# THE STUDY OF NONLINEAR DIFFERENTIAL SYSTEMS

**Valentin Irtegov**
Matrosov Institute for System Dynamics
and Control Theory SB RAS
Russia
irteg@icc.ru

**Larisa Burlakova**
Matrosov Institute for System Dynamics
and Control Theory SB RAS
Russia
burlakova.larisa@gmail.com

**Abstract**
We present our approach to the study of dynamic systems having smooth first integrals. The approach is based on the Routh-Lyapunov method for the analysis of dynamic systems of the above type and computer algebra methods.

**Key words**
dynamic systems, invariant manifolds, first integrals

## 1 Introduction

The approach can be described as follows. We find stationary sets for the differential equations of a problem, i.e., the sets of any finite dimension, on which the necessary extremum conditions of first integrals in the problem under study are satisfied. Zero dimension sets having this property are called stationary solutions, and nonzero-dimension sets are called invariant manifolds (IMs). Further, we investigate the properties of such sets (stability in the sense of Lyapunov, bifurcations and etc.).

In this talk, our approach is demonstrated by the study of two problems: Euler's equations on Lie algebras and the problem of motion of a rigid body in two force fields. Similar problems arise, e.g., in space dynamics [Sarychev, Gutnik , 2015], quantum mechanics [Adler, Marikhin, Shabat, 2012], [Smirnov, 2008].

## 2 Euler's Equations on Lie Algebras

The differential equations of the problem can be written as [Borisov, Mamaev, and Sokolov, 2001]:

$$\left.\begin{array}{l}\dot{s}_1=\alpha(r_1s_2-\alpha r_2r_3)-(\beta r_3-s_2)(\beta r_2+s_3),\\ \dot{s}_2=\beta\,(\beta r_1r_3-r_2s_1)+(\alpha r_3-s_1)\,(\alpha r_1+s_3),\\ \dot{s}_3=(\beta r_1-\alpha r_2)s_3,\\ \dot{r}_1=r_2(\alpha r_1+\beta r_2+2s_3)-r_3s_2\\ \quad -x\left((\alpha^2+\beta^2)r_3s_2+\beta s_3^2\right),\\ \dot{r}_2=r_3s_1-r_1(\alpha r_1+\beta r_2+2s_3)\\ \quad +x\left((\alpha^2+\beta^2)r_3s_1+\alpha s_3^2\right),\\ \dot{r}_3=r_1s_2-r_2s_1+x(\beta s_1-\alpha s_2)s_3.\end{array}\right\}\quad (1)$$

Here $r_i, s_i$ are the phase variables, $\alpha, \beta, x$ are some constants.

Equations (1) can be interpreted as the Kirchhoff equations for the motion of a rigid body in ideal fluid for $x = 0$, as the Poincaré-Zhukowskii equations for a rigid body with an ellipsoidal cavity filled with a liquid for $x = 1$, and as the Euler equations on the Lie algebras $so(4)$ and $so(3,1)$ for $x > 0$ and $x < 0$, respectively.

Equations (1) have the following first integrals:

$$\left.\begin{array}{l}2H=s_1^2+s_2^2+2(\alpha r_1+\beta r_2)s_3+2s_3^2\\ -(\alpha^2+\beta^2)r_3^2=2h,\\ V_1=r_1s_1+r_2s_2+r_3s_3=c_1,\\ V_2=r_1^2+r_2^2+r_3^2+x(s_1^2+s_2^2+s_3^2)=c_2,\\ V_3=x(\beta s_1-\alpha s_2)^2s_3^2+(r_1s_1+r_2s_2)\big((\alpha^2\\ \;+\beta^2)(r_1s_1+r_2s_2)+2(\alpha s_1+\beta s_2)s_3\big)\\ +s_3^2\big(s_1^2+s_2^2+(\alpha r_1+\beta r_2+s_3)^2\big)=c_3.\end{array}\right\}\quad (2)$$

The problem is to find the stationary sets (both zero and non-zero dimension) for equations (1) and to investigate their stability.

### 2.1 Finding Stationary Sets

The method of obtaining the stationary sets for equations (1), which is used in this work, reduces this problem to solving a system of polynomial algebraic equations with parameters. In order to find the desired solutions, we construct the linear combination from first integrals (2)

$$2K = 2\lambda_0 H - 2\lambda_1 V_1 - \lambda_2 V_2 - \lambda_3 V_3, \quad (\lambda_i = \text{const}) \quad (3)$$

and write down the conditions of stationarity for the integral $K$ with respect to the phase variables $r_i, s_i$:

