STABILIZATION OF SYSTEMS WITH SECTOR BOUNDED NONLINEARITY BY A SAWTOOTH SAMPLED-DATA FEEDBACK

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Abstract

The paper considers a nonlinear Lur'e type system with a sector bounded nonlinearity. The zero equilibrium of the system may be unstable, so it is stabilized by a periodically sampled feedback signal. Such stabilization problems were previously explored by a number of researches with the help of the zero-order hold (ZOH) control that is kept constant between successive sampling times. The main disadvantage of this method is that the time delay introduced by ZOH has a destabilizing impact on the closed feedback system, especially in the case when the sampling frequency is sufficiently low and the feedback gain is high. To reduce this effect it is proposed to modify the form of the stabilizing signal. In this paper the reverse sawtooth control is introduced instead of ZOH. The stability criterion is obtained in the form of a feasibility problem for some linear matrix inequalities (LMI). A numerical example demonstrates how the new stabilization method allows to reduce the sampling frequency required for stabilization.

Key words

Nonlinear system, sampled-data stabilization, linear matrix inequalities, integral quadratic constraints

1 Introduction

Stability problems for a continuous-time system under a sampled-data feedback attracted much attention in the last decade. This study was largely motivated by applications to networked control systems (see, e. g., [Hespanha et al., 2007]). Sampled-data stabilization of a sector bounded nonlinear system was treated in [Seifullaev and Fradkov, 2015a; Seifullaev and Fradkov, 2015b; Seifullaev and Fradkov, 2015c; Seifullaev and Fradkov, 2016; Zhang et al., 2017; Bryntseva and Fradkov, 2018].

For networked control systems it is desired that the sampling rate of a stabilizing feedback signal should be sufficiently low. As for the intersample control, most of researchers employ the zero-order hold (ZOH) scheme, when the control function is kept constant throughout the sampling interval. It is well known that a system with a ZOH feedback can be rewritten as a system with a linearly increasing time-varying delay (see, e. g., [Fridman, 2010]), and this delay is the greater, the greater the length of the sampling interval. Such long delays have a destabilizing effect on the system's behavior. It looks reasonable to replace ZOH for something more sophisticated, which allows to reduce the negative influence of the delay. A reverse sawtooth picewise-linear control was introduced to this end.

This paper continues a series of works [Churilov, 2018; Churilov, 2019a; Churilov, 2019b] devoted to an application of the absolute stability theory to sampled-data stabilization. For the stability analysis of the obtained hybrid system we use the Gelig's averaging method [Gelig, 1982; Gelig and Churilov, 1993b; Gelig and Churilov, 1998] reformulated in terms of linear matrix inequalities [Boyd et al., 1994]. Besides the averaging, we employ such methods as S-procedure for multiple quadratic forms [Fradkov and Yakubovich, 1979; Yakubovich, 1992] and integral quadratic constraints (see, e. g., [Yakubovich, 1968; Yakubovich, 1988; Megretski and Rantzer, 1997; Yakubovich, 2002]).

The paper is organized as follows. First the system equations are given. Then we present quadratic and integral quadratic constraints that will be needed to prove the main theorem. Further, the main result of the paper is formulated and proven. Finally, we provide an illustrative numerical example. The upper bounds of the sampling period obtained by the theorem of this paper are compared with the similar bounds for ZOH stabilization.

2 Problem Setting

Consider a nonlinear system under a sampled feedback

$$\dot{x} = Ax(t) + B_0 f_0(t) + Bu(t), \tag{1}$$

$$\sigma_0(t) = C_0 x(t), \quad \sigma(t) = Kx(t). \tag{2}$$

Here A, B, B_0, K, C_0 are constant matrix coefficients of sizes $p \times p$, $p \times 1$, $p \times 1$, $1 \times p$, and $1 \times p$, respectively, K is a vector of feedback gains.

The function $f_0(t)$ is defined as $f_0(t) = \varphi(\sigma_0(t), t)$, where the nonlinearity $\varphi(\cdot, \cdot)$ satisfies a sectoral constraint

$$\mu_1 \leqslant \frac{\varphi(\sigma_0, t)}{\sigma_0} \leqslant \mu_2 \quad \text{for all} \quad \sigma_0 \neq 0,\ t, \tag{3}$$

μ_1, μ_2 are some given numbers (see Figure 1(a)). We also assume that there exists a number μ_0 such that the function $\varphi(\sigma_0, t) - \mu_0 \sigma_0$ is bounded for all σ_0, t.

