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AVERAGE CONSENSUS IN SYMMETRIC NONLINEAR MULTI-AGENT NETWORKS WITH NON-HOMOGENEOUS DELAYS.

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Abstract
We study nonlinear continuous-time decentralized consensus algorithms for networks of single-order agents. The network topology is undirected and may switch, and the non homogenuous communication and measurement delays present. Using the absolute stability methods, we obtain effective condition for convergence of such consensus protocols, given the couplings to satisfy sectorial inequality and the network topology satisfies some connectivity properties.

Key words
Consensus, synchronization, networked systems, delayed systems

1 Introduction

Recent years the problems of decentralized cooperative control in networked multi-agent systems have attracted enormous attention. These problems deal with complex dynamical systems constituted by autonomous simpler units, or *agents*, that are able to make decisions independently. An agent may obtain information about the states of some other agents (referred as its *neighbors* in the network), and the set of neighbors is determined by the network topology. This topology may be unknown and time-varying. The goal of decentralized control is to make the population of independently controlled agents to demonstrate some collective behavior, e.g. move in parallel, follow the target, bypass a given area, etc. Below we address the consensus or synchronization problem (referred also as problem of agreement or averaging) which concerns the design of algorithms enabling the agents to get their states synchronized.

Mathematically rigorous study of the consensus dynamics takes its origin in applied statistics, probability theory and theory of positive matrices (DeGroot, 1974),(Seneta, 1981) on the one hand, and in the computer science on the other hand (see (J.N. Tsitsiklis and Athans, 1986)).

Nowadays the consensus dynamics is known to arise in numerous applications coming from the natural and social sciences, among them are the celebrated problem of Kuramoto coupled oscillators synchronization (Kuramoto, 1984),(Strogatz, 2000),(Yeung and Strogatz, 1999),(Earl and Strogatz, 2003)), the Vicsek problem of heading alignment in the flows of self-driven particles (Vicsek *et al.*, 1995), the Krause model of opinion dynamics in social networks (Krause, 2000), etc. One of the most important application of the consensus algorithms is the formation control and analysis of flocks, schools and swarms motion, pioneered by the famous model of Reynolds (Reynolds, 1987) who proposed the three empirical rules of flocking: to avoid collisions with nearby flockmates, attempt to match velocity with nearby flockmates, attempt to stay close to nearby flockmates. The second of Reynolds' requires the agreement between the agents' velocities. The history and deep results on convergence of the consensus algorithms, as well as further applications, can be found in (Jadbabaie *et al.*, 2003),(Olfati-Saber and Murray, 2004),(Blondel *et al.*, 2005),(Moreau, 2005),(Ren and Beard, 2005),(Olfati-Saber *et al.*, 2007) to mention a few.

Despite that the consensus protocols were seriously investigated, a number of questions still remain open even for the networks of simplest dynamical agents governed by the single integrator dynamics. One of such open problems is a convergence of consensus protocols in presence of communication and measurements delays. A consensus protocols with delays were investigated mainly for the case of fixed topology and linear couplings between the agents (see paragraph 2.2 below for detalized discussion and overview of results). Below we give sufficient conditions for convergence of the nonlinear consensus protocols and time-variant network topologies.

2 Problem statement.

Throughout the paper $\mathbb{G}_N$ stands for the set of all undirected graphs (possibly disconnected) with common set of vertices $V_N = \{1, 2, \ldots, N\}$, having no loops (arcs with coincident ends). For any $G \in \mathbb{G}_N$ and $j \in V_N$ let $N_j(G)$ stand for the set of all *neighbors* (adjacent vertices) the node j has in the graph G. By definition of undirected graph, $k \in N_j(G)$ implies that $j \in N_k(G)$.

Let $a_{ij}(G) = a_{ji}(G)$ ($1 \le i, j \le N$, $G \in \mathbb{G}_N$) be 1 if the nodes i, j if they are connected with an arc in G and 0 otherwise. The matrix $(a_{ij}(G))$ is called the *adjacency matrix* of the graph G.

