CYBERNETICS AND PHYSICS, VOL. 1, NO. 4, 2012 , 266–273

ON SOLUTION OF OPTIMAL CONTROL PROBLEMS WITH LIPSCHITZ DATA

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Abstract
The paper deals with the value function and the optimal synthesis in the optimal control problems with Lipschitz continuous payoffs. New sufficient optimality conditions are obtained for the problems. Numerical algorithms are proposed and results of numerical solution of the model problems are presented.

Key words
Value function, optimal synthesis, Bellman equations, method of characteristics.

1 Introduction

The paper is devoted to the construction of the value function and the optimal synthesis in the optimal control problem with the Lipschitz continuous payoff. The key role in the researches presented in the paper plays the value function. Our researches are based on the fact that the value function coincides with the generalized (minimax/viscosity) solution of Hamilton—Jacobi—Bellman equation.

Construction of the value function is obtained with the help of generalized method of characteristics. This method was suggested in the papers by A.I. Subbotin, N.N. Subbotina.

Applications of the generalized method of characteristics to the considered optimal control problem are based on the fact that necessary optimality conditions in the Hamiltonian form [Clarke, 1983] are expressed in terms of characteristics of Bellman equation.

The paper continues the works [Kolpakova,2010; Subbotina, 2006a; Subbotina and Tokmantsev, 2006]. We introduce new tools of the nonsmooth analysis, namely, partial subdifferentials in the direction. The new sufficient optimality conditions in terms of these subdifferentials are obtained and applied to the construction of the optimal synthesis.

This approach is close to the approach propsed by B. Mordukhovich [Mordukhovich, 2006].

The numerical algorithm for the considered problem is created. The results of simulation are presented.

2 Statement

Let us consider a control system defined by the equation

$$\dot{x}(t) = f(t, x, u), x(t_0) = x_0, \quad (1)$$

where $(t, x) \in \Pi_T = [0, T] \times \mathbb{R}^n$.

We want to minimize the Boltza payoff:

$$I_{t_0,\, x_0}(u(\cdot)) = \sigma(x(T)) + \int\limits_{t_0}^{T} g(t, x(t), u(t))dt, \quad (2)$$

The control $u \subset \mathbb{R}^k$ satisfies the geometric restriction

$$u \subset U \subset \mathbb{R}^k,$$

where U is a compact.

Let us define the set of admissible controls by the rule

$$\widetilde{U} = \{u(\cdot) : [0, T] \to U \text{ are measurable functions}\}.$$

The problem is considered under the following assumptions.

A1 Functions $f(t, x, u), g(t, x, u), \frac{\partial g}{\partial x_i}, \frac{\partial f}{\partial x_i}, i = 1, \ldots, n$ are defined and continuous on the set $\Pi_T \times U$.

A2 There exists constant $K_1 > 0$ such that

$$\Big|\frac{\partial f_i(t, x', u)}{\partial x_j} - \frac{\partial f_i(t, x'', u)}{\partial x_j}\Big| \le K_1||x' - x''||,$$

$$\Big|\frac{\partial g(t, x', u)}{\partial x_j} - \frac{\partial g(t, x'', u)}{\partial x_j}\Big| \le K_1||x' - x''||,$$

$i, j = 1, \ldots, n$, for any $x', x'' \in \mathbb{R}^n$.

A3 There exists constant $K_2 > 0$ such that

$$||\sigma(x') - \sigma(x'')|| \leq K_2||x' - x''||,$$

for any $x', x'' \in \mathbb{R}^n$.

A4 The set

$$\arg \min_{(f,g)\in F(t,x)} \langle s, f\rangle + g = \{(f^0, g^0)\}$$

is a singleton for any $s \in \mathbb{R}^n$, $(t, x) \in \Pi_T$. Here $F(t, x) = \{(f(t, x, u), g(t, x, u)) : u \in U\}$.

Define the Hamiltonian of problem (1), (2)

$$H(t, x, s) = \min_{u\in U}[\langle s, f(t, x, u)\rangle + g(t, x, u)].$$

We suppose additionally that

A5 Functions $D_xH(t, x, s)$ and $D_sH(t, x, s)$ exist and have sublinear growth with respect to s, that is

$$\Big|\Big|D_xH(t, x, s)\Big|\Big| \leq K_2(1 + ||s||),$$

$$\Big|\Big|D_sH(t, x, s)\Big|\Big| \leq K_2(1 + ||s||).$$

Here $D_xH(t, x, s) = \left(\frac{\partial H}{\partial x_1}, \ldots, \frac{\partial H}{\partial x_n}\right)$, $D_sH(t, x, s) = \left(\frac{\partial H}{\partial s_1}, \ldots, \frac{\partial H}{\partial s_n}\right)$.