Consider a system with a uniform sampling with the period T. The feedback control function $u(\cdot)$ is defined as

$$u(t + nT) = \sigma(nT)\left(1 - \tfrac{t}{T}\right), \quad 0 \leqslant t < T, \tag{4}$$

$n = 0, 1, \ldots$. The function $u(t)$ is a reverse sawtooth wave that linearly ramps downwards (for $\sigma(nT) > 0$) or upwards (for $\sigma(nT) < 0$) until it reaches zero (see Figure 1(b)). We are interested in obtaining effectively verifiable conditions for asymptotic to zero of the solutions of system (1)–(4).

3 The Idea of Averaging

The pulse averaging technique that we use here originates from the principle of equivalent areas (PEA) [Andeen, 1960a; Andeen, 1960b]. For sector bounded constraints it was firstly obtained in [Gelig, 1982] and the refined in [Gelig and Churilov, 1993a; Gelig and Churilov, 1998]. Following the style of that time, the mathematical statements in [Gelig and Churilov, 1998] were formulated in terms of frequency-domain inequalities, however they can be techically reformulated as LMI by using the celebrated Kalman–Yakubovich–Popov (KYP) lemma (see, e. g., [Popov, 1973; Boyd et al., 1994; Yakubovich et al., 2004]).

Following [Gelig, 1982], compute the average of the nth control pulse

$$v_n = \frac{1}{T}\int_{nT}^{nT+T} u(t)\,dt. \tag{5}$$

Let the function $w(t)$ be the integrated error of replacing $u(t)$ for its averages:

$$\begin{aligned} \dot{w}(t) &= u(t) - v_n, \quad nT \leqslant t < nT + T, \\ w(nT) &= 0 \end{aligned} \tag{6}$$

for any $n \geqslant 0$. From (5) it follows that the function $w(t)$ is continuous and turns to zero at all the sampling instants nT. The main mathematical technique used in [Gelig and Churilov, 1993a; Gelig and Churilov, 1998] was the construction of quadratic Lyapunov fuctions that depend on $x(t)$ and $w(t)$. The functions $x(t)$, $w(t)$ and the sequence v_n were linked by quadratic and integral quadratic constraints. When the sampling frequency tends to infinity, the Gelig's criteria reduce to classical criteria of absolute stability, such as circle or Popov.

The physical aspects of PEA were discussed in [Churilov, 2019a].

Notice that some distant similarity in technique can be found in later works [Briat and Seuret, 2012; Seuret, 2012; Briat, 2013], where *loop functionals* were introduced and explored. These functionals also become zero at sampling times. However, looped functionals were not related to averaging and were usually added to Lyapunov–Krasovskii functionals.

4 Quadratic and Integral Quadratic Constraints

For brevity, denote $t_n = nT$, $n = 0, 1, \ldots$. Besides the sectoral bound (3), we will need a number of auxiliary constraints. Following the averaging scheme described in the previous section, from (4), (5) we get

$$v_n = \tfrac{1}{2}\,\sigma(t_n). \tag{7}$$

Define a piecewise constant function

$$v(t) = v_n, \quad t_n \leqslant t < t_{n+1}. \tag{8}$$

By a straightforward calculation, from (6) we have

$$w(t) = \frac{v_n}{T}(t_{n+1} - t)(t - t_n), \quad t_n \leqslant t \leqslant t_{n+1}. \tag{9}$$

Notice that $|w(t)| \leqslant \frac{1}{4}T\,|v(t)|$. Additionally, a quadratic constraint

$$v(t)w(t) \geqslant 0, \quad t_n < t < t_{n+1}, \tag{10}$$

is valid. By a direct calculation we get

$$\begin{aligned} \int_{t_n}^{t_{n+1}} w(t)\,dt &= \tfrac{1}{6}\,v_n T^2 \\ \int_{t_n}^{t_{n+1}} w(t)^2\,dt &= \tfrac{1}{30}\,v_n^2 T^3. \end{aligned} \tag{11}$$

