Abstract

The results of a numerical investigation of dynamical states and bifurcation transitions of a frequency-phase-feedback oscillator’s model are presented in the paper. The behavior of the examined model is described by nonlinear six-dimensional set of differential equations with periodical nonlinearities. It is shown that the model demonstrates complex behavior including periodic and chaotic self-modulation oscillations, and various transitions to chaotic modes. Results are presented using one-parameter bifurcation diagrams, phase portraits of attractors, time realizations of oscillations, and Poincare maps.

Key words

Frequency-phase-feedback oscillator, regular and chaotic modes, synchronous regime, bifurcations and chaotic behavior, phase space, attractors, bifurcation diagrams, phase portraits, Poincare map.

1 Introduction

The systems with phase and frequency control are widely utilized in many electronic systems for solving the problems of stabilization, synchronization, and tracking. In these problems such systems provide for and automatically maintain the synchronous state, when the phase difference of reference and controlled signals becomes constant or, equivalently, the frequency difference of these signals is equal to zero [Shakhgil’dyan and Lyakhovkin, 1972]. The synchronous mode corresponds to a stable equilibrium state in the phase space of the corresponding system’s dynamical model. The systems may also operate in nonsynchronous modes (modes with variable phase and frequency mismatches) corresponding to regimes of controlled oscillator with regular and chaotic frequency modulation. Feedback loops, when out-of-lock, provide broad possibilities for excitation of various nonsynchronous modes. By varying the parameters of control circuits, one can effectively govern the properties of generated oscillations and the domains where these oscillations exist. Thus, the system with phase and frequency control may be promising for devices where information is transmitted via chaotic signals. Therefore, knowledge of the specific features of the nonlinear dynamics for various systems versions is of significant importance.

The purpose of this paper is to present new results concerning the nonlinear dynamics of system with frequency-phase control combining phase-lock (PL) and frequency-lock (FL) partial subsystems. The system incorporates the controlled oscillator with two feedback loops including nonlinear discriminators of phase and frequency errors, low-frequency filters (LFFs) with transfer functions $K_1(p)$ and $K_2(p)$, and a frequency modulator [Kapranov, 1958]. We investigate nonsynchronous modes and its bifurcations of the frequency-phase lock (FPL) system in the case when each separate partial subsystem exhibits both regular and chaotic behavior.

2 Dynamical model of FPL system

The dynamic equation for phase difference $\varphi$ between the controlled and reference signals in the FPL system is generally written in the form [Kapranov, 1958]:

$$p\varphi + \Omega K_1(p)F(\varphi) + \Omega_1 K_2(p)\Phi(p\varphi) = \delta \omega,$$  \hspace{1cm} (1)

Here $p=\frac{d}{dt}$, $F(\varphi)$ and $\Phi(p\varphi)$ are the characteristics of phase and frequency discriminators normalized to unity; $\Omega$ and $\Omega_1$ are the control circuit gains; and $\delta \omega$ is the initial frequency mistuning. Assume that the functions $F(\varphi) = \sin \varphi$ and $\Phi(p\varphi) = 2\beta_1 p\varphi / (1 + \beta_1^2 (p\varphi)^2)$ ($\beta_1^{-1}$ is the frequency mistuning providing for the maximum value of $\Phi(p\varphi)$) approximate the phase and frequency discriminator characteristics respectively.

We will consider, according to problem formulation, FPL system with the second-order filter in PL subsystem and third-order filter in FL subsystem with transfer functions $K_1(p) = \frac{1}{1 + (T_1 + T_2)p + T_1 T_2 p^2}$, $K_2(p) = \frac{1}{1 +}$
+\( T_1 + T_2 + T_3 \phi + (T_1 + T_2 + T_3 + T_4) \phi^2 + T_2 T_1 \phi^3 \), where \( T_i \), \( T_2, T_3, T_4, \) and \( T_5 \) are the inertia parameters. This case is of a special interest because the isolated partial FL subsystem with the third-order filter can possess a mode of two-spiral type chaotic attractor in its phase space, and the partial PL subsystem with the second-order filter individually exhibits chaotic modes of oscillatory (without rotation of \( \phi \)), rotatory, and oscillatory-rotatory types.