Note, that the partial derivatives of the Hamiltonian have the form

$$D_xH(t, x, s) = \left\langle s, \frac{\partial f^0(t, x, s)}{\partial x}\right\rangle + \frac{\partial g^0(t, x, s)}{\partial x},$$

$$D_sH(t, x, s) = f^0(t, x, s) + \frac{\partial g^0(t, x, s)}{\partial s}.$$

It is well known [Subbotin, 1991], that functions $f^0(t, x, s), g^0(t, x, s)$ are continuous with respect to all variables.

2.1 Properties of the Value Function

The map

$$(t_0, x_0) \to V(t_0, x_0) = \inf_{u(\cdot)\in\tilde{U}} I_{t_0,x_0}(u(\cdot))$$

is called the value function.

Assertion 1 [Subbotina and Tokmantsev, 2006].

If assumptions $A1$–$A4$ are hold then

$$V(t_0, x_0) = \min_{u(\cdot)\in\tilde{U}} I_{t_0,x_0}(u(\cdot)).$$

It is well known that under assumptions $A1$–$A5$ the value function is continuous, but it can be nonsmooth.

Recall the notions of nonsmooth analysis.

Definition 1. [Rockafellar and Wets, 1983]

The lower Dini derivative $\frac{d^-\varphi(y)}{h}$ of a function $\varphi : \mathbb{R}^m \to \mathbb{R}$ at the point y in the direction $h \in \mathbb{R}^m$ is defined as follows:

$$\frac{d^-\varphi(y)}{h} = \liminf_{\delta\to 0, h'\to h} \frac{\varphi(y + \delta h') - \varphi(y)}{\delta}.$$

Similarly the upper Dini derivative $\frac{d^+\varphi(y)}{h}$ is defined by means of $\limsup$.

Definition 2. [Subbotin, 1995]

A function $\varphi(\cdot, \cdot) : \Pi_T \to \mathbb{R}$ is called the minimax solution of (3), (4), if

$$\varphi(T, x) = \sigma(x), \quad \forall x \in \mathbb{R}^n,$$

$\sup_{s\in\mathbb{R}^n} \inf_{h\in\mathbb{R}^n} \{\frac{d^-\varphi(t,x)}{1,h} - \langle s, h\rangle + H(t, x, s)\} \leq 0,$
$\inf_{s\in\mathbb{R}^n} \sup_{h\in\mathbb{R}^n} \{\frac{d^+\varphi(t,x)}{1,h} - \langle s, f\rangle + H(t, x, s)\} \geq 0,$
for all $(t, x) \in (0, T) \times \mathbb{R}^n$.

Assertion 2 [Crandall and Lions, 1983; Subbotin, 1995].

If assumptions $A1$–$A4$ are true, then the value function $V(t, x)$ in optimal control problem (1), (2) coincides with the unique minimax/viscosity solution of the problem

$$\frac{\partial V(t,x)}{\partial t} + H(t, x, D_xV(t, x)) = 0, (t, x) \in \Pi_T, \quad (3)$$

$$V(T, x) = \sigma(x), \quad x \in \mathbb{R}^n. \quad (4)$$

Assertion 3. [Subbotina, 2006b]

Under assumptions $A1$–$A4$ the value function of the problem (1), (2) is local Lipschitz continuous.

Definition 3. [Clarke, 1983]

The set

$$\partial\psi(y) = \text{co}\{q \in \mathbb{R}^m : \ q = \lim_{y_k\to y} D\psi(y_k)\}$$

is called the subdifferential of the function $\psi(\cdot) : \mathbb{R}^m \to \mathbb{R}$ at the point $y \in \mathbb{R}^m$.

Here $D\psi(y_k)$ is the gradient of the function $\psi(\cdot)$ at the points y_k. The symbol co denotes the convex hull.

We shall construct the value function with the help of the generalized method of characteristics.