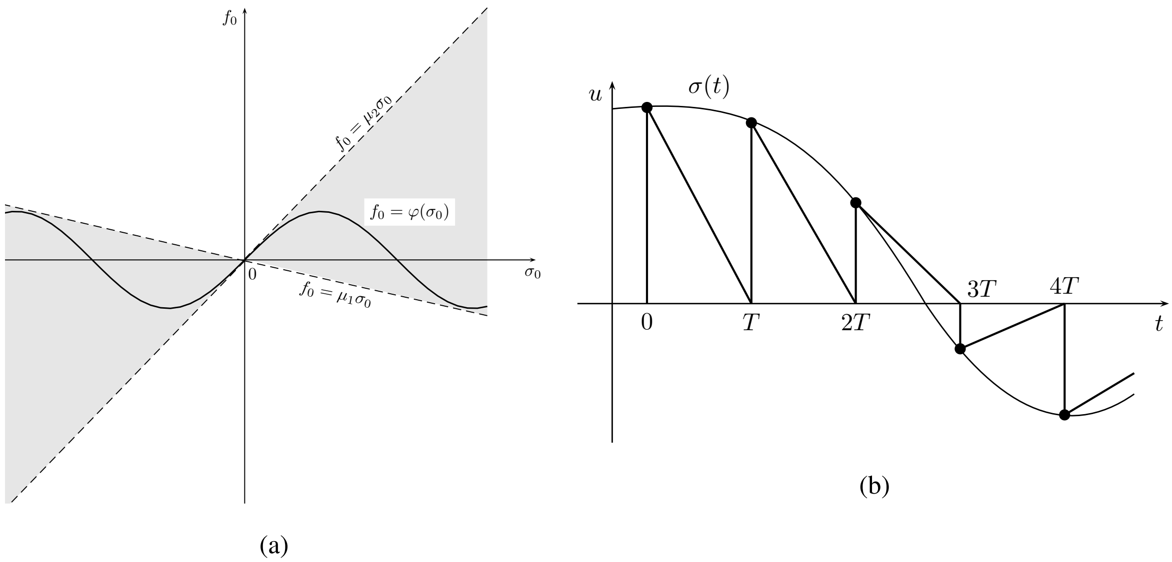

Figure 1. (a) An example of a bound sector for $\varphi(\sigma_0, t) \equiv \varphi(\sigma_0)$. (b) The sampling scheme.

Equalities (8) and (11) imply

$$T \int_{t_n}^{t_{n+1}} v(t)w(t)\,dt = 5 \int_{t_n}^{t_{n+1}} w(t)^2\,dt. \tag{12}$$

Equation (12) can be rewritten as an integral quadratic constraint

$$\int_{t_n}^{t_{n+1}} \left(Tv(t)w(t) - 5w(t)^2\right)\,dt = 0 \tag{13}$$

for all $n \geqslant 0$.

Introduce a picewise-continuous function

$$\xi(t) = \sigma(t) - \sigma(t_n) - KB\,w(t)$$

for $t_n \leqslant t < t_{n+1}$, $n \geqslant 0$. Hence

$$\xi(t) = Kx(t) - 2v(t) - KB\,w(t), \quad t \geqslant t_0. \tag{14}$$

Obviously, $\xi(t_n^+) = 0$, $n \geqslant 0$ and

$$\dot\xi(t) = KAx(t) + KB_0\,f_0(t) + KB\,v(t) \tag{15}$$

for $t \neq t_n$. Then Wirtinger inequality [Gelig and Churilov, 1993b; Gelig and Churilov, 1998] implies

$$\int_{t_n}^{t_{n+1}} \xi(t)^2\,dt \leqslant \frac{4T^2}{\pi^2} \int_{t_n}^{t_{n+1}} \dot\xi(t)^2\,dt \tag{16}$$

for $n \geqslant 0$. Additionally, we have an obvious constraint

$$\int_{t_n}^{t_{n+1}} \xi(t)\dot\xi(t)\,dt \geqslant 0. \tag{17}$$

If we introduce a vector column

$$X(t) = \mathrm{col}\{x(t),\, f_0(t),\, v(t),\, w(t)\}, \tag{18}$$

then from (14), (15) we obtain

$$\xi(t) = DX(t), \quad \dot\xi(t) = D_1X(t), \tag{19}$$

where

$$\begin{aligned} D &= [\;K \quad 0 \quad -2 \quad -KB\;], \\ D_1 &= [\;KA \quad KB_0 \quad KB \quad 0\;]. \end{aligned} \tag{20}$$

