Singular Linear-Quadratic Problem for Distributed Delay Systems

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***Abstract*— A singular linear-quadratic optimization problem for a linear system of differential equations with aftereffect is considered. The characteristic of the problem is that the optimal control contains impulsive components concentrated on the boundaries of the control time interval. In contrast to [1], more general form of the system that includes terms with distributed delay and more complicated functional is considered here. Regular linear-quadratic problems for systems with aftereffect were investigated in [2]. Singular linear-quadratic problem without aftereffect was considered in [3], [4].**

I. Introduction

Singular linear-quadratic optimization problems are of great practical importance [5]. Models of such structure describe physical problems in space flight dynamics, robotics, electrophysics etc. That concerns systems with aftereffect as well. It is shown in [3], [4] that in case without aftereffect such problems have no solutions in a class of ordinary controls and it is necessary to extend the set of admissible controls allowing impulsive controls for providing the existence of solution. It should be mentioned that for practical problems, the singularity of a functional is often met [5] and, therefore, the investigation of such problems is actual.

II. Statement of the problem and its reduction

A. Statement of the problem

Consider the problem of minimizing the functional

$$J[v(\cdot)] = x^T(T)Sx(T) + \int_{t_0}^{T}\Big[x^T(t)\Phi_0(t)x(t)$$

$$+x^T(t)\int_{-\tau}^{0}\Phi_1(t,\theta)x(t+\theta)d\theta + \int_{-\tau}^{0}x^T(t+\theta)\Phi_1^T(t,\theta)d\theta\, x(t)$$

$$+\int_{-\tau}^{0}x^T(t+s)\Phi_2(t,s)x(t+s)ds+$$

$$+\int_{-\tau}^{0}\int_{-\tau}^{0}x^T(t+\theta)\Phi_3(t,\theta,\rho)x(t+\rho)d\theta d\rho$$

$$+x^T(t-\tau)\Phi_4(t)x(t-\tau)\Big]\,dt, \qquad (1)$$

where $\Phi_0(\cdot)$, $\Phi_1(\cdot,\cdot)$, $\Phi_2(\cdot,\cdot)$, $\Phi_3(\cdot,\cdot,\cdot)$, $\Phi_4(\cdot)$ are symmetric (excluding Φ_3 and Φ_1) continuous $n\times n$-dimensional matrix functions, S is a symmetric nonnegative definite $n\times n$ matrix with constant elements along the trajectories of the system of differential equations

$$\dot{x}(t) = A(t)x(t) + A_\tau(t)x(t-\tau)$$

$$+\int_{-\tau}^{0}G(t,\theta)x(t+\theta)d\theta + B(t)\dot{v}(t) \qquad (2)$$

with initial condition

$$x(t) = \varphi(t), \qquad t_0-\tau\le t\le t_0,$$

which will be called the problem **1**. Here, $x(t)$, $\varphi(t)$ are n-dimensional vector functions , $v(t)$ is a m-dimensional vector function, $A_\tau(t)$, $A(t)$, $G(t,\theta)$ are continuous $n\times n$ matrix functions, $B(t)$ is a continuously differentiable $n\times m$ matrix function.

Since the problem **1** have no solution in a class of absolutely continuous functions, it is necessary to extend the problem by introducing impulsive controls. Let $v(t)$ and, therefore, $x(t)$ are functions of bounded variation with derivatives regarded as distributional derivatives [6]. It is reasonable to suppose that $\varphi(t)$ is also a function of bounded variation. Further, for distinctness we assume that the functions $x(t)$ and $v(t)$ are continuous from the left on the interval (t_0,T) and $v(t_0)=0$.

In this case under a solution of the equation (2) we imply a solution of the corresponding integral equation

$$x(t) = \varphi(t_0) + \int_{t_0}^{t}A(s)x(s)ds + \int_{t_0}^{t}A_\tau(s)x(s-\tau)ds$$

$$+\int_{t_0}^{t}\int_{-\tau}^{0}G(s,\theta)x(s+\theta)d\theta ds + \int_{t_0}^{t}B(s)dv(s)$$

with Riemann-Stieltjes integrals.