Consider the Hamiltonian system for the problem (3), (4)

$$\begin{aligned} \dot{\tilde{x}} &= D_{\tilde{s}} H(t, \tilde{x}, \tilde{s}), \\ \dot{\tilde{s}} &= -D_{\tilde{x}} H(t, \tilde{x}, \tilde{s}), \\ \dot{\tilde{z}} &= \langle D_{\tilde{s}} H(t, \tilde{x}, \tilde{s}), \tilde{s} \rangle - H(t, \tilde{x}, \tilde{s}) \end{aligned} \tag{5}$$

with the boundary conditions

$$\begin{aligned} &\tilde{x}(T, \xi) = \xi, \tilde{s}(T, \xi) \in \partial\sigma(\xi), \\ &\tilde{z}(T, \xi) = \sigma(\xi), \xi \in \mathbb{R}^n. \end{aligned} \tag{6}$$

Solutions $\tilde{x}(\cdot, \xi), \tilde{s}(\cdot, \xi), \tilde{z}(\cdot, \xi)$ of the problem (5), (6) are called the characteristics of the problem (3), (4).

Assertion 4.

If assumptions $A1$–$A5$ are true, then for any $\tilde{x}(T, \xi)$, $\tilde{s}(T, \xi)$, $\tilde{z}(T, \xi)$, $\xi \in \mathbb{R}^n$ there exists the unique solution of the characteristic system (5)-(6), and it is defined on the interval $[0, T]$.

Recall the Pontryagin's maximum principle [Pontryagin *et al*, 1962] in the Hamiltonian form.

Assertion 5 [Clarke, 1983].

Let conditions $A1$–$A5$ be satisfied, $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$, $u^0(\cdot) \in \tilde{U}$ and

$$I_{t_0,x_0}(u^0(\cdot)) = V(t_0, x_0),$$

then there exists such a function $s^*(\cdot) : [t_0, T] \to \mathbb{R}^n$, that the following conditions are valid for all $t \in [t_0, T]$

$$\begin{aligned} &\tfrac{dx^0}{dt} = D_s H(t, x^0(t), s^*(t)) \\ &\tfrac{ds^*}{dt} = -D_x H(t, x^0(t), s^*(t)), \\ &x^0(t_0) = x_0;\ s^*(T) \in \partial\sigma(x^0(T)); \end{aligned} \tag{7}$$

The following statement is valid.

Theorem 1.

Let assumptions $A1$–$A3$ hold, $(t_0, x_0) \in \Pi_T$, then

$$V(t_0, x_0) = \min_{\xi : \tilde{x}(t_0, \xi) = x_0} \tilde{z}(t_0, \xi), \tag{8}$$

where $\tilde{x}(\cdot, \xi), \tilde{s}(\cdot, \xi), \tilde{z}(\cdot, \xi)$ are characteristics (5), (6).

Remark.

Formula (8) is proven in the paper [Subbotina, 2006a] under assumptions of the smooth data in the optimal control problem. The proof of theorem 1 is similar to one of smooth case.

3 Structure of the Minimax Solution

According to assertion 2,3 the value function can be studied with the help of minimax/viscosity solutions of the problem (3), (4).

Below we recall the statements first proved in [Subbotina and Kolpakova,2009] and introduce new tools of the nonsmooth analysis.

Assertion 6. [Subbotina and Kolpakova,2009]

Let assumptions $A1$–$A5$ be true and let function σ be continuous differentiable. The minimax solution $\varphi(t, x)$ of the problem (1) is not differentiable at (t_0, x_0), iff there exist $\xi_1, \xi_2 \in \mathbb{R}^n$, $\xi_1 \neq \xi_2$ such that

$$\begin{aligned} &\tilde{x}(t, \xi_1) = \tilde{x}(t, \xi_2) = x, \\ &\tilde{z}(t, \xi_1) = \tilde{z}(t, \xi_2) = \varphi(t, x), \\ &\tilde{s}(t, \xi_1) \neq \tilde{s}(t, \xi_2). \end{aligned} \tag{9}$$

Assertion 7 [Kolpakova,2010].