$$
\left.\begin{array}{l}
\quad\partial K/\partial s_1 = s_1\lambda_0 - r_1\lambda_1 - x s_1\lambda_2 - \\
\left(\left(\alpha^2+\beta^2\right) r_1\left(r_1 s_1 + r_2 s_2\right)\right. \\
+\left(\alpha r_2 s_2 + r_1\left(2\alpha s_1 + \beta s_2\right)\right) s_3 + \\
\left.\left(\left(1+x\beta^2\right) s_1 - x\alpha\beta s_2\right) s_3^2\right)\lambda_3 = 0, \\
\quad\partial K/\partial s_2 = s_2\lambda_0 - r_2\lambda_1 - x s_2\lambda_2 - \\
\left(\left(\alpha^2+\beta^2\right) r_2\left(r_1 s_1 + r_2 s_2\right)\right. \\
+\left(\beta r_1 s_1 + r_2\left(\alpha s_1 + 2\beta s_2\right)\right) s_3 + \\
\left.\left(s_2 + x\alpha\left(-\beta s_1 + \alpha s_2\right)\right) s_3^2\right)\lambda_3 = 0, \\
\partial K/\partial s_3 = \left(\alpha r_1 + \beta r_2 + 2 s_3\right)\lambda_0 - r_3\lambda_1 - \\
x s_3\lambda_2 - \left(\left(\alpha s_1 + \beta s_2\right)\left(r_1 s_1 + r_2 s_2\right) + \right. \\
\left(\left(\alpha r_1 + \beta r_2\right)^2 + \left(1+x\beta^2\right) s_1^2 - 2x\alpha\beta s_1 s_2\right. \\
\left. + \left(1+x\alpha^2\right) s_2^2\right) s_3 + \\
\left. 3\left(\alpha r_1 + \beta r_2\right) s_3^2 + 2 s_3^3\right)\lambda_3 = 0, \\
\quad\partial K/\partial r_1 = \alpha s_3\lambda_0 - s_1\lambda_1 - r_1\lambda_2 - \\
\left(\left(\alpha^2+\beta^2\right) s_1\left(r_1 s_1 + r_2 s_2\right) + s_1\left(\alpha s_1\right.\right. \\
\left.\left.+\beta s_2\right) s_3 + \alpha\left(\alpha r_1 + \beta r_2\right) s_3^2 + \alpha s_3^3\right)\lambda_3 = 0, \\
\quad\partial K/\partial r_2 = \beta s_3\lambda_0 - s_2\lambda_1 - r_2\lambda_2 - \\
\left(\left(\alpha^2+\beta^2\right) s_2\left(r_1 s_1 + r_2 s_2\right) + s_2\left(\alpha s_1\right.\right. \\
\left.\left.+\beta s_2\right) s_3 + \beta\left(\alpha r_1 + \beta r_2\right) s_3^2 + \beta s_3^3\right)\lambda_3 = 0, \\
\partial K/\partial r_3 = \left(\alpha^2+\beta^2\right) r_3\lambda_0 + s_3\lambda_1 + r_3\lambda_2 = 0
\end{array}\right\} \quad (4)
$$

These equations allow one to determine both the stationary solutions and the IMs for equations (1).

## 2.2 Solving Stationary Equations with respect to Phase Variables

For equations (4), we find both general solutions (existing without any restrictions on the parameters) and particular solutions (existing under some conditions on the parameters). For this purpose, we construct a Gröbner basis for system (4) with respect to the phase variables. After a factorization the basis has the form:

$$
\left.\begin{array}{l}
\left(a_1 s_1 + a_2 s_2 + a_3 s_3\right)\left(a_4 + a_5 s_1^2 + a_6 s_2^2\right. \\
\left. + a_7 s_1 s_3 + a_8 s_2 s_3 + a_9 s_3^2\right) = 0, \\
s_3\left(a_{10} s_1 + a_{11} s_3\right)\left(a_{12} + a_{13} s_1^2 + a_{14} s_2^2\right. \\
\left. + a_{15} s_1 s_3 + a_{16} s_2 s_3 + a_{17} s_3^2\right) = 0, \\
s_3\, f_1(s_1, s_2, s_3) = 0, \quad f_2(s_1, s_2, s_3) = 0, \\
f_3(s_1, s_2, s_3) = 0, \quad s_3\, f_4(s_1, s_2, s_3) = 0, \\
f_5(s_1, s_2, s_3) = 0, \quad s_3\, f_6(s_1, s_2, s_3) = 0, \\
a_{18} r_2 + f_7(s_1, s_2, s_3) = 0, \\
a_{19} r_1 + f_8(s_1, s_2, s_3) = 0, \\
a_{20} r_3 + a_{21} s_3 = 0.
\end{array}\right\} \quad (5)
$$

Here $f_i$ are the polynomials of the 4th-6th degrees, $a_j$ are the polynomials of $\lambda_0, \lambda_1, \lambda_2, \lambda_3, x, \alpha, \beta$.

System (5) is decomposed into several subsystems. We have computed a lexicographic Gröbner basis for each of the subsystems. Below, some of these bases are represented.

$$
\left.\begin{array}{l}
(\alpha^2+\beta^2)^2\lambda_3\chi\, r_2^4 - 2\alpha^2\chi\,\kappa\, r_2^2 \\
+\alpha^4\kappa\left(\lambda_1^2 + \lambda_2\kappa\right) = 0, \\
\alpha r_1 + \beta r_2 = 0, \ r_3 = 0, s_3 = 0, \\
\alpha^3\lambda_1\kappa\, s_1 - \alpha^2\beta\lambda_2\kappa\, r_2 + \beta\chi\, r_2^3 = 0, \\
\alpha^2\lambda_1\,\kappa s_2 + \alpha^2\lambda_2\kappa\, r_2 - \chi\, r_2^3 = 0, \\
\text{where } \kappa = \lambda_0 - x\lambda_2, \ \chi = (\alpha^2+\beta^2)^2\lambda_2\lambda_3.
\end{array}\right\} \quad (6)
$$