B. Reduction of the problem to a regular one

Transforming the problem **1** by a change of variables

$$y(t) = x(t) - B(t)v(t), \qquad (3)$$

we find that $y(t)$ satisfies the system of differential equations

$$\dot{y}(t) = A(t)y(t) + A_\tau(t)y(t-\tau) + \int_{-\tau}^{0}G(t,\theta)y(t+\theta)d\theta$$

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$$+\Big(A(t)B(t)-\dot{B}(t)\Big)v(t)+A_\tau(t)B(t-\tau)v(t-\tau)$$

$$+\int\limits_{-\tau}^{0}G(t,\theta_1)B(t+\theta_1)v(t+\theta_1)d\theta_1 \tag{4}$$

with initial conditions

$$y(t)=\varphi(t),\ v(t)=0, \qquad t_0-\tau\le t\le t_0, \tag{5}$$

and the functional (1) takes the form

$$J^*[v(\cdot)]=\Big(y(T)+B(T)v(T)\Big)^TS\Big(y(T)+B(T)v(T)\Big)$$

$$+\int\limits_{t_0}^{T}\Big[\Big(y(t)+B(t)v(t)\Big)^T\Phi_0(t)\Big(y(t)+B(t)v(t)\Big)$$

$$+\Big(y(t)+B(t)v(t)\Big)^T$$

$$\times\int\limits_{-\tau}^{0}\Phi_1(t,\theta)\Big(y(t+\theta)+B(t+\theta)v(t+\theta)\Big)d\theta$$

$$+\int\limits_{-\tau}^{0}\Big(y(t+\theta)+B(t+\theta)v(t+\theta)\Big)^T\Phi_1^T(t,\theta)d\theta$$

$$\times\Big(y(t)+B(t)v(t)\Big)+\int\limits_{-\tau}^{0}\Big(y(t+s)+B(t+s)v(t+s)\Big)^T$$

$$\times\Phi_2(t,s)\Big(y(t+s)+B(t+s)v(t+s)\Big)ds$$

$$+\int\limits_{-\tau}^{0}\int\limits_{-\tau}^{0}\Big(y(t+\theta)+B(t+\theta)v(t+\theta)\Big)^T$$

$$\times\Phi_3(t,\theta,\rho)\Big(y(t+\rho)+B(t+\rho)v(t+\rho)\Big)d\theta d\rho$$

$$+\Big(y(t-\tau)+B(t-\tau)v(t-\tau)\Big)^T$$

$$\times\Phi_4(t)\Big(y(t-\tau)+B(t-\tau)v(t-\tau)\Big)\Big]dt. \tag{6}$$

If the terminal part in the functional (6) depends on $v(T)$ (i.e. when $SB(T)\neq 0$), it is possible to minimize it with respect to $v(T)$. Since the matrix S is nonnegative definite, the terminal part reaches it's minimal value at points of a manifold described by the system of equations

$$B^T(T)S\Big[B(T)v(T)+y(T)\Big]=0.$$

For the case $\det\Big(B^T(T)SB(T)\Big)\neq 0$ the last system has the solution of the form

$$v(T)=-\Big(B^T(T)SB(T)\Big)^{-1}B^T(T)Sy(T).$$

If $\det\Big(B^T(T)SB(T)\Big)=0$, to solve the system one can use the technique of semiinverse matrices [7] and obtain

$$v(T)=-\Big(B^T(T)SB(T)\Big)^-B^T(T)Sy(T)$$

$$+\Big(E-\Big(B^T(T)SB(T)\Big)^-B^T(T)SB(T)\Big)p, \tag{7}$$

where U^- is the semiinverse matrix for a matrix U, p is an arbitrary m-dimensional vector. Note if U is a quadratic matrix with $\det U\neq 0$, then $U^-=U^{-1}$.

Substituting (7) into the functional (6) and taking into account that by definition of semiinverse matrix

$$\Big(B^T(T)SB(T)\Big)$$

$$\times\Big(E-\Big(B^T(T)SB(T)\Big)^-B^T(T)SB(T)\Big)=0,$$

we obtain the following functional to be minimized

$$J^{**}[v(\cdot)]=y(T)^TNy(T)+$$

$$+\int\limits_{t_0}^{T}\Big[\Big(y(t)+B(t)v(t)\Big)^T\Phi_0(t)\Big(y(t)+B(t)v(t)\Big)$$

$$+\Big(y(t)+B(t)v(t)\Big)^T$$

$$\times\int\limits_{-\tau}^{0}\Phi_1(t,\theta)\Big(y(t+\theta)+B(t+\theta)v(t+\theta)\Big)d\theta$$

$$+\int\limits_{-\tau}^{0}\Big(y(t+\theta)+B(t+\theta)v(t+\theta)\Big)^T\Phi_1^T(t,\theta)d\theta$$