5 The Main Statement

Theorem 1. *Assume that there exist a symmetric positive definite $p \times p$ matrix H, nonnegative numbers ε_0, ε_2, ε_3 and a number ε_1 of any sign, such that*

$$\begin{aligned} &\Pi + \varepsilon_2\big(\Delta^2 D_1^\top D_1 - D^\top D\big) \\ &\quad + \varepsilon_3\big(D^\top D_1 + D_1^\top D\big) < 0, \end{aligned} \tag{21}$$

where $\Delta = 2T/\pi$, Π is a symmetric $(p+3) \times (p+3)$ matrix with the components

$$\begin{aligned} &\Pi_{11} = HA + A^\top H - \varepsilon_0\mu_1\mu_2 C_0^\top C_0, \\ &\Pi_{12} = HB_0 + \tfrac{1}{2}\,\varepsilon_0(\mu_1 + \mu_2)C_0^\top, \\ &\Pi_{13} = HB, \quad \Pi_{14} = -A^\top HB, \\ &\Pi_{22} = -\varepsilon_0, \quad \Pi_{23} = 0, \quad \Pi_{24} = -B^\top HB_0, \\ &\Pi_{33} = 0, \quad \Pi_{34} = T\varepsilon_1 - B^\top HB, \\ &\Pi_{44} = -10\varepsilon_1 \end{aligned} \tag{22}$$

and D, D_1 are defined by (20). *Then any solution of system* (1)–(4) *satisfies $x(t) \to 0$ as $t \to +\infty$ and $x(t) \in L_2[t_0, +\infty)$.*

Proof. We will apply the S-procedure with multiple quadratic forms [Fradkov and Yakubovich, 1979; Yakubovich, 1992]. Consider a $(p+3)$-dimensional vector column

$$X = \mathrm{col}\{x,\, f_0,\, v,\, w\}, \tag{23}$$

and introduce a quadratic form

$$\begin{aligned}\Phi(X) &= \varepsilon_0(\mu_2 C_0 x - f_0)(f_0 - \mu_1 C_0 x)\\ &+ 2\varepsilon_1\left(Tvw - 5\,w^2\right)\\ &+ \varepsilon_2\left(\Delta^2(D_1X)^2 - (DX)^2\right)\\ &+ 2\varepsilon_3\, D_1X\, DX.\end{aligned} \tag{24}$$

(The numbers ε_i, $0 \leqslant i \leqslant 3$, play the role of Lagrange multipliers.)

Inequality (21) can be rewritten as

$$\begin{aligned}2(x - Bu)^\top H(Ax + B_0 f_0 + Bv)&\\ + \Phi(X) \leqslant -\delta_0\|X\|^2&\end{aligned} \tag{25}$$

for all vector columns X with coordinates (23). Here δ_0 is a sufficiently small positive number. Let us take a Lyapunov function $V(x, w) = (x-Bw)^\top H(x-Bw)$. Then along the solutions of system (1)–(4) inequality (25) implies

$$\dot V(x(t), w(t)) + \Phi(X(t)) \leqslant -\delta_0\|X(t)\|^2 \tag{26}$$

for any sampling interval $t_n < t < t_{n+1}$. From quadratic bounds (3), (10) and integral quadratic constraints (13), (16), (17) we obtain

$$\int_{t_n}^{t_{n+1}} \Phi(X(t))\, dt \geqslant 0 \tag{27}$$

for all $n \geqslant 0$. Integrating (26) and taking (27) into account, we get

$$\begin{aligned}&V(x(t_{n+1}), 0) - V(x(t_n), 0)\\ &\qquad \leqslant -\delta_0 \int_{t_n}^{t_{n+1}} \|x(t)\|^2\, dt - \delta_0 T v_n^2\end{aligned}$$

for all $n \geqslant 0$. The rest of the proof reproduces the proof of Theorem 1 [Churilov, 2018]. □

6 Necessary Conditions for the Fulfillment of the Main Statement

Preposition 1. *Let conditions of Theorem 1 be satisfied. Then the matrix*

$$A_\mu = A + \mu B_0 C_0 + \tfrac{1}{2}\, BK$$

is Hurwitz stable for any number μ, $\mu_1 \leqslant \mu \leqslant \mu_2$.

