LOCOMOTION OF FRICTION COUPLED SYSTEMS

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Abstract: This note provides the general set-up and solution method for a friction-coupled system (the 'body') moving in the landscape. This separates the shape (or configuration) space from the state space. The quotient space is the space where the motion of the body can be envisioned as a point mass. In particular we are interested in cases where the inertial forces are small in comparison to applied forces, and propose a simple perturbation expansion. Next we consider a periodic regime for the motion in shape space and pose and solve the optimal control problem via Fourier techniques. This is illustrated on a simple toy system: the two-piece worm with differential friction. Alternative friction models are proposed.

Keywords: Locomotion, Periodic Control, Biomimetic worm

1. FRICTION COUPLED SYSTEMS

In this section we consider the general framework of the locomotion of an object coupled to the landscape by friction. By locomotion, it is understood that the forces causing the motion of the object originate within the object itself (Holmes *et al.*, 2006; Ross, 2006).

Let the object be described in its own configuration space Θ, and let the rate of change of configuration be ω. We further assume the following model for the configuration dynamics

$$\dot{\theta} = \omega \tag{1}$$

$$\epsilon\dot{\omega} = f(\omega, \theta, u, c). \tag{2}$$

Here u is the vector of inputs (forces and/or torques) internally applied to the body, and c is the *coupling* vector, coupling the body to the environment (landscape) via sliding or viscous friction. The n-dimensional vector θ encodes all positions (prismatic or angular) coordinates that comprise the *configuration* or *shape* of the body. In the underactuated problem, the dimension of u is less than n.

Our ineterest is in the case where the parameter ϵ is small, meaning that the motion is such that inertia is almost negligible. In an underactuated system, the number of actuators is less than number of degrees of freedom (DOF), which. are governed by springs and mechanical limits. No special control is needed. This is manifest in many biological locomotion systems (Alexander, 2003). We model the body in the landscape from a *macroscopic view* as a point mass, obeying dynamical equations

$$\dot{x} = v \tag{3}$$

$$\epsilon\dot{v} = F(x, v, c). \tag{4}$$

where it is noted that - as the name suggests - the coupling term c appears in both sets of equations. This coupling itself is determined by a static equation

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$$c = \Gamma(\theta, \omega; x, v). \quad (5)$$

The components of x involve position and possibly the orientation. Hence its dimension is at most 6. In this paper we apply the above setup for the modeling and optimal control of some systems and make a few remarks about the necessity and approach towards stochastic models. We close with a high level motion planning.

Instead of considering the most general case directly, we focus here first on systems with nice symmetry properties. The first class of systems to be discussed are the ones for which $f(\omega, \theta, u, c)$ is semi-linear, meaning that for each θ, $f(k\omega, \theta, ku, kc) = kf(\omega, \theta, u, c)$. Likewise we shall assume semi-linearity of $F(x, v, c)$ for each x, and semi-linearity of Γ for each θ and x.
Finally we discuss the modeling and optimal control of some systems and make a few remarks about the necessity and approach towards stochastic models. We close with some remarks on high level motion planning.

2. INERTIA FREE SOLUTION AND APPROXIMATION

The inertia free solution is the solution assuming that the mass of the object is zero. The real solution will approximate this one if the inertial force is small compared to the other forces considered. This is for instance applicable for motion in fluids at low Reynolds number (Shapere and Wilczek, 1987). Hence, we solve the above system for $\epsilon = 0$ in a systematic way. First, $F(x, v, c) = F(x, v, \Gamma(\theta, \omega; x, v)) = 0$ yields by the implicit function theorem

$$v = \Omega(x, \omega, \theta), \quad (6)$$

provided $\frac{\partial F}{\partial v} + \frac{\partial F}{\partial c}\frac{\partial \Gamma}{\partial v} \neq 0$ is nonsingular. Then, $f(\omega, \theta, u, \Gamma(\theta, \omega; x, \Omega(x, \omega, \theta))) = 0$ yields the requisite body controls, provided that in turn the matrix $\frac{\partial f}{\partial u}$ has full rank. Indeed, since $\dim c$ may be smaller than $\dim f = n$, arbitrary parameters, p, may need to be be introduced here. We shall refer to a particular choice of parameter as a *gauge*, consistent with its usage in physics. Let