$$\times\Big(y(t)+B(t)v(t)\Big)+\int\limits_{-\tau}^{0}\Big(y(t+s)+B(t+s)v(t+s)\Big)^T$$

$$\times\Phi_2(t,s)\Big(y(t+s)+B(t+s)v(t+s)\Big)ds$$

$$+\int\limits_{-\tau}^{0}\int\limits_{-\tau}^{0}\Big(y(t+\theta)+B(t+\theta)v(t+\theta)\Big)^T$$

$$\times\Phi_3(t,\theta,\rho)\Big(y(t+\rho)+B(t+\rho)v(t+\rho)\Big)d\theta d\rho$$

$$+\Big(y(t-\tau)+B(t-\tau)v(t-\tau)\Big)^T$$

$$\times\Phi_4(t)\Big(y(t-\tau)+B(t-\tau)v(t-\tau)\Big)\Big]dt, \tag{8}$$

where

$$N=\Big(E-B(T)\Big(B^T(T)SB(T)\Big)^-B^T(T)S\Big)^TS$$

$$\times\Big(E-B(T)\Big(B^T(T)SB(T)\Big)^-B^T(T)S\Big).$$

Further, the optimization problem of minimizing the functional (8) along the trajectories of system (4), (5) is called the problem **2** . The distinguishing feature of the problem **2** is that the control $v(t)$ is a function of bounded variation here and, therefore, the trajectory $y(t)$ is an absolutely continuous function. Thus, we have obtained the auxiliary problem which is the regular problem that can be solved using well known methods of optimal control theory.

III. Solution of the auxiliary problem

Theorem 1: Let for $t \in [t_0, T]$ the following conditions hold

1) $\det\Big(B^T(t)\Phi_0(t)B(t) + P_6(t,0)\Big) \neq 0$,

2) the matrix

$$\begin{pmatrix} \frac{1}{\tau}\Phi_0(t) & \Phi_1(t,\theta) \\ \Phi_1^T(t,\theta) & \Phi_2(t,\theta) \end{pmatrix}$$

is nonnegative definite for $\theta \in [-\tau, 0]$,

3) the matrix $\Phi_3(t,\theta,\rho)$ has of the structure

$$\Phi_3(t,\theta,\rho) = \widetilde{\Phi_3}(t,\theta)\,\widetilde{\Phi_3}(t,\rho),$$

4) the matrix $\Phi_4(t)$ is nonnegative definite.

Then the optimal control for the problem **2** has the form

$$v(t) = W_0(t)y(t) + \int_{-\tau}^{0} W_1(t,\theta)y(t+\theta)\,d\theta + \int_{-\tau}^{0} W_2(t,\theta_1)v(t+\theta_1)\,d\theta_1. \qquad (9)$$

Here

$$W_0(t) = -H^{-1}(t)\,F_0(t)$$
$$W_1(t,\theta) = -H^{-1}(t)\,F_1(t,\theta),$$
$$W_2(t,\theta_1) = -H^{-1}(t)\,F_2(t,\theta_1),$$

where

$$F_0(t) = P_1^T(t,0) + B_1^T(t)P(t) + P_5^T(t,0) + B^T(t)\Phi_0(t),$$
$$F_1(t,\theta) = B_1^T(t)Q(t,\theta) + P_2^T(t,\theta,0) + B^T(t)\Phi_1(t,\theta),$$
$$F_2(t,\theta_1) = B_1^T(t)P_1(t,\theta_1) + P_3^T(t,\theta_1,0) + B^T(t)\Phi_1(t,\theta_1)B(t+\theta_1),$$
$$H(t) = B^T(t)\Phi_0(t)B(t) + P_6(t,0),$$
$$B_1(t) = A(t)B(t) - \dot{B}(t)$$

and the matrices

$$P(t),\ P_4(t,s) = P_4^T(t,s),\ P_6(t,r),\ P_1(t,\theta_1),\ R(t,\theta,\rho) = R^T(t,\rho,\theta),\ P_2(t,\theta,\theta_1),\ P_5(t,p),\ Q(t,\theta),\ P_3(t,\theta_1,\theta_2) = P_3^T(t,\theta_2,\theta_1)$$

satisfy the equations

$$\frac{dP(t)}{dt} + Q(t,0) + Q^T(t,0) + P^T(t)A(t) + A^T(t)P(t) + P_4(t,0) + \Phi_0(t) = F_0^T(t)H^{-1}(t)F_0(t),$$

