone and AL-Maaitah, 1992], it is possible to suggest the schema of solution classes retrieval of perturbation equation with the aids of solution of nonperturbed equation. It is allow purposeful finding of analytical expressions for matrix propagators.

If fringe field can be presented analytically, parameters of the function $k(s)$ can be control parameters. In other case, one can approximate fringe field using piecewise constant, piecewise liner and even more smoothness functions. Then, as the control parameters one can consider the parameters, which form some approximation modeling function (e. g. for piecewise constant model such parameters are height and length of "steps"). It is allow formulating the corresponding optimal control tasks in terms of nonlinear programming tasks.

4.4 Fringe fields in linear model

In linear model one can find a set of optimal parameters of FPS in terms of required demagnification value.

Some numerical experiments (see e. g. [Berz, Erdélyi and Makino, 2000; Tereshonkov, Andrianov, 2008]) show, that in linear model fringe fields always make worse the demagnification value of the beam on target. In other words, optimal parameters in linear model are no more optimal if some control elements have fringe fields. However, the proposed methodology allow finding a set of optimal FPS parameters with a glance of fringe fields.

4.5 Fringe fields simulation

On the initial stage of modeling it is easy to use the piecewise constant model for fringe fields approximation, which allow varying number of segmentation intervals. Also, it provides the initial solution of the focusing and transport tasks. On the basis of piecewise constant model the first variant of FPS is synthesized (see e. g. [Andrianov, 2004]). This is the very popular approach, which can be found in many papers according to modeling of FPS. However, it is known, that similar variant is just the fist stage of modeling process before constructing a real FPS. If it is necessary to consider some fringe field effects in detail, one can use piecewise linear approximation of steering fields, which is better than piecewise constant model. In this case, matrix propagator (linear propagator is called matrizant) consists of Airy functions. For the real calculations with the aids of symbolic matrizants one can represent Airy functions as the series and use only several first terms of series. The similar models are sufficient only for a small class of tasks, but it is more interesting to consider more extensive class of modeling functions (which can be replenished). For example, one can use bell-shaped function, Enge functions [Berz, Erdélyi and Makino, 2000] and so on.

On the next stage it is required to find modeling functions, which can approximate fringe field on the whole interval or on its parts. It is useful to approximate fringe fields with modeling functions, for which it is known an analytical solutions (matrizants). Modeling functions can be composed and consists two or more parts. Meanwhile the "sewing together" conversion allows taking into account many specific features of real steering fields. One can use "sewing together" conversion for $F(s)$, determined on $s \in (s_1, s_2)$ in order to get new function which consists of $F_1(s)$ and $F_2(s)$ functions. They could be found using the following equations

$$F_1(s) = F(2s)/2, s \in (s_1, s_c) \;\; F_2(s) = -F\big(2(-s + s_2 - s_1)\big)/2 + F(s_2), s \in (s_c, s2), \tag{9}$$

where $s_c = (s_1 + s_2)/2$. Modeling functions satisfied sewing together conditions and some regular conditions (7)–(8) It is preferred to select modeling functions with free parameters in order to approximate real steering field more accurately with a glance of experimental data and find some optimal solutions for FPS.

In the present paper, modeling functions for fringe fields are considered as analytical formulas, which include many control parameters. Similar approach is useful on modeling stage for performing investigations before constructing real facilities. For example, an input fringe field $f_{\text{in}}(s)$ can be presented as function of s: $f_{\text{in}}([f_{\text{mod}}, \mathbf{A}], s)$, where f_{mod} is a model function variant and $\mathbf{A}$ is a vector of parameters, which characterized all features of fringe field modeling function.

As above mentioned the selection of fringe field modeling functions are determined by some conditions and parameters. One can reject some conditions or parameters if it is necessary. Varying some parameters of selected modeling function for fringe field, one can change its form or length in order to get more suitable variant.