For considered LFFs equation (1) can be written in the form

\[
d\phi/dt = u, \quad d\xi/dt = \varepsilon, \quad d\psi/dt = \psi, \quad d\upsilon/dt = \psi, \quad d\upsilon/dt = \xi, \quad d\eta/dt = \eta, \quad d\xi/dt = \eta,
\]

\[
\mu A d\eta/dt = \gamma - \sin\theta - b \Phi'(\upsilon)(\gamma + (1 + \varepsilon_2 \cos\theta)u - \varepsilon_1 \cos\theta \Phi' (\gamma) - (\mu_1 + \varepsilon_3 \cos\theta - \mu_2))(\mu_1 \varepsilon_2 \cos\theta + b \Phi' (\gamma))(\mu_1 \mu_2 + \mu_3) + \mu_2 \cos\theta \Phi' (\gamma) - (\mu_1 \varepsilon_2 \cos\theta + b \Phi' (\gamma)) \eta + \mu_5 \sin\theta + \mu_6 \cos\theta + 3 \mu_2 \sin\theta - b \Phi' (\gamma) \xi^2,
\]

In equations (2) \( \tau = \Omega t \) is the dimensionless time, \( \gamma = \delta \Omega, \beta = \beta \Omega, \mu = T_1 + T_2 + T_3 + T_4 \Omega^2, \varepsilon_1 = T_1 + T_2 + T_3 \Omega^2, \mu = T_1 T_2 T_3 \Omega^2, \Phi(\upsilon) = 2\sqrt(1 + \upsilon^2), \Phi' (\upsilon) = 2(1 - \upsilon^2)/(1 + \upsilon^2)^{3/2}, \Phi'' (\upsilon) = -4\upsilon(3 - \upsilon^2)/(1 + \upsilon^2)^{5/2}, \nu = \text{mod}(2\pi, \xi, \eta, \upsilon, \psi, \xi). \) Because of the system (2) nonlinearity its investigations are performed using qualitative-numerical methods of analysis of nonlinear dynamical systems [Belustina, Kiveleva and Fraiman, 1982; Anishchenko, 1990] and employing the software developed in [Ponomarenko and Matrosov, 1997]. In the course of analysis, we examine time realizations, projections attractor’s phase portraits, Poincare maps, and one-parameter bifurcation diagrams.

3 Stability of the synchronous mode

First, let us establish stability conditions for the synchronous mode. At \( 0 \leq \gamma < 1 \), system (2) has two equilibrium states: \( A_1 (\text{arcsin} \gamma, 0, 0, 0, 0) \) and \( A_2 (\pi - \text{arcsin} \gamma, 0, 0, 0, 0) \). The equilibrium state \( A_1 \) may be both stable and unstable, and \( A_2 \) is an unstable saddle type equilibrium state. We find stability conditions for equilibrium state \( A_1 \) by analyzing the roots of the characteristic equation for the eigenvalues of the linearized system near \( A_1 \)

\[
\lambda^6 + \alpha_1 \lambda^5 + \alpha_2 \lambda^4 + \alpha_3 \lambda^3 + \alpha_4 \lambda^2 + \alpha_5 \lambda + \alpha_6 = 0.
\]

In equation (3)

\[
\alpha_1 = (\mu_1 + \mu_2 + \mu_3) / \mu_1 \mu_2, \quad \alpha_2 = (\mu_1 \varepsilon_2 + \mu_5 + \mu_3) / \mu_1 \mu_2, \quad \alpha_3 = (\mu_1 \varepsilon_2 + \mu_5 + \mu_3) (1 - \gamma^2)^{1/2}, \quad \alpha_4 = 2b \mu_1 / \mu_1 \mu_2, \quad \alpha_5 = (\mu_1 + \mu_2 + \mu_3) (1 - \gamma^2)^{1/2}, \quad \alpha_6 = (\mu_1 \mu_2 + \mu_3) (1 - \gamma^2)^{1/2} / \mu_1 \mu_2.
\]