2.1 Problem in question.

Consider a group of N independent *agents* governed by the first order equations

$$\dot{x}_j(t) = u_j(t) \in \mathbb{R}^d, \quad j = 1, 2, \ldots, N. \qquad (1)$$

Here $x_j, u_j \in \mathbb{R}^d$ stand respectively for the state vector and the control input of j-th agent.

We assume the pattern of communication links between the agents to be bidirectional and described by a graph-valued function $G(\cdot) : [0; +\infty] \to \mathbb{G}_N$. That is, the data transmission between the agent j, k is possible at time $t \ge 0$ if and only if $G(t)$ has an edge $\{i, j\}$ or, equivalently, $k \in N_j(G(t))$ (and thus $j \in N_k(G(t))$). The graph $G(t)$ is referred as communication or underlying graph (or topology) of the multi-agent network in question at time $t \ge 0$. Throughout the paper we assume the function $G(\cdot)$ to be the Lebesgue measurable, i.e. for any $\Gamma \in \mathbb{G}_N$ the set $G^{-1}(\Gamma) = \{t : G(t) = \Gamma\}$ is Lebesgue measurable.

Below we investigate distributed control policies or *protocols* as follows

$$u_j(t) = \sum_{k \in N_j(G(t))} \varphi_{jk}(t, z_{jk}(t - \tau_{jk}(t)), \qquad (2)$$

where by definition

$$z_{jk}(t) = x_k(t) - x_j(t), \quad 1 \le j, k \le N. \qquad (3)$$

Here $\{\varphi_{jk}(t, y)\}_{j \ne k}$ is a family of functions (with arguments $t \ge 0$, $y \in \mathbb{R}^d$) referred as *couplings* and describing the interaction strength between the agents. The delays $\tau_{jk} = \tau_{kj} \ge 0$ are assumed to be constant. Such control strategies are typical for decentralized coordination and synchronization problems without global reference frame in presence of communication and measurement delays. Each agent makes the decision based on the delayed measurements (made in its own reference frame) of the neighbors states.

To provide the unique solvability of the closed loop system (1), (2) one has to specify initial data:

$$x_j(t) = \alpha_j(t), t < 0, \quad \lim_{t \downarrow 0} x_j(t) = \alpha_j^0 \qquad (4)$$

We assume that $\alpha_j \in L_2([-\max_k \tau_{jk}; 0] \to \mathbb{R}^d)$ but do not suppose the solutions to be continuous at $t = 0$, so α_j^0 may be chose independently of α_j.

We say the protocol (2) to provide the *asymptotic consensus* if for any i, j, $1 \le i, j \le N$ and arbitrary initial data set $\{\alpha_j(\cdot)\}$, $\{\alpha_j^0\}$ one has

$$\lim_{t \to +\infty} |x_i(t) - x_j(t)| = 0. \qquad (5)$$

If additionally the states x_j have common limit

$$\lim_{t \to +\infty} x_j(t) = \frac{1}{N} \sum_{k=1}^{N} \alpha_j^0, \; \forall j = 1, 2, \ldots, N, \qquad (6)$$

the protocol (2) is said to provide *average consensus*.

Under the Assumption 1 below (symmetry of the protocol) the asymptotic consensus and average consensus conditions are equivalent since the sum $\sum_{j=1}^N x_j(t)$ remains constant.

The aim of the paper is to find easily verifiable conditions which guarantee the achievement of average consensus (6) for the wide class of control algorithms (2) with the couplings φ_{jk} being nonlinear and uncertain, but satisfy the following assumptions.

The first of those assumptions is symmetry of the delays and couplings.

Assumption 1. For any pair $1 \le k, j \le N$, $i \ne j$ and any $t \ge 0$, $y \in \mathbb{R}^d$ one has $\varphi_{jk}(t, y) = -\varphi_{kj}(t, -y)$ and $\tau_{jk} = \tau_{kj}$.□

Assumption 2. (Sector condition). A constant $\gamma > 0$ exists such that $\varphi_{ij}(t, x)^T x \ge \gamma^{-1} |\varphi_{ij}(t, x)|^2$ for any $i \ne j$.□

For the scalar case ($x_j(t) \in \mathbb{R}$) the sector condition means that the graph of the function $\varphi_{ij}(\cdot)$, i.e. the set $\{(x, y) : y = \varphi_{ij}(x)\}$ lies between the lines $y = 0$ and $y = \gamma x$. Sometimes nonlinearities satisfying the sector conditions above are called *passive*.