$$u = U(\omega, \theta, x; p) \quad (7)$$

exist. Finally, if the matrix $\frac{\partial U}{\partial}\omega$ has full rank the inverse function theorem gives

$$\omega = K(\theta, x, u; p), \quad (8)$$
$$\dot{\theta} = \omega, \quad (9)$$
$$\dot{x} = v. \quad (10)$$

The work done, $W(t) = \int_0^t u(\tau)\,\mathrm{d}\theta(\tau)$, can now easily be computed as a nonlinear *output*.

2.1 Matching expansions

The solution to the original system may now be approximated by the technique of matching expansions. We set $z = z_0 + \epsilon z_1 + \epsilon^2 z_2 + \ldots$, where z is any of the variables, x, v, u, θ, ω and c.

Restricted to the first perturbation only, we find for the zeroth order term

$$\dot{\theta}_0 = \omega_0 \quad (11)$$
$$0 = f(\omega_0, \theta_0, u_0, c_0) \quad (12)$$
$$\dot{x}_0 = v_0 \quad (13)$$
$$0 = F(x_0, v_0, c_0) \quad (14)$$
$$c_0 = \Gamma(\theta_0, \omega_0; x_0, v_0) \quad (15)$$

Of course its solution is the inertia free solution, described above. The first order terms are

$$\dot{\theta}_1 = \omega_1 \quad (16)$$
$$\dot{\omega}_0 = \frac{\partial f}{\partial \omega}\omega_1 + \frac{\partial f}{\partial u}u_1 + \frac{\partial f}{\partial c}c_1 \quad (17)$$
$$\dot{x}_1 = v_1 \quad (18)$$
$$\dot{v}_0 = \frac{\partial F}{\partial x}x_1 + + \frac{\partial F}{\partial v}v_1 + \frac{\partial F}{\partial c}c_1 \quad (19)$$
$$c_1 = \frac{\partial \Gamma}{\partial \theta}\theta_1 + \frac{\partial \Gamma}{\partial \omega}\omega_1 + \frac{\partial \Gamma}{\partial x}x_1 + + \frac{\partial F}{\partial v}v_1. \quad (20)$$

All functions and their derivatives are computed at the zero-th order solution. Hence,

$$\dot{\theta}_1 = \omega_1 \quad (21)$$
$$\dot{\omega}_0 = f_1(\omega_1, \theta_1, u_1, c_1) \quad (22)$$
$$\dot{x}_1 = v_1 \quad (23)$$
$$\dot{v}_0 = F_1(x_1, v_1, c_1) \quad (24)$$
$$c_1 = \Gamma_1(\theta_1, \omega_1, x_1, v_1). \quad (25)$$

This is solved as before: $F_1(x_1, v_1, \Gamma_1(\theta_1, \omega_1, x_1, v_1)) - \dot{v}_0 = 0$ yields by the implicit function theorem

$$v_1 = \Omega_1(x_1, \omega_1, \theta_1, \dot{v}_0). \quad (26)$$

Then $f_1(\omega_1, \theta_1, u_1, \Gamma_1(\theta_1, \omega_1; x_1, \Omega_1(x_1, \omega_1, \theta_1, \dot{v}_0))) - \dot{\omega}_0 = 0$ yields the first order perturbation in the requisite body controls for a suitable gauge.

$$u_1 = U_1(\omega_1, \theta_1, x_1, \dot{v}_0, \dot{\omega}_0). \quad (27)$$

Finally, the inverse function theorem gives

$$\omega_1 = K_1(\theta_1, x_1, u_1, \dot{v}_0, \dot{\omega}_0), \quad (28)$$
$$\dot{\theta}_1 = \omega_1, \quad (29)$$
$$\dot{x}_1 = \Omega_1(x_1, \omega_1, \theta_1, \dot{v}_0). \quad (30)$$