The stability conditions can be written in the form

\[
a_1, a_2, \ldots, a_6 > 0, \quad a_1 (a_2 - a_1) > a_2 (a_3 - a_2) > 0, \quad (a_1 a_2 a_3) (a_1 a_2 a_3) - (a_1 a_2 a_3) > 0, \quad (a_4 a_5 a_6) (a_4 a_5 a_6) - (a_4 a_5 a_6) > 0.
\]

If conditions (4) are satisfied, the studied FL system has a synchronous mode corresponding to equilibrium state \( A_1 \). The domain of parameters \( C_1 \), where conditions (4) are fulfilled corresponds to the domain where synchronous mode persists. When as a result of parameters varying, conditions (4) are violated limit cycle of oscillatory type during which the \( \phi \) coordinate is not exceed \( 2\pi \) arises in the phase space \( U \). This limit cycle corresponds to a quasi-synchronous mode in the FL system where periodic oscillations of phase variables are observed around equilibrium state \( A_1 \) that has become unstable.

4 Dynamical modes and bifurcations of model (2)

Now let us consider nonsynchronous modes of the FL system that develops for the parameter values out of domain \( C_1 \). For this purpose, we use the results of numerical simulation of model (2).

Fig.1 displays one-parameter bifurcation diagram \( \{\mu, \phi\} \) for the point Poincare mapping produced by the trajectories of model (2). The diagram is calculated at the parameter values \( \gamma = 0.55, \beta = 0.55, \varepsilon_1 = 1.0, \mu_1 = 2.0, \mu_2 = 3.2 \). Fig.2 shows \( (\phi, \upsilon) \) projections of the phase portraits, \((\phi, v)\) and \((\zeta, v)\) projections of the Poincare map, and time realization \( u(t) \) corresponding to the attractors of model (2). The \( \{\mu, \phi\} \) diagram depicts evolution of quasi-synchronous modes of limit cycles \( S_0 \) (Fig.2a) and \( S_2 \) (Fig.2d) as parameter \( \mu_1 \) varies from 1.8 to 8.11. Note, that in the interval 1.8 < \( \mu_1 < 3.0 \) the bistable behavior of the system is observed; i.e., the attractors based on limit cycles \( S_1 \) and \( S_2 \) exist in the phase space simultaneously. One can see that the mode of cycle \( S_2 \) becomes chaotic (Fig.2h) via period-doubling bifurcations; then, the mode of chaotic attractor \( P_2 \) is destroyed, and the system passes to the mode of complex limit cycle \( S^* \) (Fig.2g).
The evolution of the mode of cycle $S_1$, as $\mu_1$ is increased, starts when a stable limit cycle $S_1$ becomes unstable and gives rise to a stable 2D torus $T_1$ (Fig.2b). In Fig.2c close invariant curve $\Gamma_1$ of the Poincare map corresponding to the torus $T_1$ is given. With increasing $\mu_1$, the mode of torus $T_1$ collapses, and the system passes to the mode of oscillatory limit cycle $S_0$ (Fig.2e). With a further increase of $\mu_1$, the mode of torus $T_0$ from the limit cycle $S_0$ appears (Figs.2f, 2g) which than, is rigidly replaced by the mode of complex oscillatory limit cycle $S^*$ (fig.2i). Then, the cycle $S^*$ breaks down and the system rigidly enters a mode of oscillatory-rotatory type chaotic attractor $P_4$ (Figs.2j, 2k). As $\mu_1$ is increased attractor $P_4$ is softly transformed to the torus $T_0$, which further is transformed to limit cycle $S_0$, and the system again returns to the periodic quasi-synchronous mode. When $\mu_1 > 3.54$ the mode of cycle $S_0$ is rigidly replaced by the mode of oscillatory-rotatory type chaotic attractor $P_5$ (Figs.2l-2n). In interval $4.69 < \mu_1 < 8.3$ irregular alternation of chaotic, periodic, and quasi-synchronous modes is observed. When $\mu_1$ passes through the value $\mu_1 = 8.33$, the system passes to the asynchronous (with rotation of $\varphi$) mode of oscillatory-rotatory type chaotic attractor via intermittency.