Assumption 3. If $x \in \mathbb{R}^d$ is separated from 0, the same is true for $\varphi_{ij}(t, x)$: for any $\varepsilon > 0$ one has

$$\eta_{ij}(\varepsilon) = \inf\{\varphi_{ij}(t, x) : t \ge 0, |x| \ge \varepsilon\} > 0. \qquad (7)$$

2.2 Discussion and known results.

The undelayed ($\tau_{jk} = 0$) protocols of the type (2) with linear and nonlinear couplings have been investigated in numerous papers, see e.g. (Olfati-Saber and Murray, 2003),(Olfati-Saber and Murray, 2004),(Moreau, 2005),(Chopra and Spong, 2006), (Olfati-Saber *et al.*, 2007),(Lin *et al.*, 2007), (Ajorlou *et al.*, 2010) just to mention a few. The same nonlinear dynamics arises in problems of coupled oscillators synchronization, including the celebrated Kuramoto models (Kuramoto, 1984), (Strogatz, 2000), (Chopra and Spong, 2009), (Jadbabaie *et al.*, 2004),(Mallada and Tang, 2010) and some problems of flocking (Olfati-Saber, 2006).

The convergence of the *delayed* protocol (2) was investigated mainly for the case of linear couplings. The most exhaustive result concerns the case of fixed topology $G(t) = G_0$ and linear time-invariant couplings: $\varphi_{ij}(t,y) = w_{ij}y$ with $w_{ij} = w_{ji} > 0$ for i,j being neighbors in G_0 and $w_{ij} = 0$ otherwise. The gain w_{ij} may be treated as a weight of an arc connecting i and j if such an arc exists. Let L_0 be the Laplacian matrix of the obtained weighted graph:

$$L_0 = \begin{bmatrix} \sum_{j=1}^N w_{1j} & -w_{12} & \dots & -w_{1N} \\ -w_{21} & \sum_{j=1}^N w_{2j} & \dots & -w_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -w_{N1} & -w_{N2} & \dots & \sum_{j=1}^N w_{Nj} \end{bmatrix} \geq 0,$$

and $\lambda_N = \lambda_{max}(L_0)$ be its maximal eigenvalue. The following brilliant result gives easily verifiable consensus criterion in terms of λ_N and maximal delay magnitude:

Theorem 1. *For the protocol (2) with $G(t) = G_0$ and $\varphi_{ij}(t,y) = w_{ij}y$ with w_{ij} specified above the following propositions are valid:*

1. *if $\tau_{ij} < \bar{\tau} = \frac{\pi}{2\lambda_{max}(L_0)}$ for any i,j then the protocol provides consensus with exponential rate of convergence;*
2. *the consensus is not achieved for $\tau_{ij} = \bar{\tau}\ \forall i,j$ so $\bar{\tau}$ can not be replaced with a greater number.*

The statement 2 and the statement 1 for the case of equal delays was obtained in (Olfati-Saber and Murray, 2004) and the general case of the statement 1 was proved in (Ferrari-Trecate, 2008) (Theorem 2).

The linear protocol (2) with switching topology (possibly directed but balanced and connected) was studied in (Qin *et al.*, 2009) where a sufficient condition for consensus expressed in terms of LMI solvability was obtained.

Below we investigate nonlinear protocols (2). Unlike the previous paper (Proskurnikov, 2010), the delays in communication links may be different and sector condition is weakened, as well as the connectivity assumptions.

A number of other measurement delay models considered recently should be mentioned, among which a dynamics with undelayed self state

$$u_i(t) = \sum_{j \in N_i(G(t))} \varphi_{ij}(x_j(t-\tau) - x_i(t)) \qquad (8)$$

seems to be most exhaustively examined (Earl and Strogatz, 2003),(Yeung and Strogatz, 1999),(Moreau, 2004),(Chopra and Spong, 2006), (Papachristodoulou *et al.*, 2010). Notice that unlike the protocol (2) studied in the present paper, (8) typically does not provide the average consensus. It also can not be implemented in the coordination problems without global reference frame. Investigation of more complicated delayed protocols with linear couplings (mainly for the case of fixed topology), including the models with multiple and distributed delays can be found in (Michiels *et al.*, 2009),(Münz, 2010), and in references therein.