If conditions $A1$-$A5$ are valid, function σ is continuous differentiable and the state space is one-dimensional, then all points of nondifferentiability of the minimax solution $\varphi(t, x)$ lie on at most denumerable family of lines $t \to x_*(t) : 0 \leq t_* < t \leq T$ satisfying the Rankine-Hugoniot condition:

$$\frac{dx_*(t)}{dt} = \frac{H(t,x_*(t),D_+\varphi(t,x_*(t))) - H(t,x_*(t),D_-\varphi(t,x_*(t)))}{D_+\varphi(t,x_*(t)) - D_-\varphi(t,x_*(t))}, \tag{10}$$

$$D_+\varphi(t, x_*(t)) = \lim_{x \to x_*(t)+0} \nabla\varphi(t, x),$$

$$D_-\varphi(t, x_*(t))) = \lim_{x \to x_*(t)-0} \nabla\varphi(t, x),$$

and the inequality

$$D_-\varphi(t, x_*(t))) < D_+\varphi(t, x_*(t))).$$

Here $\nabla\varphi = \left(\frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial x}\right)$.

We introduce the new tool of nonsmooth analysis to describe properties of the local Lipschitz continuous value function in the problem (1), (2).

Definition 4. [Subbotina, 2006b]

The set

$$\partial_h \psi(y) = \text{co}\{\xi \in \mathbb{R}^m :\ \xi = \lim_{\delta_k \downarrow 0} \frac{\psi(y_k) - \psi(y)}{\delta_k}\}$$

is called the partial subdifferential of the local Lipschitz continuous function $\psi(\cdot)$ at the point y in the direction h.

Here $y_k = y + h\delta_k$ are the points of differentiability of function $\psi(\cdot)$.

The main result of the paper is following.

Theorem 2.

Let assumptions $A1$–$A5$ hold, $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$, and there exist characteristics $\tilde{x}(\cdot, \xi_*), \tilde{s}(\cdot, \xi_*), \tilde{z}(\cdot, \xi_*)$

of the problem (1), (2) such that $\tilde{x}(t_0, \xi_*) = x_0$, and the following relations are valid for all $t \in [t_0, T]$

$$\begin{array}{l} \alpha^* = -H(t, \tilde{x}(t, \xi_*), \tilde{s}(t, \xi_*)),\ p^* = \tilde{s}(t, \xi_*) \\ (\alpha^*, p^*) \in \partial_{h(t)} V(t, \tilde{x}(t, \xi_*)), \end{array} \quad (11)$$

$h(t) = (1,\ D_s H(t, \tilde{x}(t, \xi_*), \tilde{s}(t, \xi_*)))$.

Then $\tilde{x}(\cdot, \xi_*)$ is optimal trajectory of the problem (1), (2) and $h(t)$ is the optimal direction.

Proof.

We shall show, that the derivative of the value function V in the direction $D_s H(t, x, s)$, satisfying (11) is equal to 0. It is the necessary and sufficient optimality condition [Subbotina, 2006a].

According to the formula, proved in [Subbotina, 2006b], the following inequalities are valid for any $(t, x) \in \Pi_T$

$$\min_{(\alpha,p)\in\partial_h V(t,x)} \langle(\alpha, p), h\rangle \le \frac{d^- V(t, x)}{h} \le$$

$$\frac{d^+ V(t, x)}{h} \le \max_{(\alpha,p)\in\partial_h V(t,x)} \langle(\alpha, p), h\rangle.$$

Let us consider the expression

$$\langle(\alpha, p), h\rangle,\ \text{where } (\alpha, p) \in \partial_h V(t, x).$$

Remind that

$$\alpha = \lim_{k\to\infty} -H(t_k, x_k, D_x V(t_k, x_k)),$$

$p = \lim\limits_{k\to\infty} D_x V(t_k, x_k)$. Then $\langle(\alpha, p), h\rangle =$

$$\lim_{k\to\infty} \langle(-H(t_k, x_k, D_x V(t_k, x_k)), D_x V(t_k, x_k)), h\rangle.$$

The set $\partial_{h(t)} V(t, x)$ is convex and closed. Hence

$$\arg \min_{(\alpha,p)\in\partial_{h(t)} V(t,x)} \langle(\alpha, p), h\rangle = (\alpha^0(h), p^0(h)).$$

The function $(\alpha^0(h), p^0(h))$ is continuous [Subbotin, 1991]. This assertion is valid also for function $\arg\max$.