Fig.3 represents bifurcation diagram $\{\gamma, \varphi\}$ corresponding to the parameter values $b=1$, $\beta=20.5$, $\epsilon_1=1.25$, $\epsilon_2=15$, $\mu=2$, $\mu_2=6.25$, $\mu_3=3.2$. It characterizes evolution of quasi-synchronous modes that develop in the system on the base of the mode of limit cycle $S_1$ when $\gamma$ decreases. It is seen from the diagram that alternating modes of torus $T_1$, limit cycle $S_2$, torus $T_3$, chaotic attractor $P_3$, limit cycle $S_4$, torus $T_5$, chaotic attractor $P_3$, and limit cycle $S_5$ are observed in the model (2). Fig.4 displays $(\varphi,u)$ projections of the phase portraits, time realizations $\varphi(\tau)$ and $u(\tau)$, and $(\varphi,v)$ and $(z,u)$ projections of the Poincare map corresponding to these modes.

The bifurcation diagram $\{\mu_1,u\}$ represented in Fig.5a shows transformation of quasi-synchronous mode of the limit cycle $S_1$ into asynchronous mode of oscillatory-rotatory type chaotic attractor $W_0$. The diagram calculated at the parameter values $\gamma=0.5$, $b=0.5$, $\beta=1$, $\epsilon_1=0.9$, $\epsilon_2=80$, $\mu=2$, $\mu_2=2$. Figs.5b-5i display $(\varphi,u)$ projections of the phase portraits, and time realizations $\varphi(\tau)$ corresponding to oscillatory and oscillatory-rotatory attractors of the system. With increasing of $\mu_1$, limit cycle $S_2$ is softly transformed into chaotic attractor.
With further increasing of $\mu_1$, attractor $P_2$ is transformed into chaotic attractor $W_0$ (Figs.5g, 5h). Then there is a “window” of values of $\mu_1$ at $\{\mu_1, u_1\}$ diagram corresponding to the mode of oscillatory-rotatory type limit cycle $L_0$ (Fig.5i).

$\text{Fig. 6. Transition to the chaotic quasi-synchronous mode via torus distortion}$

Formation of chaotic quasi-synchronous mode may be realized in the system considered via distortion of the oscillatory torus mode. This effect is illustrated by $(\varphi, u)$ projections of the phase portraits, time realizations $\varphi(\tau)$ and $u(\tau)$, and $(\varphi, u)$ projections of the Poincare map represented at Fig.6 for the parameter values $\gamma=0$, $b=1$, $\beta=20.5$, $c_2=15$, $\mu_2=2$, $\mu_3=3.7$, $\mu_5=3.2$. Fig.6 shows evolution of quasi-synchronous mode of limit cycle $S_1$ (Fig.6a) when parameter $\varepsilon_1$ decreases from 1.2 to 0.72. When $\varepsilon_1$ decreases the mode of oscillatory torus $T_2$ (Figs.6b-6d) appears from limit cycle $S_1$. After torus $T_2$ is formed, the phase portrait of the Poincare map (Fig.6d) is characterized by the presence of stable closed invariant curve $\Gamma$. With decreasing of $\varepsilon_1$ the distortion of the curve $\Gamma$ and gradual transformation of the mode of torus-chaos type chaotic attractor (Figs.6e-6f) takes place.

$\text{Fig. 7. Formation of oscillatory type chaotic attractor via torus-doubling bifurcation}$

The model (2) exhibits such interesting phenomena as formation of oscillatory type chaotic attractor via torus-doubling bifurcation, and rise of 3D torus in the phase space $U$. Fig.7 illustrates the first of these phenomena. It shows $(\varphi, u)$ projections of the phase portraits, $(\varphi, v)$ and $(u, v)$ projections of the Poincaré map corresponding to the attractors of model (2) for the parameter values $\gamma=0$, $b=1$, $\beta=20.5$, $c_2=15$, $\mu_2=2$, $\mu_3=3.7$, $\mu_5=3.2$. Parameter $\varepsilon_1$ decreases from 1.65 to 1.45. For $\varepsilon_1=1.65$ the mode of oscillatory limit cycle $S_1$ (Fig.7a) is the initial state of the system. We see that, as $\varepsilon_1$ decrease, cycle $S_1$ loses its stability, and torus $T_1$ (Fig.7b) corresponding to closed invariant curve $\Gamma$ (Fig.7c) appears in the phase space. Then period doubling bifurcations of curve $\Gamma$ adequate to torus-doubling bifurcations occurs with formation of closed invariant curves $\Gamma_3^2$ and $\Gamma_3^3$ (Figs.7d, 7e). If $\varepsilon_1$ continues to decrease, the mode of chaotic attractor $V_1$ rises.