2.2 Invariance under time scaling

The semi-linear case is defined by: for all $k > 0$,

$$F(x,v,c)=0 \Rightarrow F(x,kv,kc)=0 \quad (31)$$
$$f(\omega,\theta;u,c)=0 \Rightarrow f(k\omega,\theta;ku,kc)=0 \quad (32)$$
$$\Gamma(\theta,k\omega;x,kv) = k\Gamma(\theta,\omega;x,v). \quad (33)$$

Introduce the time scaling operator $\mathbf{S}_\alpha$, defined via $(\mathbf{S}_\alpha x)(t)=x(\alpha t)$ for all t, Using the fact that $\mathbf{D}x=v$ implies $\mathbf{D}\mathbf{S}_{1/k}x=k\mathbf{S}_{1/k}v$, gives from $c=\Gamma(\theta,\omega;x,v)$ that $kc=\Gamma(\theta,k\omega;x,kv)$. Combining with $F(x,kv,kc)=0$ yields $kv=\Omega(x,k\omega,\theta)$ and then

$$f(k\omega,\theta,ku,\Gamma(\theta,k\omega,x,\Omega(x,k\omega,\theta)))=0,$$

and by inverse function theorem $ku=U(k\omega,\theta,x)$. Thus speeding up the velocity requires speeding up the coupling and internal forces. Finally, this is consistent with a time scale, $\mathbf{S}_{1/k}$ of all (generalized) coordinates.

The work done by the internal forces is of the form $W(t)=\int_0^t u(\tau)\,\mathrm{d}\theta(\tau)$. To speed up the motion by a factor k, requires that the forces must be scaled by a factor k. while the time to reach a certain fixed distance is proportional to $1/k$. Hence, the work required to reach this distance will be proportional to k. We conclude that the smaller the speed, the less energy is required to reach a given distance. Expressed another way, the product $T\mathcal{E}(x)$, where $\mathcal{E}$ is the energy to reach x remains constant under scaling of speed. In principle, *a transfer to the desired position is possible with zero energy, but requires an infinite time.* An optimization problem of the form: "Find the control requiring minimum energy to reach a given distance x" will then only make sense if we restrain the total time somehow, i.e., the only problem that makes sense in this case is the one for determining the optimal *profile* of the force over some dimensionless time.

3. THE TWO-PIECE WORM

Consider a toy model for a worm, consisting of two blocks separated by a spring (spring constant k), and an actuator, exerting a force u to the block on the right, and $-u$ to the one on the left (Fig. 1). The configuration space for the worm consists

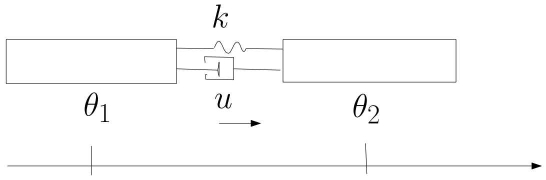

Fig. 1. The Two Piece Worm

of the two excursions θ_1 and θ_2. Consequently, the object (configuration) dynamics is given by

$$\dot\theta_1=\omega_1 \quad (34)$$
$$\epsilon\dot\omega_1=-k(\theta_1-\theta_2)-u+c_1 \quad (35)$$
$$\dot\theta_2=\omega_2 \quad (36)$$
$$\epsilon\dot\omega_2=-k(\theta_2-\theta_1)+u+c_2 \quad (37)$$

The landscape is modeled for simplicity in one

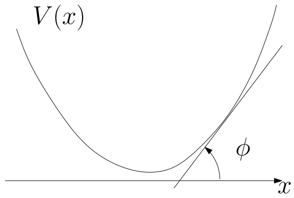

Fig. 2. The Landscape V(x)

dimension with topography given by some function $V(x)$. With g the gravitational acceleration at the surface, we obtain for the point mass in the landscape

$$\dot x = v \quad (38)$$
$$(2\epsilon)\dot v=-2\epsilon g\frac{V'(x)}{\sqrt{1+V'(x)^2}}+c_1+c_2. \quad (39)$$