Note that

$$\lim_{k\to\infty} h_k = \lim_{k\to\infty} (1, D_s H(t_k, x_k, D_x V(t_k, x_k))) = h.$$

Then

$$\lim_{h_k\to h} \langle(\alpha^0(h_k), p^0(h_k)), h_k\rangle =$$

$$\langle(-H(t_k, x_k, D_x V(t_k, x_k)), D_x V(t_k, x_k)), h_k\rangle = 0$$

$$= \langle(\alpha^0(h), p^0(h)), h\rangle$$

because t_k, x_k are the points of differentiability of function V. Therefore we get

$$\frac{d^- V(t, x)}{h} = \frac{d^+ V(t, x)}{h} = 0.$$

□

This theorem provides sufficient optimality conditions for the case of Lipschitz continuous terminal function σ in the problem (1), (2).

Let us introduce the following notion.

Definition 5. The set

$$\partial_h^M \varphi(t, x) = \{\lim_{k\to\infty} \nabla\varphi(t_k, x_k)\}$$

is called nonconvex subdifferential at the point (t, x) in direction $h \in \mathbb{R}^n$. Here $\lim\limits_{k\to\infty} \delta_k = 0$, $\lim\limits_{k\to\infty} h_k = h$, $(t_k, x_k) = (t + \delta_k, x + \delta_k h_k)$ are the points of differentiability of the function φ.

We can prove the sufficient condition of optimality in terms of nonconvex subdifferential in the direction h of the value function.

Theorem 3.

Let conditions $A1$–$A5$ hold. Assume that there exist such characteristics $\tilde{x}(\cdot, \xi_*), \tilde{s}(\cdot, \xi_*), \tilde{z}(\cdot, \xi_*)$ of the problem (1), (2) such that $\tilde{x}(t_0, \xi_*) = x_0$, and the following relations hold for all $t \in [t_0, T]$

$$\begin{array}{l} \alpha^* = -H(t, \tilde{x}(t, \xi_*), \tilde{s}(t, \xi_*)),\ p^* = \tilde{s}(t, \xi_*), \\ (\alpha^*, p^*) \in \partial^M_{h(t)} V(t, \tilde{x}(t, \xi_*)), \end{array} \quad (12)$$

$h(t) = (1,\ D_s H(t, \tilde{x}(t, \xi_*), \tilde{s}(t, \xi_*)))$.

Then the trajectory $\tilde{x}(t, \xi_*)$ is optimal in the problem (1), (2).

Proof.

We shall prove that $\frac{dV^{\pm}(t, \tilde{x}(t, \xi_*))}{h} = 0$, satisfying (12).

Note that

$$\min_{(\alpha,p)\in\partial_h V(t,x)} \langle(\alpha, p), h\rangle = \min_{(\alpha,p)\in\partial_h^M V(t,x)} \langle(\alpha, p), h\rangle =$$

$$\max_{(\alpha,p)\in\partial_h V(t,x)} \langle(\alpha, p), h\rangle = \max_{(\alpha,p)\in\partial_h^M V(t,x)} \langle(\alpha, p), h\rangle$$

for any $(t, x) \in \Pi_T$.

Hence, we obtain $\frac{dV^{\pm}(t, \tilde{x}(t, \xi_*))}{h} = 0$.

□

4 Optimal Synthesis

Let us consider the optimal control problem in the class of feedbacks

$$[0,T]\times\mathbb{R}^n(t,x)\mapsto u(t,x)\in U$$

and allow them to be discontinuous. We use a formalization of discontinuous feedbacks, proposed by N.N. Krasovskii ([Krasovskii and Subbotin, 1988]).

Recall the main notions of the approach.

Consider a partition

$$\Gamma=\{t_i,\ i=0,1,\dots,N\}\subset[t_0=0,t_N=T]$$

with the fineness

$$\text{diam }(\Gamma)=\max_{i=1,\dots,N}(t_i-t_{i-1}).$$

Definition 6.

The step-by-step motion $x_\Gamma(\cdot)$ of the system (1) is defined in the following way

$$\begin{array}{l} x_\Gamma(\cdot):[t_{i-1},t_i]\mapsto\mathbb{R}^n,\ i=1,\dots,N;\\ u_\Gamma(t)=u^{i-1}=u\big(t_{i-1},x_\Gamma(t_{i-1})\big)=\text{ const};\\ \forall t\in[t_{i-1},t_i),\\ \frac{dx}{dt}=f(t,x,u^{i-1}),\forall t\in[t_{i-1},t_i);\\ x_\Gamma(t_0)=x_0. \end{array}\tag{13}$$

Define

$$C_\Gamma\big(t_0,x_0;u(t,x)\big)=I_{t_0,x_0}\big(u_\Gamma(\cdot)\big).$$

Definition 7.