$$\Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial\theta}\Big)Q(t,\theta) + R(t,0,\theta) + P(t)G(t,\theta) + A^T(t)Q(t,\theta) + \Phi_1(t,\theta) = F_0^T(t)H^{-1}(t)F_1(t,\theta),$$

$$\Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial\theta} - \frac{\partial}{\partial\rho}\Big)R(t,\theta,\rho) + G^T(t,\theta)Q(t,\rho) + Q^T(t,\theta)G(t,\rho) + \Phi_3(t,\theta,\rho) = F_1^T(t,\theta)H(t)^{-1}F_1(t,\rho),$$

$$\Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial\theta_1}\Big)P_1(t,\theta_1) + P_2(t,0,\theta_1) + A^T(t)P_1(t,\theta_1) + P(t)G(t,\theta_1)B(t+\theta_1) + \Phi_1(t,\theta_1)B(t+\theta_1) = F_0^T(t)H^{-1}(t)F_2(t,\theta_1),$$

$$\Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial\theta} - \frac{\partial}{\partial\theta_1}\Big)P_2(t,\theta,\theta_1) + Q^T(t,\theta)G(t,\theta_1)B(t+\theta_1) + G^T(t,\theta)P_1(t,\theta_1) + \Phi_3(t,\theta,\theta_1)B(t+\theta_1) = F_1^T(t,\theta)H^{-1}(t)F_2(t,\theta_1),$$

$$\Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial\theta_1} - \frac{\partial}{\partial\theta_2}\Big)P_3(t,\theta_1,\theta_2) + B^T(t+\theta_1)G^T(t,\theta_1)P_1(t,\theta_2) + P_1^T(t,\theta_1)G(t,\theta_2)B(t+\theta_2) + B^T(t+\theta_1)\Phi_3(t,\theta_1,\theta_2)B(t+\theta_2) = F_2^T(t,\theta_1)H^{-1}(t)F_2(t,\theta_2),$$

$$\Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial s}\Big)P_4(t,s) + \Phi_2(t,s) = 0,$$

$$\Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial p}\Big)P_5(t,p) + \Phi_2(t,p)B(t+p) = 0,$$

$$\Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial r}\Big)P_6(t,r) + B^T(t+r)\Phi_2(t,r)B(t+r) = 0 \qquad (10)$$

with the boundary conditions

$$P(T) = N, \quad Q(T,\theta) = R(T,\theta,\rho) = P_1(T,\theta_1) = P_2(T,\theta,\theta_1) = P_3(T,\theta_1,\theta_2) = P_4(T,s) = P_5(T,p) = P_6(T,r) = 0,$$

$$P(t)A_\tau(t)B(t-\tau) = P_1(t,-\tau),$$
$$B^T(t-\tau)A_\tau^T(t)Q(t,\theta) = P_2^T(t,\theta,-\tau),$$
$$A_\tau^T(t)P(t) = Q^T(t,-\tau),$$
$$A_\tau^T(t)Q(t,\theta) + Q^T(t,\theta)A_\tau(t) = R^T(t,\theta,-\tau) + R(t,-\tau,\theta),$$
$$A_\tau^T(t)P_1(t,\theta_1) = P_2(t,-\tau,\theta_1),$$
$$B^T(t-\tau)A_\tau^T(t)P_1(t,\theta_1) + P_1^T(t,\theta_1)A_\tau(t)B(t-\tau) = P_3^T(t,\theta_1,-\tau) + P_3(t,-\tau,\theta_1),$$
$$\Phi_4(t) = P_4(t,-\tau),$$
$$\Phi_4(t)B(t-\tau) = P_5(t,-\tau),$$
$$B^T(t-\tau)\Phi_4(t)B(t-\tau) = P_6(t,-\tau),$$

for $t < T$, $-\tau \le \theta, \theta_1, \theta_2, \rho, p, r, s \le 0$.

Proof: The conditions (2)–(4) guarantee nonnegative definiteness of the functional (8). The validity of the theorem is easy to establish using the technique from [2]. ■

IV. Optimal control for the initial problem

The optimal program control for the initial problem is determined by differentiating the control (9) in distribution sense [8].

Since the value of the function $v(t)$ equal to zero for $t \le t_0$ and defined by formula (7) for $t \ge T$, the distributional derivative of this function has the form

$$\dot v(t) = \Delta v(t_0, \varphi(\cdot))\,\delta(t - t_0) + \dot v_r(t)$$

$$+\Delta v(T, x(T-0))\,\delta(t-T). \qquad (11)$$

Here, $\dot v_r(t)$ is the regular part of the generalized optimal control $\dot v(t)$. I.e. the optimal program $\dot v(t)$ for the problem **1** generates impulses only at the initial and the terminal points of the control time interval and is a summable function in the interior of this interval.