Here $V'(x)=\frac{\mathrm{d}V(x)}{\mathrm{d}x}$ (more generally, the gradient). $\tan\phi=V'(x)$ yields $\sin(x)=\frac{V'(x)}{\sqrt{1+V'(x)^2}}$. This system was analyzed for Coulomb friction in (Chernousko, 2002). Here we depart from this situation and consider the motion with some lubrication, implying a viscous friction, but with a friction coefficient μ that may be a function of the position in the landscape (x-variables) and the absolute velocity, $v+\omega_i$. In addition, if the viscous friction coefficient is independent of mass, one may take a quasi static approach, (neglecting inertia). The friction coupling gives

$$c_i=-\mu(x,\omega_i+v)[\omega_i+v], \quad i=1,2. \quad (40)$$

The solution $v(\omega_1,\omega_2,\theta_1,\theta_2,x)$ is obtained from [2]

$$c_1+c_2=0$$

Integration gives x, and the requisite force u can be calculated.

3.1 Motion on a flat surface with differential friction

Consider the simplest case, where the potential function is constant, and with viscous friction coefficient depending on the direction of the motion. This means,

$$\mu(x,\omega_1+v)[\omega_1+v]+\mu(x,\omega_2+v)[\omega_2+v]=0.$$

[2] The author is indebted to Deryck Yeung (Georgia Tech) for pointing out a mistake in the original manuscript.

This problem bears some similarity to (Chernousko, 2006). Since the friction coefficient is always positive, the velocities $v_1 = v + \omega_1$ and $v_2 = v + \omega_2$ must have opposite signs. By symmetry, it suffices to study the case $v + \omega_1 < 0$, for which $\mu(v + \omega_1) = \mu_{FW}$ and $\mu(v + \omega_2) = \mu_{BW}$. We assume that the positive direction is towards the right, but the natural motion of the worm is towards the left. It follows from this that

$$v = -\frac{\mu_{FW}\omega_1 + \mu_{BW}\omega_2}{\mu_{FW} + \mu_{BW}}. \tag{41}$$

This gives in turn

$$0 = -u - \mu_{FW}[\omega_1 + v] - k(\theta_1 - \theta_2)$$
$$0 = u - \mu_{BW}[\omega_2 + v] - k(\theta_2 - \theta_1).$$

Note that only one of these relations is needed, as can be seen by substituting the expression (41) for v. This leads to

$$u = -\frac{\mu_{FW}\mu_{BW}}{\mu_{FW} + \mu_{BW}}(\omega_1 - \omega_2) - k(\theta_1 - \theta_2), \tag{42}$$

This is only one equation in two unknowns, but we bring in a special *gauge* (Shapere and Wilczek, 1989), which would appropriately be called the *momentum* gauge, by requiring that $\omega_1 + \omega_2 = 0$. This gives

$$\omega_1 = -\omega_2 = -\frac{u + k(\theta_1 - \theta_2)}{2}\left(\frac{1}{\mu_{FW}} + \frac{1}{\mu_{BW}}\right).$$

In addition, (42) implies $\omega_1 + v = \frac{2\mu_{BW}}{\mu_{FW}+\mu_{BW}}\omega_1$. The condition $\omega_1 + v < 0$ is thus equivalent to $\omega_1 < 0$. Since $\omega_1 + \omega_2 = 0$ also implies that $\theta_1 + \theta_2$ is constant, $2\theta_0$ say, in this gauge we can write:

$$u = -\frac{2\mu_{FW}\mu_{BW}}{\mu_{FW} + \mu_{BW}}\omega_1 - 2k(\theta_1 - \theta_0).$$