4.6 Some function classes with analytical solution

Using the methodology from [Antone and AL-Maaitah, 1992] let us show, for example, three classes of functions and some their entries, for which on can found the analytical solution (matrizants): $f_1(s) = \psi(s)e^{\varphi}(s)$, $f_2(s) = \varphi(s)\cos\psi(s)$, $f_3(s) = \psi^n\dot{\psi}^m$. Examples from the first class can be the following functions:
$(a + bs)^{-4}, \alpha^2 + (a\sin\alpha s + b\cos\alpha s)^{-4}$, $1 + 2n - s^2 + (e^{2s^2})/[H_n(s)]^4$, where $H_n(s)$ is a n degree Hermitian polynomial. On the Fig. 2 examples of the first and the last functions are presented, where F_1 and F_2 are constructed using "sewing together" conversion for above mentioned basic functions. For the second class one can find the following examples $n^2s^{2n-2} -$

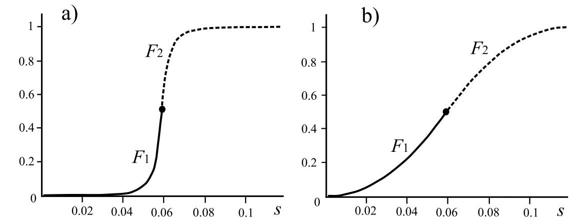

Figure 2. Examples of piecewise smooth fringe field modeling functions.

$(n^2-1)/(4s^2)$, $1/2-\cos^2\alpha s+3/4\tan^2\alpha s$. Finally, third class consists of e. g. the following functions $n-n(n-1)\tan^2\alpha s$, $n(3-s^2)-n(n-1)(s-1/s)^2$, $(1+4s^2-4n^2)/(4s^2)-2J_n(s)/\left(sJ_{n+1}(s)\right)$, where $J_n(s)$ is a n degree Bessel function.

4.7 Optimization tasks in terms of functionals

Optimization tasks in beam physics are known to be multicriterion. Some criterions can be antagonistic. In the present paper is is required to find optimal structure of FPS with a glance of fringe fields. In terms of similar task it is taking into account some technological restrictions on control elements construction and their possible arrangement. In this paper the task of getting nanosize beams is considered with the aids of standard control elements of mainstream production. Similar task leads to functional minimization, which describe beam phase portrait on target. In terms of linear model it is possible to consider the following functional $\inf\limits_{\mathbf{A}}\max(\sup\limits_{\mathfrak{M}} x, \sup\limits_{\mathfrak{M}} y)$, where x, y are particle beam dimensions in $\{x,s\}$ and $\{y,s\}$ planes correspondingly, $\mathbf{A}$ is a vector of parameters of fringe field modeling function. All the rest demands can be selected as equality and inequality.

One of the most essential problem of beam physics is beam focusing. Similar task can be described with the aids of different functionals, for example, minimal size of radius of beam phase portrait on target, minimal area of circle, which contains the whole beam phase portrait of maximum area of circle, which located in beam phase portrait and so on. In linear model all similar functionals can be reduced to minimization on beam size on target in $\{x,s\}$ or $\{y,s\}$ planes, or both simultaneously.

5 Optimization problem in nonlinear model

For the quadrupole lenses the next aberration order after first is third, which is corresponding to spherical and geometrical aberrations. The most harmful are spherical aberrations, which is bind with coefficient before ${x'}^3$ and $x'{y'}^2$ in entries of corresponding propagators. The matrix propagator $R^{13}(s_N|s_0)$ with the glance of third order aberrations can be calculated using the following formula (see e. g. [Andrianov, 2004]):

$$\mathbb{R}^{13}(s_N|s_0) = \int\limits_{s_0}^{s_N}\mathbb{R}^{11}(s_N|s)\mathbb{P}^{13}(s)\mathbb{R}^{33}(s|s_0)ds,$$

where s_0, s_N are locations of beam source and target correspondingly. In order to find analytically the integral it is required to select specific approximation functions of steering fields or use expansion procedure. With the increasing of complexity of selected modeling functions, propagator $\mathbb{R}_{13}(s_N|s_0)$ entries can be very bulky.

6 Conclusion

In the present paper the methodology for mathematical and computer modeling of fringe fields are presented in detail. Some parts of fringe fields can be presented as virtual control elements with the number of parameters. It allows transferring from control functions to the representation with control parameters only. Moreover, several examples of fringe fields modeling functions, which have analytical solutions are presented. For functions which do not have analytical propagators it is proposed to approximate them with piecewise constant, piecewise linear or even more smoothness functions in order to get analytical propagator in common case. Similar procedure can be applied to any modeling functions. In the present paper some examples of propagator entries are given for some modeling functions. The proposed approaches allows finding optimal working modes and structure of FPS with a glance of fringe fields effect. Moreover, it is possible to investigate fringe fields influence on beam characteristics.

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