$$
\left.\begin{array}{l}
b_{30} s_3^4 s_1^4 + (b_{31} s_3^2 + b_{35}) s_3^3 s_1^3 + (b_{32} s_3^4 + b_{22} s_3^2 \\
+b_{10}) s_3^2 s_1^2 + (b_{33} s_3^6 + b_{21} s_3^4 + b_9 s_3^2 + b_{16}) s_3 s_1 \\
+b_{34} s_3^8 + b_{20} s_3^6 + b_5 s_3^4 + b_7 s_3^2 + b_{14} = 0, \\
(b_{28} s_3^2 + b_4) s_3 s_2 + b_{37} s_3^3 s_1^3 + (b_{38} s_3^2 + b_{41}) s_3^2 s_1^2 \\
+(b_{39} s_3^4 + b_{19} s_3^2 + b_{12}) s_3 s_1 + b_{40} s_3^6 \\
+b_{23} s_3^4 + b_1 s_3^2 + b_{17} = 0, \\
b_{45} s_3 r_2 + b_{42} s_3^2 s_1^2 + s_3(b_{43} s_3^2 + b_{46}) s_1 \\
+b_{44} s_3^4 + b_{29} s_3^2 + b_{13} = 0, \\
s_3(b_{27} s_3^2 + b_3) r_1 + b_{36} s_3^3 s_1^3 \\
+(b_{25} s_3^2 + b_{18}) s_3^2 s_1^2 + (b_{24} s_3^4 + b_2 s_3^2 \\
+b_{11}) s_3 s_1 + b_{26} s_3^6 + b_6 s_3^4 + b_8 s_3^2 + b_{15} = 0, \\
b_{47} r_3 + b_{48} s_3 = 0,
\end{array}\right\} \quad (7)
$$

where $b_i$ are polynomials of $\lambda_j, x, \alpha, \beta$.

We can obtain the information on dimension of the solutions and find the solutions directly from the above bases.

System (6) has the finite number of solutions: 4 general solutions. Below, some of them are represented.

$$
\left.\begin{array}{l}
r_1 = \pm\frac{\beta\sqrt{(\rho+\lambda_1\sqrt{d})/\lambda_3}}{\alpha^2+\beta^2}, \\
r_2 = \mp\frac{\alpha\sqrt{(\rho+\lambda_1\sqrt{d})/\lambda_3}}{\alpha^2+\beta^2}, \ r_3 = 0, \\
s_1 = \mp\frac{\beta\sqrt{(\rho+\lambda_1\sqrt{d})/(d\,\lambda_3)}}{(\alpha^2+\beta^2)}, \\
s_2 = \pm\frac{\alpha\sqrt{(\rho+\lambda_1\sqrt{d})/(d\,\lambda_3)}}{(\alpha^2+\beta^2)}, \ s_3 = 0; \\
\text{where } \rho = \lambda_0 - x\lambda_2, \ d = -\rho/\lambda_2.
\end{array}\right\} \quad (8)
$$

These are the families of stationary solutions of equations (1) parametrized by $\lambda_i$.

System (7) has infinitely many solutions. For finding the general solutions of equations (7), it is necessary to solve an equation of 4th degree. In this case, the solutions will be bulky.

Here, we restrict ourselves the particular solutions of this system, which were obtained for $\lambda_1 = 0$. Below, some of these solutions are represented.

$$
\left.\begin{array}{l}
r_1 = (\beta\sigma + \alpha\varrho)\varrho/((\alpha^2+\beta^2)\lambda_3 s_3), \\
r_2 = -(\alpha\sigma - \beta\varrho)\varrho/((\alpha^2+\beta^2)\lambda_3 s_3), \\
r_3 = 0, \\
s_1 = \pm(\alpha\sigma - \beta\varrho)\sqrt{\lambda_2}/((\alpha^2+\beta^2)\lambda_3 s_3), \\
s_2 = \pm(\beta\sigma + \alpha\varrho)\sqrt{\lambda_2}/((\alpha^2+\beta^2)\lambda_3 s_3),
\end{array}\right\} \quad (9)
$$

where $\varrho = \sqrt{\lambda_0 - \lambda_2 x - \lambda_3 s_3^2}$, $\sigma = \sqrt{(\alpha^2+\beta^2)\, x\lambda_3 s_3^2 - \varrho^2}$. Solutions (9) are the families of one-dimensional IMs of equations (1).

## 2.3 Solving Stationary Equations with respect to Some Part of Phase Variables and Parameters

Let us consider another technique for finding the solutions of equations (4). Using this technique, it is possible to obtain IMs together with the first integrals of differential equations on these IMs. The latter allows us to set the problem for finding and analysing the stationary sets of these differential equations. Following this technique, we have computed a lexicographical

Gröbner basis for equations (4) with respect to the variables $r_3, s_2, \lambda_1, \lambda_2, \lambda_0$. As a result, we have obtained a basis which is decomposed into two subsystems:

$$\left.\begin{array}{l}\lambda_0-\lambda_3\,(\alpha r_1+\beta r_2+s_3)\,s_3=0,\ \lambda_2=0,\\ \alpha\lambda_1+(\alpha^2+\beta^2)\lambda_3\,(\alpha r_1 s_1+\beta r_2 s_1\\ +s_1 s_3)=0,\\ \alpha s_2-\beta s_1=0,\ \alpha r_3-s_1=0.\end{array}\right\}\quad(10)$$

$$\left.\begin{array}{l}b_{12}\lambda_0^2+b_2\lambda_0+b_1=0,\\ b_{15}\lambda_2+b_7\lambda_0+b_4=0,\\ b_{11}\lambda_1+b_6\lambda_0+b_8=0,\\ b_{10}s_2+b_{14}\lambda_0+b_5=0,\\ b_9 r_3+b_{13}\lambda_0+b_3=0,\end{array}\right\}\quad(11)$$

where $b_i$ are polynomials of $s_1,\ s_3,\ r_1,\ r_2,\ \lambda_3, x, \alpha, \beta$.