3 Main results.

The aim of this section is to give a simply verifiable consensus conditions (expressed in terms of the topology $G(\cdot)$, sector bound $\gamma > 0$ and maximal delay magnitude) for the protocol (2) with the couplings satisfying Assumptions 1-3.

Let $\xi_{ij}(t) = a_{ij}(G(t))\varphi_{ij}(t, x_j(t-\tau_{ij}) - x_i(t-\tau_{ij}))$, where $a_{ij}(G)$ stands for the adjacency matrix of the graph $G \in \mathbb{G}_N$. In particular, for any j one has

$$\dot{x}_j(t) = u_j(t) = \sum_{k=1}^N \xi_{jk}(t) \qquad (9)$$

(by definition we have $\xi_{jj}(t) = 0$). Notice that all functions ξ_{ij} are Lebesgue measurable due to the measurability of the graph-valued function $G(\cdot)$.

Our main results are based on the following key lemma stating that for sufficiently small delays all of the functions ξ_{ij} are uniformly bounded in L_2-norm.

Lemma 1. *Suppose the couplings φ_{ij} and delays τ_{ij} to satisfy Assumptions 1,2 above and*

$$2\gamma(N-1)\tau_{ij} < 1. \qquad (10)$$

Then a constant $C > 0$ exists depending on the delays τ_{ij} and sector bound γ only such that

$$\sum_{j,k=1}^N \int_0^{+\infty} |\xi_{jk}(t)|^2 dt < C \sum_{j=1}^N \left(|\alpha_j^0|^2 + \int_{-\tau}^0 |\alpha_j(t)|^2 dt \right). \qquad (11)$$

Here $\alpha_j(\cdot)$,α_j^0 are initial data from (4), $\tau = \max_{i,j} \tau_{ij}$ and we take by defintion $\alpha_j(t) = 0$ for $t < -\max_k \tau_{jk}$.

The proof of Lemma 1 is based upon absolute stability theory techniques extending the celebrated Popov method (Popov, 1973),(Yakubovich, 2000) and can be found in the Appendix.

Notice that none of the Lemma 1 assumptions concerns properties of the network topology, in particularly, the topology may be disconnected. This is not surprising, since the inequality (11) evidently remains valid even for $G(t)$ with empty set of arcs. In the same time the control algorithm (2) can not provide consensus unless the underlying graph $G(\cdot)$ possess some connectivity properties.

Below we prove the convergence of consensus protocols under rather weak connectivity assumption as follows.

Let $E(t)$, $t \geq 0$ denotes for the set of edges (arcs) of the graph $G(t)$. For any unordered pair of vertices $e = \{i, j\}$ (with $1 \leq i, j \leq N$, $i \neq j$) let $M_e = \{t \geq 0 : e \in E(t)\} \subset \mathbb{R}$ be the set of time instants when the link e exists. For an interval $\Delta = [t_1; t_2] \subset [0; +\infty)$ consider the set $S_\varepsilon(\Delta)$ of all possible edges with duration of existense on Δ greater than $\varepsilon > 0$, i.e. $S_\varepsilon(\Delta) = \{e : mes(\Delta \cap M_e) > \varepsilon\}$. We refer the graph $(V_N, S_\varepsilon(\Delta)) \in \mathbb{G}_N$ as *ε-skeleton* of the topology $G(\cdot)$ on the interval Δ. At last we call the topology $G(\cdot)$ *jointly ε-connected* on Δ if its ε-skeleton on Δ is a connected graph.

Assumption 4. There exist $\varepsilon > 0$, $T > 0$ such that the topology $G(\cdot)$ is jointly ε-connected on the interval $[t; t + T]$ for any $t \geq 0$.