Next, from this

$$\dot\theta_1 = -\dot\theta_2 = -\frac{u + k(\theta_1 - \theta_2)}{2}\left(\frac{1}{\mu_{FW}} + \frac{1}{\mu_{BW}}\right).$$

It is however simpler to leave it in terms of the *symmetric components*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\theta_1 - \theta_2) = -[u + k(\theta_1 - \theta_2)]\left(\frac{1}{\mu_{FW}} + \frac{1}{\mu_{BW}}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\theta_1 + \theta_2) = 0. \tag{43}$$

The case $\omega_1 + v > 0$ is analogous, but in that case $\omega_1 + v = \frac{\mu_{FW}}{\mu_{FW}+\mu_{BW}}(\omega_1 - \omega_2) = \frac{2\mu_{FW}}{\mu_{FW}+\mu_{BW}}\omega_1$. Again, $\omega_1 + v > 0$ is equivalent to $\omega_1 > 0$. From here on we shall denote ω_1 and θ_1 respectively by ω and θ. The expression for v is, for all t

$$v = -\frac{\mu_{BW} - \mu_{FW}}{\mu_{BW} + \mu_{FW}}|\omega|. \tag{44}$$

It is interesting to see that despite the hybrid nature of the friction coupling, the systems (43) producing the symmetric components from the input are *smooth*. Finally, integration of v gives x. If u is periodic, then $|\omega|$ is periodic. In one period, the distance x traveled is then $4\frac{\mu_{BW}-\mu_{FW}}{\mu_{BW}+\mu_{FW}}\Delta\theta$, where $\Delta\theta$ is the *amplitude* of θ.

3.2 Optimal Periodic Control

In this section we obtain the optimal periodic steady state using *Fourier techniques*. We shall only illustrate the works for the zero-th order term. Assuming that u is periodic (Shiriaev *et al.*, 1995) with period 2π (recall the similitude with respect to time scaling), we assume the existence of Fourier series expansions

$$u(t) = \sum_{n=-\infty}^{\infty} u_n\, \mathrm{e}^{jnt} \quad \text{and} \quad \theta(t) = \sum_{n=-\infty}^{\infty} \theta_n\, \mathrm{e}^{jnt}.$$

Defining $\frac{1}{\mu} = \frac{1}{\mu_{BW}} + \frac{1}{\mu_{FW}}$, we get from (43) that

$$\theta_n = -\frac{1}{2(jn\mu + k)}u_n. \tag{45}$$

Also

$$\omega_n = -\frac{jn}{2(jn\mu + k)}u_n. \tag{46}$$

Since $\omega(t)$ is positive in an interval of half the period, we may set $|\omega(t)| = \omega(t)\mathrm{sgn}(\sin t)$, from which the Fourier series expansion of $\omega(t)$ follows as the convolution

$$|\omega|_n = (\omega * s)_n = \sum_{\ell=-\infty}^{\infty} \omega_{n-\ell}s_\ell,$$

where $\{s_n\}$ is the sequence of Fourier series coefficients of the block wave $s(t) = \mathrm{sgn}(\sin t)$. But the latter gives $s_n = -\frac{2j}{n\pi}$ for odd, and 0 for even n.

$$|\omega_n| = -\sum_m \frac{2j}{(2m+1)\pi}\omega_{n-2m-1}. \tag{47}$$

Hence, it follows also from (44) that

$$v_n = -\kappa|\omega|_n \tag{48}$$

where we defined the parameter $\kappa = \frac{\mu_{BW}-\mu_{FW}}{\mu_{BW}+\mu_{FW}}$.