It is easy to see that system (10) has one solution, and system (11) has two solutions.

The latter two expressions of (10) determine the invariant manifold (IM) of equations (1).

The differential equations of vector field on this IM are given by:

$$\left.\begin{array}{l}\dot s_1=(\beta r_1-\alpha r_2)s_1,\ \dot s_3=(\beta r_1-\alpha r_2)s_3,\\ \dot r_1=\alpha r_1 r_2+2r_2 s_3-\beta s_1^2(\beta^2 x+1)/\alpha^2\\ +\beta(r_2^2-(s_1^2+s_3^2)x),\\ \dot r_2=-\alpha r_1^2-r_1(\beta r_2+2s_3)+\alpha(s_1^2+s_3^2)x\\ +s_1^2(\beta^2 x+1)/\alpha.\end{array}\right\}\quad(12)$$

The first three expressions of (10) are the first integrals of equations (12).

The general solutions of system (11) are bulky, here we represent the particular solutions obtained when $s_3=0$:

$$\begin{array}{l}r_3=\dfrac{\sigma\, s_1}{r_2(\alpha r_2-\beta r_1)},\ s_2=-\dfrac{r_1 s_1}{r_2},\ s_3=0,\\ \lambda_0=0,\ \lambda_1=0,\ \lambda_2=0;\\ r_3=0,\ s_2=\dfrac{r_2 s_1}{r_1},\ s_3=0,\ \lambda_0=\dfrac{\lambda_3\sigma\, s_1^2}{r_1^2},\\ \lambda_1=\dfrac{\lambda_3\sigma s_1(s_1^2-(\alpha^2+\beta^2)(r_1^2-s_1^2 x))}{r_1(r_1^2-s_1^2 x)},\\ \lambda_2=-\dfrac{\lambda_3\sigma\, s_1^4}{r_1^4-r_1^2 s_1^2 x},\ (\text{where } \sigma=r_1^2+r_2^2).\end{array}$$

The first three expressions of each of the above solutions define the IMs of equations (1), and the latter three expressions of each of the solutions are the first integrals of differential equations on these IMs.

### 2.4 Parametric Analysis of Stationary Sets

Using Gröbner bases technique, we have found a series of the solutions of equations (4) under some conditions imposed on the parameters.

For the case $\lambda_0 = x\,\lambda_2,\ \lambda_1=0,\ x=-1/(\alpha^2+\beta^2)$, the solution

$$r_1=-\frac{\alpha s_3}{\alpha^2+\beta^2},\ r_2=-\frac{\beta s_3}{\alpha^2+\beta^2},\ r_3=0\quad(13)$$

has been obtained. It represents the 3-dimensional IM of equations (1).

For the case $\lambda_0=0,\ \lambda_1=0,\ \lambda_2=0$, the solution

$$\beta r_2+\alpha r_1+s_3=0,\ \beta s_1-\alpha s_2=0$$

has been found. It represents the 4-dimensional IM of equations (1).

It is easily verified that the above solutions pass through the zero solution. The elements of the families of one-dimensional IMs (9) also pass through the zero solution when $\lambda_0=\lambda_1=\lambda_2=0$. So, the zero solution is a bifurcation point.

#### 2.4.1 Stability of Stationary Sets.

Let us investigate the stability of both the zero solution and the IM which passes through this solution by the Routh-Lyapunov method [Lyapunov, 1954]. In simple cases, the problem is reduced to verifying the sign-definiteness conditions for the 2nd variation of integral $K$ (3) obtained in the neighbourhood of the solution under study.

The 2nd variation of the integral $K$ in the neighbourhood of the zero solution can be written as:

$$\left.\begin{array}{l}2\,\delta^2 K=-\lambda_2 y_1^2-\lambda_2 y_2^2-((\alpha^2+\beta^2)\lambda_0\\ +\lambda_2)y_3^2-2\lambda_1 y_1 y_4+(\lambda_0-\lambda_2 x)y_4^2\\ -2\lambda_1 y_2 y_5+(\lambda_0-\lambda_2 x)y_5^2+2\alpha\lambda_0 y_1 y_6\\ +2\beta\lambda_0 y_2 y_6-2\lambda_1 y_3 y_6+(2\lambda_0-\lambda_2 x)y_6^2.\end{array}\right\}\quad(14)$$

Here $y_i$ are the deviations of the perturbed solution from the unperturbed one.

Using Sylvester's criterion, we can write down the conditions for the positive definiteness of the quadratic form $\delta^2 K$ as

$$\left.\begin{array}{l}\lambda_2<0,\ D_1<0,\ (\alpha^2+\beta^2)\lambda_0^2(\lambda_0-x\lambda_2)\\ +D_1(2\lambda_0-x\lambda_2)<0,\\ (D_1+D_2\lambda_0)(\lambda_1^2+D_2(\lambda_0-x\lambda_2))>0,\end{array}\right\}\quad(15)$$

where $D_1=\lambda_1^2+\lambda_2(\lambda_0-x\lambda_2),\ D_2=(\alpha^2+\beta^2)\lambda_0+\lambda_2$.