Remark. Often the switching topology is piecewise-constant with the *dwell time* (infimum of distances between consequent switchings) being positive. For this case Assumption 4 means that for some $T > 0$ all of the graphs $(V_N, \bigcup\limits_{t=t_0}^{t_0+T} E(t))$ are connected. The latter condition is a "UQSC (uniform strong quasi connectedness) property" proposed in (Lin *et al.*, 2007) (see also (Moreau, 2004)). Therefore Assumption 4 may be treated as a generalization of UQSC property for the case of arbitrary Lebesgue measurable underlying graph $G(\cdot)$. Notice that, as shown in (Lin *et al.*, 2007), Theorem 3.8 for the case of positive dwell-time and undelayed protocols, the UQSC property is almost necessary for achieving consensus (and becomes necessary if one requires the uniform convergence).

Now we present the main result of the paper.

Theorem 2. *Suppose that Assumptions 1-4 and (10) hold. Then the protocol (2) provides consensus (and thus average consensus).*

Proof. Suppose on the contrary that the consensus (5) is not achieved, so a number $\delta > 0$, indices i, j and a sequence $t_n \uparrow +\infty$ exist such that $|x_i(t_n) - x_j(t_n)| \geq \delta$. Bounding ourselves with an appropriate subsequence $\{t_{n_k}\}$, we may suppose that $t_{n+1} - t_n > T$, thus intervals $\Delta_n = [t_n - T/2; t_n + T/2]$ are disjoint (here T is the number from Assumption 4). Since ε-skeleton of $G(\cdot)$ on Δ_N is a connected graph, for some indices i_n, j_n one has: 1) the existense time of the arc arc $e_n = (i_n, j_n)$ on Δ_n is greater then ε; 2) $|x_{i_n}(t_n) - x_{j_n}(t_n)| > \delta' = \delta/(N-1)$. From Lemma 1 one may conclude that $u_j \in L_2$, thus $\int_{t_n}^{+\infty} |u_j(t)|^2 dt \to 0$ as $n \to +\infty$. This implies that for sufficiently large n the inequality $|x_{i_n}(t) - x_{j_n}(t)| > \delta'$ holds for any $t \in [t_n - T/2 - \tau; t_n + T/2]$, where $\tau = \max\limits_{j,k} \tau_{jk}$. Using Assumption 3, one obtains that $\varphi_{i_n j_n}(t, x_{i_n}(t - \tau_{i_n j_n}) - x_{j_n}(t - \tau_{i_n j_n})) \geq \eta_0 = \eta(\delta')$ whenever $t \in \delta_n$. Therefore for n sufficiently one has

$$\sum_{j,k=1}^{N} \int\limits_{\Delta_n} |\xi_{jk}(t)|^2 dt \geq \int\limits_{\Delta_n} |\xi_{i_n j_n}(t)|^2 dt \geq \varepsilon \eta_0,$$

which obviosly contradicts (11).■

4 Conclusion.

We investigate convergence of continuous-time distributed consensus protocols for networks of first-order agents with nonlinear uncertain couplings satisfying sector inequalities. The network topology may be switching and even lose its connectedness, and each communication link between the agents may be delayed. The delays are assumed to be time-invariant, but may be not coincident for different links. We obtain effective convergence criteria for such agreement protocols using absolute stability methods, in particular the extension of the celebrated Popov method which was proposed by V.A.Yakubovich.

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Appendix A Proof of Lemma 1.

The proof is divided on two stages: the first stage (summarized by Lemma 2) is to prove uniform bound (11) (with common constant $C > 0$) for the case of "stable" (in L_2 sense) solutions, and the second one is to prove that every solution is stable.

We start with the first part of the proof which is based on the important result of V.A.Yakubovich on quadratic functionals semiboundedness.

Let Z and Ξ be two complex Hilbert spaces of finite dimension. Consider a linear stabilizable system

$$\dot{z}(t) = Az(t) + B\xi(t), t \geq 0 \tag{12}$$

For any $a \in Z$ denote by $\mathfrak{M}_a \subset L_2([0; +\infty) \to Z \times \Xi)$ the set of all pairs $w(\cdot) = [z(\cdot), \xi(\cdot)]$ such that $|w(\cdot)| \in L_2[0; +\infty]$, (12) is satisfied and $z(0) = a$.