$\text{Fig. 8. Rise the mode of 3D torus and transition to a chaotic mode}$

(Figs.8c, 8d). Then, transition to the mode of chaotic attractor $V_2$ is observed (Fig.8e).
characterized by irregular switches of a phase variable \( \phi \). Fig. 9a represents bifurcation diagram \( \{ \mu_1, \phi \} \) corresponding to \( \gamma = 0.15, \ b = 5, \ \beta = 1, \ \epsilon_1 = 1, \ \epsilon_2 = 35, \ \mu_2 = 13.5 \). Figs. 9b-9n shows \( (\varphi, u) \) projections of the phase portraits, time realizations \( \varphi(\tau) \), and \( (\varphi, v) \) and \( (z, v) \) projections of the Poincare map corresponding to the system’s attractors, when parameter \( \mu_1 \) varies from 2.75 to 4.98. The modes of limit cycles \( S_1 \) and \( S_2 \) (Fig. 9b) are the system’s starting state for \( \mu_1 = 2.75 \). The \( \{ \mu_1, \phi \} \) diagram shows how the modes of cycles \( S_1 \) and \( S_2 \) become chaotic via period doubling bifurcations corresponding to attractors \( P_1 \) and \( P_2 \) (Figs. 9c-9e). When \( \mu_1 \) increases the system passes to the mode of chaotic attractor \( V_0 \), characterized by irregular alternating oscillations at the attractors \( P_1 \) and \( P_2 \) (Figs. 9f-9h). The dynamic \( \varphi \) range of attractor \( V_0 \) exceeds \( 2\pi \). Formation of attractor \( V_0 \) indicates that, in the phase space, there are two domains where the system chaotically oscillates and irregularly passes from one domain into the other. As \( \mu_1 \) increases the duration of oscillations in these domains decrease, while the frequency of switches from one domain to another grows (Figs. 9i-9k). Then, the system undergoes a hard transition to the asynchronous mode of oscillatory-rotatory type chaotic attractor \( W_0 \) (Figs. 9l-9n).

5 Conclusion

In this paper we have investigated the dynamical modes and transitions to the chaotic behavior in a frequency-phase locked system with the second-order filter in a phase-locked loop and the third-order filter in a frequency-locked loop. Using the dynamical model of considered FPL system we found that the system exhibits a rich variety of nonsynchronous modes: periodic modes caused by the loss of the synchronous mode stability and by saddle-node bifurcations of oscillatory limit cycles; quasi-periodic quasi-synchronous modes corresponding to oscillatory 2D and 3D tori in the phase space; chaotic modes that formed via period-doubling bifurcations, as well as rigidly formed through saddle-node bifurcations of limit cycles, via destruction of tori, via the torus-doubling bifurcations, via the creation of united chaotic attractor in the phase space with irregular switches of phase variable \( \phi \).

These phenomena, in our opinion, are of importance for both basic and applied research of nonlinear dynamics of the system with frequency and phase control. It further to understanding the behavior of the systems when the synchronous state is cut off as a result of the system parameters perturbation. The discovered effects of nonlinear dynamics and the obtained data on restructuring FPL system’s dynamical states under variation of the system’s parameters are of interest for suppression or enhancement of mode instabilities, control of behavior modes, and generation of chaotic oscillations.

The wide variety of chaotic modes offers considerable possibilities of forming various frequency-modulated signals at the output of the FPL system. Control the characteristics of generated signals can be easily realized by means of subsystems parameters.

Acknowledgements

This work was supported by the Russian Foundation for Basic Research (grants No 05-02-17409, No 06-02-16499).

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