Inequalities (15) are compatible under the following constraints imposed on the parameters $\lambda_i, \alpha, \beta, x$:

$$\left.\begin{array}{l}\alpha\neq 0 \text{ and } \beta\neq 0 \text{ and } \lambda_2<0 \text{ and } \Big(\big(\lambda_0>0\\ \text{and}\lambda_0+\frac{\lambda_2}{\alpha^2+\beta^2}<0 \text{ and } x>\frac{\lambda_1^2+D_2\lambda_2}{D_2\lambda_2}\big)\\ \text{or } \big(\lambda_0\leq 0 \text{ and } x>\frac{\lambda_1^2+(\lambda_2+D_2)\lambda_0}{\lambda_2^2}\big)\Big).\end{array}\right\}\quad(16)$$

Conditions (16) are sufficient for the stability of the zero solution.

Further, let us investigate the stability of IM (13).

The variation of the integral $\tilde{K} = 2\lambda_0 H - \lambda_2 V_2 - \lambda_3 V_3$ in the neighbourhood of this IM is

$$2\Delta\tilde{K} = -\lambda_2 y_2^2 - \lambda_2 y_3^2 - \lambda_3(\alpha y_2 + \beta y_3)^2 s_3^2 \\ -(\alpha^2+\beta^2)\lambda_3(s_1 y_2 + s_2 y_3)^2.$$

Here $y_1 = r_1 + \alpha s_3/(\alpha^2+\beta^2)$, $y_2 = r_2 + \beta s_3/(\alpha^2+\beta^2)$, $y_3 = r_3$ are the deviations of the perturbed solution from the unperturbed IM.

Next, we introduce the following variables $z_1 = (\alpha y_2 + \beta y_3)s_3$, $z_2 = s_1 y_2 + s_2 y_3$. The $\Delta\tilde{K}$ in the variables $y_2, y_3, z_1, z_2$ has the form: $2\Delta\tilde{K} = -\lambda_2\big(y_2^2 + y_3^2\big) - \lambda_3\big(z_1^2 + (\alpha^2+\beta^2)z_2^2\big)$.

The latter quadratic form is sign definite with respect to the variables $y_2, y_3, z_1, z_2$ when the following conditions $\alpha^2+\beta^2 \neq 0$ and $\lambda_2 > 0, \lambda_3 > 0$ (or $\lambda_2 < 0, \lambda_3 < 0$) hold. Hence, these conditions are sufficient for the stability of IM (13) with respect to the variables $y_2, y_3$.

## 3 A Rigid Body under the Influence of Two Force Fields

The rotation of a rigid body around a fixed point in uniform gravitational and magnetic force fields is considered. The distribution of mass in the body corresponds to the Kowalewski integrable case.

The equations of motion of the body in a coordinate system which is rigidly attached to the body and its center is at the fixed point can be written as:

$$\left.\begin{array}{l} 2\dot{p} = b\delta_3 + q\,r, \quad \dot{\gamma}_1 = \gamma_2 r - \gamma_3 q, \\ 2\dot{q} = x_0\gamma_3 - p\,r, \quad \dot{\gamma}_2 = \gamma_3 p - \gamma_1 r, \\ \dot{r} = -b\delta_1 - x_0\gamma_2, \; \dot{\gamma}_3 = \gamma_1 q - \gamma_2 p, \\ \dot{\delta}_1 = \delta_2 r - \delta_3 q, \; \dot{\delta}_2 = \delta_3 p - \delta_1 r, \\ \dot{\delta}_3 = \delta_1 q - \delta_2 p. \end{array}\right\} \quad (17)$$

Here $p, q, r$ are the projections of angular velocity vector onto the axes related to the body, $\gamma_1, \gamma_2, \gamma_3$ are the direction cosines of upward vertical, $\delta_1, \delta_2, \delta_3$ are the direction cosines of the vector of constant magnetic moment, parameters $x_0, b$ are proportional to the coordinate of the mass center of the body and the coordinate of the vector of constant magnetic moment, respectively.

The equations admit the following first integrals:

$$\left.\begin{array}{l} 2H = 2(p^2+q^2) + r^2 + 2(x_0\gamma_1 - b\,\delta_2) = 2h, \\ V_1 = (p^2 - q^2 - x_0\gamma_1 - b\,\delta_2)^2 + (2p\,q - x_0\gamma_2 \\ +b\,\delta_1)^2 = c_1, \; V_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \\ V_3 = \delta_1^2 + \delta_2^2 + \delta_3^2 = 1, \\ V_4 = \gamma_1\delta_1 + \gamma_2\delta_2 + \gamma_3\delta_3 = c_2. \end{array}\right\} \quad (18)$$

When $b = 0$, the system under consideration corresponds to the Kowalewski integrable case.

On the invariant manifold of codimension 2

$$p^2 - q^2 - x_0\gamma_1 - b\,\delta_2 = 0, \; 2pq - x_0\gamma_2 + b\,\delta_1 = 0 \quad (19)$$

system (17) has an additional cubic integral [Bogoyavlenskii, 1984] and is completely Liouville integrable. Further, we study the above differential equations written on IM (19):

$$\left.\begin{array}{ll} 2\dot{p} = q\,r + b\,\delta_3, & \dot{\delta}_1 = r\,\delta_2 - q\,\delta_3, \\ 2\dot{q} = x_0\gamma_3 - p\,r, & \dot{\delta}_2 = \delta_3 p - \delta_1 r, \\ \dot{r} = -2(p\,q + b\,\delta_1), & \dot{\delta}_3 = \delta_1 q - \delta_2 p, \\ \multicolumn{2}{l}{x_0\dot{\gamma}_3 = -((p^2+q^2)\,q + b\,(p\,\delta_1 + q\,\delta_2)).} \end{array}\right\} \quad (20)$$