The distance traveled in one period is readily computed as the integral of v, noting that the only term in the Fourier series contributing to the integral is the v_0 term. Thus, the distance moved in one period is

$$x = 2\pi v_0 = \sum_m \frac{4\kappa j}{(2m+1)}\omega_{-2m-1}. \tag{49}$$

The work done is $-\int_0^T u(t)\,\omega(t)\mathrm{d}t$. The "$-$" sign stems from the fact that the positive direction for force and excursion were taken in opposite directions. With

$$u(t) = -2\frac{\mu_{FW}\mu_{BW}}{\mu_{FW}+\mu_{BW}}\omega(t) - 2k(\theta(t)-\theta_0)$$

sthe work done during one period, $T = 2\pi$, equals

$$W(T) = -2\frac{\mu_{FW}\mu_{BW}}{\mu_{FW}+\mu_{BW}}\int_0^T \omega_1^2\,\mathrm{d}t,$$

since $\theta_1(T) = \theta_1(0)$ by periodicity. By Parseval's theorem,

$$\int_0^{2\pi} \omega^2(t)\,\mathrm{d}t = \sum_n |\omega_n|^2. \qquad (50)$$

Reconsider now the problem of moving a given distance x in one period, while minimizing the work done. This is a standard complex linear-quadratic (albeit infinite dimensional) parameter optimization problem.

$$\inf \sum_n \psi_n |u_n|^2 \quad \text{s.t.} \quad \sum_n \phi_n u_n = x. \qquad (51)$$

Standard Lagangian optimization methods yield

$$\psi_n u_n^* + \lambda\phi_n = 0. \qquad (52)$$

Hence, $u_n = -\lambda^* \frac{\phi_n^*}{\psi_n^*}$. Back substitution gives $x = -\lambda^* \sum_n \frac{|\phi_n|^2}{\psi_n^*}$, and thus the optimal Fourier coefficients of the applied force are

$$u_n = x\frac{\frac{\phi_n^*}{\psi_n^*}}{\sum_m \frac{|\phi_m|^2}{\psi_m^*}}. \qquad (53)$$

In the problem at hand, ϕ_n and ψ_n are zero for even n, and $\phi_{2m+1} = -\frac{1}{2(k+j(2m+1))}$, while $\psi_{2m+1} = -\frac{\mu_F W \mu_{BW}}{2(\mu_{FW}+\mu_{BW})}\frac{(2m_+1)^2}{k^2+\mu^2(2m+1)^2}$.
This leads to $u_n = 0$ for n even and

$$u_{2m+1} = -2\frac{k+j\mu(2m+1)}{(2m+1)^2}\frac{1}{\sum_\ell \frac{1}{(2\ell+1)^2}}.$$

Note that the summation in the denominator converges to $\pi^2/4$. so that the odd Fourier coefficients for the optimal solution are

$$u_{2m+1} = -\frac{8}{\pi^2}\frac{k+j\mu(2m+1)}{(2m+1)^2}. \qquad (54)$$

In figure 3, the Fourier approximation to the optimal periodic control for the two piece worm is given over two periods for the parameters $k = 1$, $\mu_{FW} = 0.1$ and $\mu_{BW} = 1$. This is the normalized

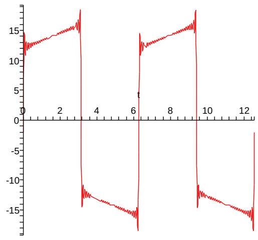

Fig. 3. The Optimal Periodic Control

solution for $x(2\pi) = 1$. It is clear that the optimal control is a piecewise linear function of t. This can also be deduced directly from the analytic form of the Fourier coefficients. The corresponding evolutions of the shape functions ω, θ are shown below. We note that since ω is a block wave, the landscape coordinate $v = |\omega|$ is constant, so that the motion is uniform. The distance traveled by the worm is then a linear function of time.

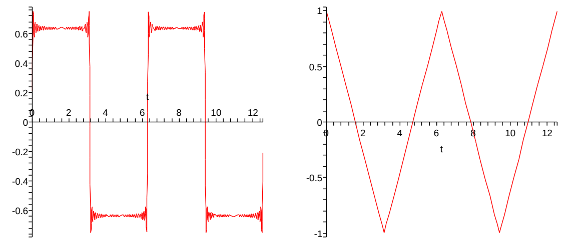

Figure 4: The Optimal Periodic Shape Change, ω, and Shape θ.