Taking into account (3), we have

$$v(t) = W_0(t)\Big(x(t) - B(t)v(t)\Big) + \int\limits_{-\tau}^{0} W_1(t,\theta)x(t+\theta)\,d\theta$$

$$+\int\limits_{-\tau}^{0}\Big[W_2(t,\theta_1) - W_1(t,\theta_1)B(t+\theta_1)\Big]v(t+\theta_1)\,d\theta_1. \quad (12)$$

From (5) follows that the equation (12) for $t = t_0$ will take the form

$$W_0(t_0)x(t_0) + \int\limits_{-\tau}^{0} W_1(t_0,\theta)x(t_0+\theta)\,d\theta = 0.$$

According to the principle of optimality the latter equality remains valid for any $t \in (t_0, T)$ and becomes

$$W_0(t)x(t) + \int\limits_{-\tau}^{0} W_1(t,\theta)x(t+\theta)\,d\theta = 0. \qquad (13)$$

The equation (13) describes the functional manifold in a space of functions of bounded variation $x(t)$ defined on $[t-\tau, t]$, which contain the optimal trajectory for $t \in (t_0, T)$. Taking into account (12), (13) and the independence of trajectories of the system (2) on control prehistory, we conclude that

$$W_0(t)B(t) = -E,$$

$$W_2(t,\theta_1) - W_1(t,\theta_1)B(t+\theta_1) = 0, \qquad (14)$$

for $t \in [t_0 - \tau, T],\ \theta_1 \in [-\tau, 0]$.
Differentiating the equation (13) in view of system (2), we obtain

$$\frac{dW_0(t)}{dt}x(t) + W_0(t)\Big[A(t)x(t) + A_\tau(t)x(t-\tau)$$

$$+\int\limits_{-\tau}^{0} G(t,\theta)x(t+\theta)\,d\theta + B(t)\dot v_r(t)\Big]$$

$$+\int\limits_{-\tau}^{0}\Big[\frac{\partial W_1(t,\theta)}{\partial t}x(t+\theta) + W_1(t,\theta)\dot x(t+\theta)\Big]\,d\theta = 0.$$

From this by (14) we can derive the following formula for $\dot v_r(t)$ which is defined on (t_0, T)

$$\dot v_r(t) = Z_1(t)x(t) + Z_2(t)x(t-\tau)$$

$$+\int\limits_{-\tau}^{0} Z_3(t,\xi)x(t+\xi)\,d\xi. \qquad (15)$$

Here

$$Z_1(t) = \frac{dW_0(t)}{dt} + W_0(t)A(t) + W_1(t,0),$$

$$Z_2(t) = W_0(t)A_\tau(t) - W_1(t,-\tau),$$

$$Z_3(t,\theta) = G(t,\theta) + \Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial\theta}\Big)W_1(t,\theta).$$

Hence, for $[t_0, T]$, the optimal control according to (11) and (15) takes the form

$$\dot v(t) = \Delta v(t_0,\varphi(\cdot))\,\delta(t-t_0) + Z_1(t)x(t) + Z_2(t)x(t-\tau)$$

$$+\int\limits_{-\tau}^{0} Z_3(t,\theta)x(t+\theta)\,d\theta + \Delta v(T, x(T-0))\,\delta(t-T).$$

By (12), (7) and (3) the first and the last terms in the righthand side of the latter are determined by the formulas

$$\Delta v(t_0,\varphi(\cdot)) = W_0(t_0)\varphi(t_0) + \int\limits_{-\tau}^{0} W_1(t_0,\theta)\varphi(t_0+\theta)\,d\theta,$$

$$\Delta v(T, x(T-0)) = -\Big(B^T(T)SB(T)\Big)^{-}B^T(T)Sx(T-0)$$

$$+\Big(E - \Big(B^T(T)SB(T)\Big)^{-}B^T(T)SB(T)\Big)p.$$

Remark. If the structure of the matrices $P_1(t,\theta_1)$, $P_2(t,\theta,\theta_1)$, $P_3(t,\theta_1,\theta_2)$, $P_5(t,p)$, $P_6(t,r)$ is put in a specific form, it is possible to simplify the system (10).

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