The first integrals of equations (20) are given by

$$\left.\begin{array}{l} 2\tilde{H} = 4p^2 + r^2 - 4b\,\delta_2 = 2\tilde{h}, \\ \tilde{V}_2 = \gamma_3^2 + \frac{(2p\,q + b\,\delta_1)^2}{x_0^2} + \frac{(q^2 - p^2 + b\delta_2)^2}{x_0^2} = 1, \\ V_3 = \delta_1^2 + \delta_2^2 + \delta_3^2 = 1, \\ \tilde{V}_4 = \frac{2p\,q\,\delta_2 + (p^2 - q^2)\,\delta_1}{x_0} + \gamma_3\,\delta_3 = \tilde{c}_2, \\ 2V_5 = (p^2+q^2)\,r - 2x_0 p\,\gamma_3 + 2bq\,\delta_3 = m. \end{array}\right\} \quad (21)$$

Within the framework of the study of the phase space of system (20), we state the problem to find IMs of this system for their simplest classification and to investigate their stability.

### 3.1 Finding Invariant Manifolds

Likewise as above, we construct the linear combination from first integrals (21)

$$2K = \lambda_0\tilde{H} - \lambda_1\tilde{V}_2 - \lambda_2 V_3 - 2\lambda_3\tilde{V}_4 - \lambda_4 V_5, \quad (22)$$

and write down the necessary conditions for the integral $K$ to have an extremum with respect to the phase variables $p, q, r, \gamma_3, \delta_1, \delta_2, \delta_3$:

$$\left.\begin{array}{l} \partial K/\partial p = 4\lambda_0 p - \frac{2\lambda_1[(p^2+q^2)\,p + b(q\,\delta_1 - p\,\delta_2)]}{x_0^2} \\ \quad - \frac{2\lambda_3(p\,\delta_1 + q\,\delta_2)}{x_0} + \lambda_4\,(x_0\gamma_3 - p\,r) = 0, \\ \partial K/\partial q = -\frac{2\lambda_1\,[(p^2+q^2)\,q + b\,(p\,\delta_1 + q\,\delta_2)]}{x_0^2} \\ \quad + \frac{2\lambda_3\,(q\,\delta_1 - p\,\delta_2)}{x_0} - \lambda_4\,(q\,r + b\,\delta_3) = 0, \\ \partial K/\partial r = 2\lambda_0\,r - \lambda_4\,(p^2+q^2) = 0, \\ \partial K/\partial\gamma_3 = -\lambda_1\gamma_3 - \lambda_3\,\delta_3 + \lambda_4 x_0 p = 0, \\ \partial K/\partial\delta_1 = -\frac{\lambda_1 b\,(2p\,q + b\,\delta_1)}{x_0^2} - \lambda_2\,\delta_1 \\ \quad - \frac{\lambda_3\,(p^2 - q^2)}{x_0} = 0, \\ \partial K/\partial\delta_2 = -2b\,\lambda_0 - \lambda_2\,\delta_2 - \frac{\lambda_1 b\,(q^2 - p^2 + b\,\delta_2)}{x_0^2} \\ \quad - \frac{2\lambda_3 p\,q}{x_0} = 0, \\ \partial K/\partial\delta_3 = -\lambda_2\,\delta_3 - \lambda_3\,\gamma_3 - \lambda_4\,bq = 0. \end{array}\right\} \quad (23)$$

We shall find the solutions of stationary equations (23) with two procedures. The 1st procedure is based on solving these equations with respect to some part of the phase variables and the family parameters of the integral $K$. This technique was already used in the given work. The 2nd procedure finds new IMs by eliminating the family parameters from the known solutions of the stationary equations. Both techniques provide a possibility to reveal embedded in one another IMs.

### 3.2 Applying First Procedure

We find the IMs of various dimension for equations (20). Since first integrals correspond to IMs of codimension 1, let us begin with IMs of codimension 2. To this end, we take, e.g., $\delta_1, \delta_2, \lambda_1, \lambda_2, \lambda_0, \lambda_4$ as unknowns, and construct a Gröbner basis with respect to the lexicographic ordering $\delta_1 > \delta_2 > \lambda_1 > \lambda_2 > \lambda_0 > \lambda_4$ for the polynomials of system (23). As a result, we have the following system:

$$\begin{aligned}
&\lambda_4\, g_1(p,q,r,\gamma_3,\lambda_3,\lambda_4)=0, g_2(p,q,r,\lambda_0,\lambda_4)=0,\\
&g_3(q,\gamma_3,\delta_3,\lambda_2,\lambda_3,\lambda_4)=0,\\
&g_4(p,\gamma_3,\delta_3,\lambda_1,\lambda_3,\lambda_4)=0,\\
&g_5(p,q,r,\gamma_3,\delta_2,\delta_3,\lambda_3,\lambda_4)=0,\\
&g_6(p,q,r,\gamma_3,\delta_1,\delta_3,\lambda_3,\lambda_4)=0,
\end{aligned}$$

where $g_j (j=1,\dots,6)$ are the polynomials of the basis. The resulting system is bulky, therefore it is not represented explicitly here.

The system can be decomposed into two subsystems represented below.