The value $C\big(t_0,x_0;u(t,x)\big)$ of the form

$$C=\limsup_{\text{diam}(\Gamma)\to0}\ C_\Gamma\big(t_0,x_0;u(t,x)\big)\tag{14}$$

is called the value for the feedback $u(t,x)$ in the system (1) at the initial state (t_0,x_0).

Definition 8.

A feedback $u(t,x)$ satisfying the equality

$$C\big(t_0,x_0;\ u(t,x)\big)=V(t_0,x_0)\tag{15}$$

is called the optimal feedback at the initial state (t_0,x_0).

Definition 9. A universal optimal feedback $u^0(t,x)$ satisfying the relations

$$\begin{array}{l} C\big(t_0,x_0;\ u^0(t,x)\big)=V(t_0,x_0),\\ \forall(t_0,x_0)\in[t_0,T]\times\mathbb{R}^n \end{array}\tag{16}$$

is called the optimal synthesis.

From theorem 3 and work [Subbotina and Tokmantsev, 2010] the following theorem is valid.

Theorem 4.

The optimal synthesis $u^0(t,x)\ :\ (t,x)\to U$ in the problem (1), (2) has the form:

$$u^0(t,x)\in\text{Arg}\min_{u\in U}r(t,x,p,u),$$

where $r(t,x,p,u)=\langle p,f(t,x,u)\rangle+g(t,x,u)$ and p satisfies the condition

$$(-H(t,x,p),p)\in\partial^M_{h(t)}V(t,x),$$

where $h(t)=(1,D_pH(t,x,p))$.

To derive an optimal synthesis in the considered problem we apply the method suggested in ([Subbotina and Tokmantsev, 2009]). The method is based on a backward procedure of integrating the characteristic system for the Bellman equation.

We use the numerical procedure, see ([Subbotina and Tokmantsev, 2010]) to provide adaptive grids in the phase space and to define a grid feedback at nodes of this grids according to theorem 4. This feedback is called an optimal grid synthesis. The step-by-step motions $x^0(\cdot)$ generated by the optimal grid synthesis are considered to solve the problem (1), (2). It is proven, see ([Subbotina and Tokmantsev, 2009]) that the value of the optimal grid synthesis is close to the optimal result at all nodes of the mentioned grids.

5 Example 1

Let the dynamics of the system be given by

$$\begin{array}{l} \dot x_1=x_2,\ \dot x_2=u-\sin x_1,\\ |u|\le1,t\in[0,T]. \end{array}\tag{17}$$

We want to minimize the payoff functional of the form

$$\begin{array}{l} I_{t_0,x_0}\big(x(\cdot),u(\cdot)\big)=\\ |x_1(T)|+|x_2(T)|+\int\limits_{t_0}^{T}\frac{\varepsilon u^2(t)}{2}dt. \end{array}\tag{18}$$

Define the Hamiltonian

$$H(x,p)=p_1x_2-p_2\sin x_1-\frac{(p_2)^2}{2\varepsilon}.$$

Note, that the Hamiltonian and the dynamics satisfy the conditions $A1$–$A5$. We construct the characteristic system

$$\begin{cases} \frac{d\tilde x_1}{dt}=\tilde x_2, & \frac{d\tilde x_2}{dt}=-\sin\tilde x_1-\frac{\tilde p_2}{\varepsilon},\\ \frac{d\tilde p_1}{dt}=\tilde p_2\cos\tilde x_1, & \frac{d\tilde p_2}{dt}=-\tilde p_1, \qquad \frac{d\tilde z}{dt}=-\frac{(\tilde p_2)^2}{2\varepsilon}. \end{cases}$$

with boundary conditions

$$\tilde{x}_1(T,y) = y_1,\ \tilde{x}_2(T,y) = y_2,$$
$$\tilde{p}_1(T,y) = \begin{cases} -1, & y_1 < 0, \\ 1, & y_1 > 0, \\ [-1,1], & y_1 = 0, \end{cases}$$
$$\tilde{p}_2(T,y) = \begin{cases} -1, & y_2 < 0, \\ 1, & y_2 > 0, \\ [-1,1], & y_2 = 0, \end{cases}$$
$$\tilde{z}(T,y) = |y_1| + |y_2|.$$

The grid in the phase space of the final time $T = 5.0$ is considered within the domain $D = [-2,2] \times [-2,2]$, the step of partition Γ is $\Delta t = 0.05$, the parameter $\varepsilon = 10^{-7}$. The numerical results are in figures 1-6.