4. OTHER FRICTION MODELS

The result of the previous sections may seem unrealistic due to the fact that a transfer from 0 to x is possible with arbitrarily small energy expenditure if unlimited time is available. A remedy to this would be to consider a performance index which combines time and energy to reach a given x. For instance, we could take

$$J = \rho T + \int_0^T u(t)\,\mathrm{d}\theta(t).$$

See for instance (Verriest and Lewis, 1991). Alternatively, we may change the friction model. Recent studies (Persson, 2000) substantiated four regimes of friction: static friction, a transition from standing to the sliding condition, a level of friction at low sliding speed known as the kinetic friction, and a regime with positive slope labeled as viscous friction. The curve representing the force of friction as function of velocity is known as the Stribeck curve. For the static friction, a whole interval of equilibria ($v = 0$) exist. The function

$\mu(v) = \sqrt{(v-a)^2+b^2}$ gives a simple approximation for a symmetric (no differential friction) Stribeck curve. The minimum occurs for $v = a$ and is $\mu(a) = b$. If the block is initially at rest, and a force F is applied, the block will remain at rest as long as $F < \sqrt{a^2+b^2}$. If $F = \sqrt{a^2+b^2}$ then a slight perturbation will jerk the block towards an equilibrium velocity $v = 2a$ according to the nonlinear equation

$$\epsilon\dot{v} + \sqrt{(v-a)^2+b^2} = \sqrt{a^2+b^2}.$$

For small ϵ, we know that when the applied force, u, increases from 0 to u_s, the static friction, the velocity will remain zero. The instant u_s is traversed, the operating point shifts to some point $v' \geq v_1$, where v_1 is the solution on the Stribeck curve, where $f(v_1) = u_s$. If $\epsilon \to 0$, then $\dot{v}(t) \to \infty$, and since the translation is instantaneous, the force equals u_s during this transition. This gives the horizontal line segment AB on the (v, u) diagram. See Figure 4. The actual trajectory for nonzero ϵ must lie above this horizontal, since $\dot{u}(t) > 0$ by assumption. This is indicated by dash-dotted line. A rough bound on the transition time

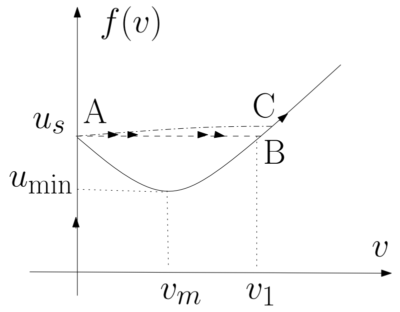

Fig. 4. Stribeck curve and trajectory for u.

for AC is $T > \epsilon\frac{v_1}{u_s - u_{\min}}$.

Singular perturbation techniques on the general equation $\epsilon\dot{v} + f(v) = u$ where u is the driving force, show that the minimum energy required to travel a given distance is bounded away from zero. For motion on dry sand, a fluid model may be appropriate (Tardos and Mort, 2005; Zhu, 2005), justifying our earlier approach with a pure viscous friction model.

5. CONCLUSIONS

We have discussed a class of locomotion systems that can be analyzed easily and effectively. In more complicated multi-link systems exhibiting symmetry, it is expected that the individual controls are easily generated by some central pattern generator (Holmes *et al.*, 2006; Iwasaki, 2006). For instance, for a snake like device, torques generate curvature of the body, but this curvature can be propagated down the snake body as a wave. There is a large literature on snake locomotion, see for instance (Chernousko, 2000; McIsaav and Ostrowski, 2003; Ostrowski *et al.*, 1995; Zhuravlev, 2002). In a continuum model such a wave can be generated as the solution to a (wave) partial differential equation. Spatial discretization of the wave equation gives then a nearest neighbor command control: The control signal is propagated from one actuator to the next (Verriest, 1989).

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