Subsystem 1:

$$\left.\begin{aligned}
&\lambda_4 b\, x_0(\varrho-2(p^2+q^2)pq)-\lambda_3(x_0\gamma_3(2p(p^2\\
&+q^2)+x_0\gamma_3 r)+b\,(b\,\delta_3 r\\
&-2q\,(p^2+q^2))\,\delta_3)=0,\\
&2\lambda_0\, b\, x_0\,(\varrho-2(p^2+q^2)\,p\,q])\,r-\lambda_3\,(p^2\\
&\quad+q^2)\,(x_0\,\gamma_3\,(2p\,(p^2+q^2)+x_0\gamma_3\,r)\\
&\quad+b\,(b\,\delta_3\,r-2q\,(p^2+q^2))\,\delta_3)=0,\\
&\lambda_2\,x_0\,(2(p^2+q^2)\,p\,q-\varrho)+\lambda_3\,b\,(2(p^2\\
&\quad+q^2)q^2-\varrho_2)=0,\\
&\lambda_1\,b\,(2(p^2+q^2)\,p\,q-\varrho)+\lambda_3\,x_0\,(2(p^2\\
&\quad+q^2)p^2+\varrho_2)=0,
\end{aligned}\right\}\quad(24)$$

$$\left.\begin{aligned}
&2b\,(p^2+q^2)\,r\,\delta_2+b\,(b\,r\,\delta_3 r+q(r^2\\
&\quad-2(p^2+q^2)))\,\delta_3-(p\,r-x_0\,\gamma_3)\\
&\quad\times(2p\,(p^2+q^2)+x_0\,\gamma_3\,r)=0,\\
&-2b\,(p^2+q^2)\,\delta_1-p\,[2q\,(p^2+q^2)+b\,\delta_3\,r]\\
&\quad-x_0\gamma_3\,q\,r=0.
\end{aligned}\right\}\quad(25)$$

Subsystem 2:

$$\left.\begin{aligned}
&\lambda_4=0,\ \lambda_0=0,\ -(\lambda_2\delta_3+\lambda_3\gamma_3)=0,\\
&\quad-(\lambda_1\gamma_3+\lambda_3\delta_3)=0,
\end{aligned}\right\}\quad(26)$$

$$\left.\begin{aligned}
&(x_0^2\gamma_3^2+b^2\,\delta_3^2)\,\delta_2-(2x_0\gamma_3 p\,q\\
&\quad+b\,(p^2-q^2)\,\delta_3)\,\delta_3=0,\\
&(x_0^2\gamma_3^2+b^2\,\delta_3^2)\,\delta_1+(2b\,\delta_3 p\,q-x_0\,\gamma_3(p^2\\
&\quad-q^2))\,\delta_3=0.
\end{aligned}\right\}\quad(27)$$

Here $\varrho=(b\,\delta_3 p-x_0\gamma_3\,q)\,r$, $\varrho_2=(b\,\delta_3 q+x_0\gamma_3 p)\,r$.

Let us analyze the equations of subsystem 1.

It can be easy verified by IM definition that equations (25) define the IM of codimension 2 for differential equations (20).

The equations of vector field on IM (25) are given by:

$$\left.\begin{aligned}
&2\dot p=q\,r+b\,\delta_3,\ 2\dot q=x_0\gamma_3-p\,r,\ \dot r=\tfrac{\varrho_2}{p^2+q^2},\\
&\dot\gamma_3=\tfrac{b\,[bq\,r\,\delta_3-(p^2+q^2)(2q^2-r^2)]\,\delta_3}{2x_0\,(p^2+q^2)\,r}+\tfrac{p\,\gamma_3\,q}{r}\\
&+\tfrac{(x_0^2\gamma_3^2-2(p^2+q^2)^2)\,q}{2x_0\,(p^2+q^2)},\\
&\dot\delta_3=\tfrac{[b\,r\,\delta_3-2(p^2+q^2)\,q]\,p\,\delta_3}{2\,(p^2+q^2)\,r}-\tfrac{1}{2b}\\
&+\tfrac{x_0\gamma_3\,p\,(2p\,(p^2+q^2)+x_0\gamma_3\,r)}{2b\,(p^2+q^2)\,r}.
\end{aligned}\right\}\quad(28)$$

From (24), we find the values for $\lambda_0, \lambda_1, \lambda_2, \lambda_4$ which are the first integrals of equations (28).

In a similar manner, we have established that equations (27) also define the IM of codimension 2 for differential equations (20), and the values of $\lambda_1, \lambda_2$ found from the two latter expressions of (26) are the first integrals for the equations of vector field on this IM. Obviously, these integrals are dependent. We have also found the families of IMs of codimension 3, 4 and 5.

Let us consider the latter. In order to obtain this family, we take $\delta_1, \delta_2, \delta_3, \gamma_3, r, \lambda_0$ as unknowns, and construct a Gröbner basis with respect to the lexicographic ordering $\delta_1 > \delta_2 > \delta_3 > \gamma_3 > r > \lambda_0$ for the polynomials of system (23). A result will be the following system:

$$\lambda_0(4\lambda_1\lambda_2-4\lambda_3^2)+\lambda_4^2\alpha_1=0,\quad(29)$$

$$\left.\begin{aligned}
&\lambda_4\,\alpha_1\,r-2\alpha_2\,(p^2+q^2)=0,\\
&-\alpha_2\,\gamma_3-\lambda_4\,(\lambda_3\,b\,q+\lambda_2 x_0 p)=0,\\
&\alpha_2\,\delta_3-\lambda_4\,(\lambda_1\,b\,q+\lambda_3 x_0\,p)=0,\\
&2\alpha_1\,\alpha_2\,\delta_2-2\lambda_1\,\alpha_2\,b\,(p^2-q^2)\\
&+4\lambda_3\,\alpha_2\,x_0 p\,q+\lambda_4^2\,\alpha_1\,b\,x_0^2=0,\\
&-\alpha_1\,\delta_1-2\lambda_1 b\,p\,q-\lambda_3 x_0\,(p^2-q^2)=0,
\end{aligned}\right\}\quad(30)$$

where $\alpha_1=\lambda_1 b^2+\lambda_2 x_0^2$, $\alpha_2=\lambda_3^2-\lambda_1\lambda_2$.