Consider a Hermitian functional $J_0(w) = \int_{-\infty}^{+\infty} \hat{w}(i\omega)^* P(i\omega) \hat{w}(i\omega) d\omega$, where $\hat{w}$ stands for the Fourier transform of w, $P(\cdot)$ bounded and analytic at any point $\omega \in \mathbb{R}$ operator-valued function such that $P(i\omega) = P(i\omega)^* : Z \times \Xi \to Z \times \Xi$. Let

$$J(w) = J_0(w) + 2\,Re \int\limits_{-\infty}^{+\infty} s(i\omega)^* \hat{w}(i\omega) d\omega \qquad (13)$$

where $s(\cdot) \in L_2(i\mathbb{R} \to Z)$. We also introduce an auxiliary operator-valued function $\Pi(i\omega)$ defined for $\omega \in \mathbb{R}$ such that $\det(i\omega I_n - A) \neq 0$:

$$\Pi(i\omega) = W(i\omega)^* P(i\omega) W(i\omega),$$
$$\text{where} \quad W(i\omega) = \begin{bmatrix} (i\omega I_n - A)^{-1} B \\ I_m \end{bmatrix}$$

The straightforward computation shows that for $w = [z, \xi] \in \mathfrak{M}_0$ one has $\hat{w}(i\omega) = W(i\omega)\hat{\xi}(i\omega)$ and thus $J_0(w) = \int\limits_{-\infty}^{+\infty} \hat{\xi}(i\omega)^* \Pi(i\omega) \hat{\xi}(i\omega) d\omega$.

Our goal is to find conditions which guarantee the quadratic function J to be bounded from above on the set $\mathfrak{M}_a$ for any a. It is evident that necessary condition is non-strict negative definiteness of J_0 on the correspondent linear space $\mathfrak{M}_0$ which is easily rewritten as $\Pi(i\omega) \leq 0$ for any $\omega \in \mathbb{R}$. This condition appears to be sufficient under certain additional assumption.

Theorem 3. $\Pi(i\omega) \leq 0$ *for any* ω *if and only if then* $J_0(w) \leq 0$ *for any* $w \in \mathfrak{M}_0$. *If* $\Pi(i\omega) \leq 0$ *and a matrix* $\Pi_\infty > 0$ *exists such that* $\Pi(i\omega) \leq -\Pi_\infty < 0$ *for sufficiently large* $|\omega|$, *then a constant* $C > 0$ *exists depending on* P, Q, R, A, B *only, such that*

$$\sup_{w \in \mathfrak{M}_a} J(w) \leq C(|a|^2 + \|s\|^2_{L_2}).$$

Proof. In the case of Hurwitz matrix A the Theorem 3 directly follows from (Arov and Yakubovich, 1981), Theorem 2, see analogous reasoning in the proof of (Likhtarnikov and Yakubovich, 1983), Theorem 2. The non-stable case reduces to the case of stable system by the variable change $\xi = \xi' + Kz$, the matrix $A + BK$ being Hurwitz.■

Consider the space Z of all matrices $z = (z_{jk})$, $1 \leq j, k \leq N$ and the space $\Xi \subset Z$ consisting of all skew-symmetric matrices. Taking $z_{jk} = x_k - x_j$, $z = (z_{jk})$, $\xi = (\xi_{jk})$, where x_j is a solution of (1), (2) and ξ_{jk} are the same as in Lemma 1, the system (9) is easily rewritten as (12) for appropriate A, B. Now we take an *integral quadratic constraint* into account which follows follows from Assumption 2. Consider a quadratic function $J(z(\cdot), \xi(\cdot))$ defined by

$$J = \int\limits_0^{+\infty} \sum_{j,k=1}^N \xi_{jk}(t)^* z_{jk}(t - \tau_{jk}) dt -$$
$$-(\gamma^{-1} - \varepsilon) \int\limits_0^{+\infty} \sum_{j,k=1}^N |\xi_{jk}(t)|^2 dt.$$

Here $\varepsilon > 0$ is a sufficiently small number to be detalized below. Assumption 2 implies that $\xi^*_{jk}(t) z_{jk}(t - \tau_{jk}) \geq \gamma |\xi_{jk}|^2$ and thus $J(z, \xi) \geq \varepsilon \sum_{j,k=1}^N \|\xi_{jk}\|^2_{L_2}$.