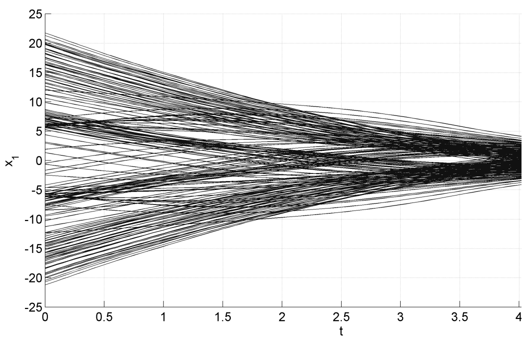

Figure 1. The optimal trajectories $x_1(t)$ of the problem (17)–(18).

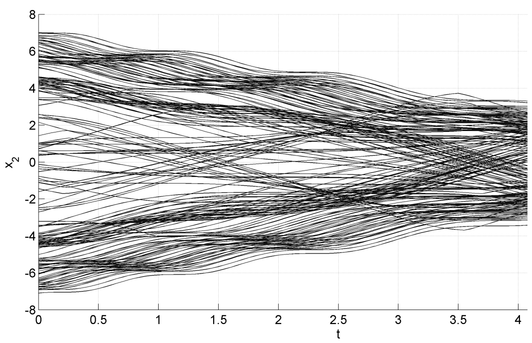

Figure 2. The optimal trajectories $x_2(t)$ of the problem (17)–(18).

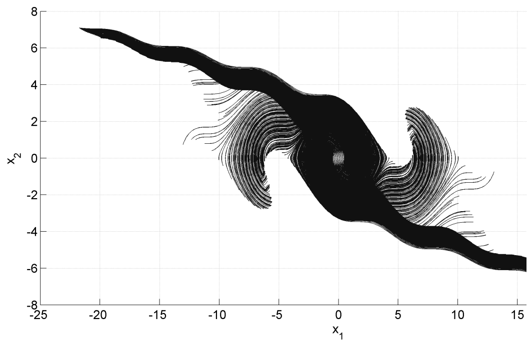

Figure 3. The optimal trajectories $(x_1(t), x_2(t))$, $t \in [0,T]$, of the problem (17)–(18).

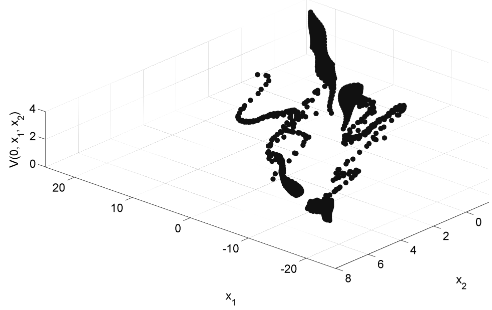

Figure 4. The graph of the value function $V(0, x_1, x_2)$ of the problem (17)–(18).

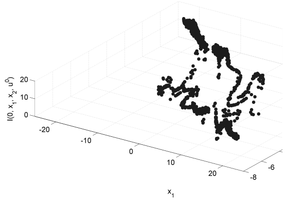

Figure 5. The graph of the functional $I(0, x_1, x_2; u^0(\cdot))$ (18).

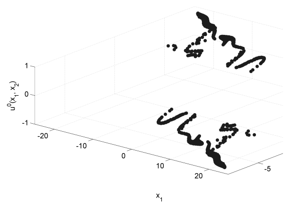

Figure 6. The graph of the optimal controls $u^0(0, x_1, x_2)$ of the problem (17)–(18) at instant $t = 0$.