Equations (30) define the family of IMs of codimension 5 for differential equations (20). The parameters of the family are $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. This family possesses an extremal property: the integral $K$ (22) takes a stationary value on the elements of the family when $\lambda_0=-\lambda_4^2\alpha_1/(4\alpha_2)$ (this value is found from equation (29)).

Obviously the solutions found by the described technique will be related. Indeed, on substituting expressions (30) (resolved with respect to $\delta_1, \delta_2, \delta_3, \gamma_3, r$) into equations (27), the latter equations become identities. Hence, we can conclude that the elements of IMs family (30) are submanifolds of IM (25).

Thus, the procedure presented above allows one to find the embedded in one another IMs families. In the case considered, the latter is caused by the technique applied. In general case, this technique enables us to classify IMs on the basis of their embedding and degree of their degeneration.

The IMs families found for the differential equations written on IM (19) can be "lifted up" as invariant into the phase space of system (17). To this end, it is sufficient to add the equations of IM (19) to the equations of the IMs families.

### 3.3 Applying 2nd Procedure

Let us eliminate the parameter $\lambda_4$ from equations (30) with the aid of one of the equations, e.g., the first. The value of $\lambda_4$ found from this equation is

$$\lambda_4 = -\frac{2\alpha_2\,(p^2+q^2)}{\alpha_2\,r}. \tag{31}$$

Next, construct a lexicographic Gröbner basis with respect to the lexicographic ordering $\delta_1 > \delta_2 > \delta_3 > \gamma_3$ for the polynomials of a resulting system (after eliminating $\lambda_4$ from equations (30)). The system obtained

$$\left.\begin{array}{l} \alpha_1\,\gamma_3\,r + 2(p^2+q^2)(\lambda_3 bq + \lambda_2 x_0 p) = 0, \\ \alpha_1\,r\,\delta_3 - 2(p^2+q^2)(\lambda_1 bq + \lambda_3 x_0 p) = 0, \\ \alpha_1^2 r^2\,\delta_2 + \alpha_1\left[\lambda_1 b\,(q^2-p^2) + 2\lambda_3\,x_0\,p\,q\right] \\ \quad \times r^2 - 2\alpha_2(p^2+q^2)^2 = 0, \\ -\alpha_1\,\delta_1 - 2b\lambda_1 p\,q + \lambda_3 x_0(q^2-p^2) = 0 \end{array}\right\} \tag{32}$$

defines the IMs family of codimension 4 for the initial differential equations, which is parameterized by $\lambda_1, \lambda_2, \lambda_3$.

Expression (31) is the first integral for the equations of vector field on the elements of IMs family (32). The latter is verified by IM definition.

The elements of IMs family (30) are submanifolds of the IMs family found. This can be verified by direct substitution of expressions (30) (resolved with respect to $\delta_1, \delta_2, \delta_3, \gamma_3, r$) into equations (32).

The above example shows that the presented procedure also provides a possibility to find embedded in one another IMs families by eliminating the family parameters from the equations of known IMs families. In this case, the resulting IMs family includes the initial one.

## 4 Conclusion

We have found and analyzed some part of the possible IMs families of the problems only. For the exhaustive analysis of the problems on the base of the presented approach, it is necessary to study in detail the algebra of the first integrals of these problems. In the given work, we restricted our consideration to linear combinations of the integrals.


### Acknowledgements

This research was supported financially by the Russian Foundation for Basic Research (16-07-00201a) and the Council on Grants of the President of the Russian Federation for State Support to Leading Scientific Schools of the Russian Federation (NSh-8081.2016.9).


## References


Adler, V. E., Marikhin, V. G., Shabat, A. B. (2012). Quantum Tops as Examples of Commuting Differential Operators. *Theoret. and Math. Phys.,* **172**(3), pp. 1187–1205.

Bogoyavlenskii, O. I.(1984). Two integrable cases of a rigid body dynamics in a force field. *USSR Acad. Sci. Doklady,* **275**(6), pp. 1359–1363.

Borisov, A. V., Mamaev, I. S. and Sokolov, V. V. (2001). A new integrable case on so(4), *Dokl. Phys.* **46**(12), pp. 888–889.

Lyapunov, A. M. (1954). *On Permanent Helical Motions of a Rigid Body in Fluid. Collected Works, vol. 1.* USSR Acad. Sci., Moscow–Leningrad. (In Russian)

Sarychev, V. A., Gutnik, S. A. ( 2015). Dynamics of a Satellite Subject to Gravitational and Aerodynamic Torques. Investigation of Equilibrium Positions. *Cosmic Research.*, **53**(6), pp. 449–457.

Smirnov, A. V. (2008). Systems of sl(2, C) tops as two-particle systems. *Theoretical and Mathematical Physics*, **157**(1), pp. 1370-1382 .