Using the Plancherel theorem, the functional J is easily seen to have the form (13) with $P(i\omega)$ bounded and analytic. A straightforward computation shows that the operator-valued function $\Pi(i\omega)$ is defined by

$$\hat{\xi}^* \Pi(i\omega) \hat{\xi} = 2 \sum_{j,k=1}^N Re \frac{\hat{\xi}^*_{jk} \hat{u}_j e^{-i\omega\tau_{jk}}}{i\omega} -$$
$$- \left(\frac{1}{\gamma} - \varepsilon\right) \sum_{j,k=1}^N |\hat{\xi}_{jk}|^2.$$

where $\hat{\xi} \in \Xi$, $\hat{u}_j = \sum_{k=1}^N \hat{\xi}_{jk}$. Since $e^{i\omega\tau_{jk}} = 1 + \beta_{jk}$ where $|\beta_{jk}| \leq |\omega|\tau$, $\tau = \max \tau_{jk}$, one has

$$\sum_{k=1}^N Re \frac{\hat{\xi}^*_{jk} \hat{u}_j e^{-i\omega\tau_{jk}}}{i\omega} \leq \tau |\hat{u}_j| \sqrt{(N-1) \sum_{k=1}^N |\hat{\xi}_{jk}|^2}$$

(the multiplier $(N - 1)$ appears here instead of N since $\hat{\xi}_{jj} = 0$). At the same time $|\hat{u}_j|^2 \leq (N - 1) \sum_{k=1}^N |\hat{\xi}_{jk}|^2$. Therefore

$$\hat{\xi}^* \Pi(i\omega) \hat{\xi} \leq \sum_{j,k=1}^N |\hat{\xi}_{jk}|^2 (2\tau(N-1) - \gamma^{-1} + \varepsilon)$$

Since $2\tau(N - 1)\gamma < 1$, for sufficiently small $\varepsilon > 0$ one has $\Pi(i\omega) \leq -\delta I$, where $\delta > 0$ is some small constant. Due to Theorem 3, one obtains the result as follows:

Lemma 2. *The conclusion of Lemma 1 is valid whenever the solution of (1), (2) satisfies* $\xi_{jk} \in L_2[0; +\infty]$, $x_k - x_j \in L_2[0; +\infty]$ *for any* j, k. *The constant* $C > 0$ *in (11) is independent on partial solution and is determined by the delays* τ_{ij} *and constant* γ *only.*

To accomplish the proof of Lemma 1, we need now a result proving absense of "unstable" in L_2-sense solutions. This will be done by the following standard trick from absolute stability theory, used for proving "minimal stability" conditions (Yakubovich, 2000),(Yakubovich, 2002). Consider arbitrary protocol (2) satisfying Assumptions 1,2 with the topology function $G(t)$. For some $T > 0$ and $\mu \in (0; \gamma)$ consider the new coupling functions

$$\tilde{\varphi}_{jk}(t, y) = \begin{cases} \varphi_{jk}(t, y), & t \leq T \\ \mu y, & t > T \end{cases}$$

and the new underlying graph function $\tilde{G}(t)$ which coincides with $G(t)$ for $t < T$ and is the complete graph for $t > T$. The protocol

$$u_j(t) = \sum_{k \in N_j(\tilde{G}(t))} \tilde{\varphi}_{jk}(t, x_k(t - \tau_{jk}(t) - x_j(t - \tau_{jk}(t)),$$

is known to provide consensus with exponential rate of convergence due to the inequality $\mu < \gamma < \frac{1}{2(N-1)} < \frac{\pi}{2N}$ (see Theorem 1). At the same time, the solution of the new closed-loop system coincides with the solutions of (1),(2) for $t < T$. Accordingly to Lemma 2, one has

$$\sum_{j,k=1}^{N} \int_0^T |\xi_{jk}(t)|^2 dt < C \sum_{j=1}^{N} \left(|\alpha_j^0|^2 + \int_{-\tau}^{0} |\alpha_j(t)|^2 dt \right).$$

with C independent of T and initail data. taking limit as $T \to +\infty$, one proves Lemma 1.