6 Example 2

Let the dynamics of the system be given by

$$\begin{gathered} \dot{x}_1 = \cos x_3, \\ \dot{x}_2 = \sin x_3, \\ \dot{x}_3 = u, \\ |u| \leq 1, t \in [0,T]. \end{gathered} \tag{19}$$

We want to minimize the payoff functional

$$I_{t_0,x_0}\big(x(\cdot),u(\cdot)\big) = |x_1(T)| - |x_2(T)| + \int_{t_0}^{T} \frac{\varepsilon u^2(t)}{2} dt. \tag{20}$$

According to the algorithm consider the characteristic system

$$\begin{cases} \frac{d\tilde{x}_1}{dt} = \cos\tilde{x}_3,\ \frac{d\tilde{x}_2}{dt} = \sin\tilde{x}_3,\ \frac{d\tilde{x}_3}{dt} = -\frac{\tilde{p}_3}{\varepsilon}, \\ \frac{d\tilde{p}_1}{dt} = 0,\ \frac{d\tilde{p}_2}{dt} = 0,\ \frac{d\tilde{p}_3}{dt} = \tilde{p}_1\sin\tilde{x}_3 - \tilde{p}_2\cos\tilde{x}_3, \\ \frac{d\tilde{z}}{dt} = -\frac{(\tilde{p}_3)^2}{2\varepsilon}. \end{cases}$$

with the boundary conditions

$$\tilde{x}_1(T,y) = y_1,\ \tilde{x}_2(T,y) = y_2,$$

$$\tilde{p}_1(T,y) = \begin{cases} -1, & y_1 < 0, \\ 1, & y_1 > 0, \\ [-1,1], & y_1 = 0, \end{cases}$$

$$\tilde{p}_2(T,y) = \begin{cases} 1, & y_2 < 0, \\ -1, & y_2 > 0, \\ [-1,1], & y_2 = 0, \end{cases}$$

$$\tilde{z}(T,y) = |y_1| - |y_2|.$$

The grid in phase space in the instant $T = 7.0$ is considered within the domain $D = [-2,2]\times[-2,2]$, the step of partition Γ is $\Delta t = 0.1$, the parameter $\varepsilon = 10^{-4}$. The numerical results are in figures 7-9.

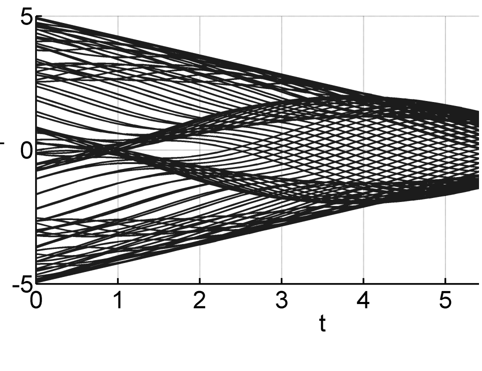

Figure 7. The optimal trajectories $x_1(t)$ of the problem (19)–(20).

Acknowledgements
This work was supported by the Russian Foundation for Basic Researches (projects 11-01-00214-a, 12-01-31300, 12-01-31394), by the Program of Government RF on the Sponsorship for Leading Scientific Schools NSh-64508.2010.1.

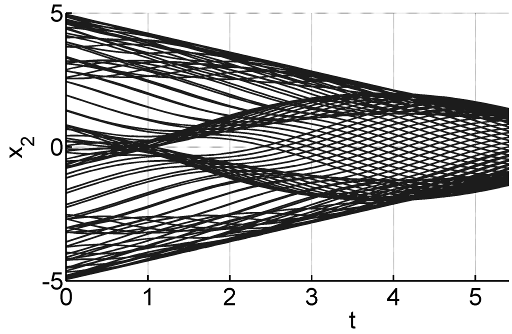

Figure 8. The optimal trajectories $x_2(t)$ of the problem (19)–(20).

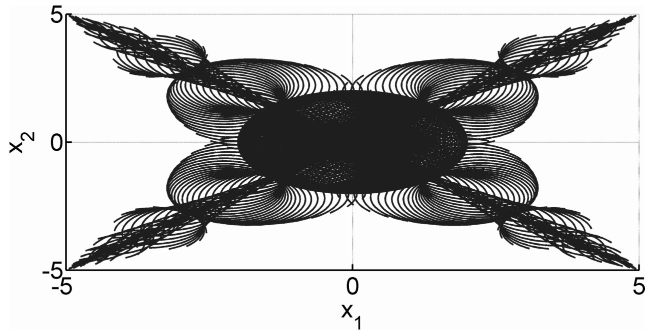

Figure 9. The optimal trajectories $(x_1(t), x_2(t))$, $t \in [0,T]$, of the problem (19)–(20).

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