Proof. Let us put

$$f_0 = \mu C_0 x, \quad v = \tfrac{1}{2}\, Kx, \quad w = 0 \tag{28}$$

in (24), (25). Then $DX = 0$ and (25) implies

$$HA_\mu + A_\mu^\top H < 0.$$

Since $H > 0$, the Hurwitz stability of A_μ follows. □

7 Numerical Example

Consider the system (see [Seifullaev and Fradkov, 2015c])

$$\begin{aligned}\dot x_1 &= -2x_1 + \sin x_2,\\ \dot x_2 &= x_1 - x_2 + 2\sin x_2 - u(t).\end{aligned} \tag{29}$$

Here $\sigma_0(t) = x_2(t)$, $\varphi_0(\sigma_0) = \sin\sigma_0$, and we can take $\mu_1 = -0.2173$, $\mu_2 = 1$. The control function $u(t)$ is defined by (4), where $\sigma(t) = kx_2(t)$ and k is a scalar feedback gain. Thus system (29) can be written in the form of (1)–(4) with

$$A = \begin{bmatrix}-2 & 0\\ 1 & -1\end{bmatrix}, \quad B_0 = \begin{bmatrix}1\\ 2\end{bmatrix}, \quad B = \begin{bmatrix}0\\ -1\end{bmatrix},$$
$$C_0 = [0 \quad 1], \quad K = [0 \quad k].$$

Feasibility of inequalities (21) of Theorem 1 was explored with the help of YALMIP software package for interface and SeDuMi solver for semidefinite programming [Löfberg, 2004; Sturm, 1999].

System (29) was previously studied in [Seifullaev and Fradkov, 2015c] with the use of ZOH control

$$u(t) = \sigma(t_n), \quad t_n \leqslant t < t_{n+1}.$$

The results of computer experiments are consolidated in Table 1. The computer experiment was produced for the values $k = 2, 3, 4, 5, 10$ (see column 1). The maximal values of T computed with the help of Theorem 1 are given in column 2. Column 3 contains the maximal values of T obtained by the Lyapunov-Krasovskii method for the case of ZOH control in [Seifullaev and Fradkov, 2015c] (the value $k = 4$ was not considered). For $k = 2, 3$ Theorem 1 gives no feasible values, but for $k = 5, 10$ the maximum of T obtained from Theorem 1 is greater than that was found in [Seifullaev and Fradkov, 2015c].

The comparison of the reverse sawtooth (RS) control and the ZOH control is presented in Table 2. The maximal values of T are obtained by a direct computer simulation under the T-periodic sampling (i. e., they are irrelevant to any stability criteria). It is seen that for the same feedback gain k the RS control provides a sampling period T approximately 1.9 times greater than that of ZOH control.

Some transients for system (29) with $k = 5$ and the T-periodic sampling are shown in Figures 2,3. With the increase of T the zero equilibrium turns from a stable node to a stable focus. With the further increase of T this focus loses stability.

Table 1. Maximal values of T computed according to Theorem 1 and their comparison with the values obtained in [Seifullaev and Fradkov, 2015c] for ZOH feedback

k	T (Theorem 1)	T (Seifullaev, Fradkov, 2015)
2	–	0.68
3	–	0.53
4	0.56	?
5	0.51	0.35
10	0.29	0.187

Table 2. Maximal values of T simulated for Reverse Sawtooth (RS) feedback and for ZOH feedback

k	Simulated T (RS)	Simulated T (ZOH)
2	–	1.21
3	1.38	0.71
4	0.98	0.51
5	0.77	0.40
10	0.38	0.20

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8 Conclusion

The paper introduces a new control function based on the reverse sawtooth signal wave, which can significantly increase the sampling period compared with the zero-order hold. Based on the Gelig–Yakubovich mathematical technique, the paper suggests an easily verifiable stability criterion formulated in terms of linear matrix inequalities. Simulation shows its reasonable conservatism and an agreement with the previously obtained results.

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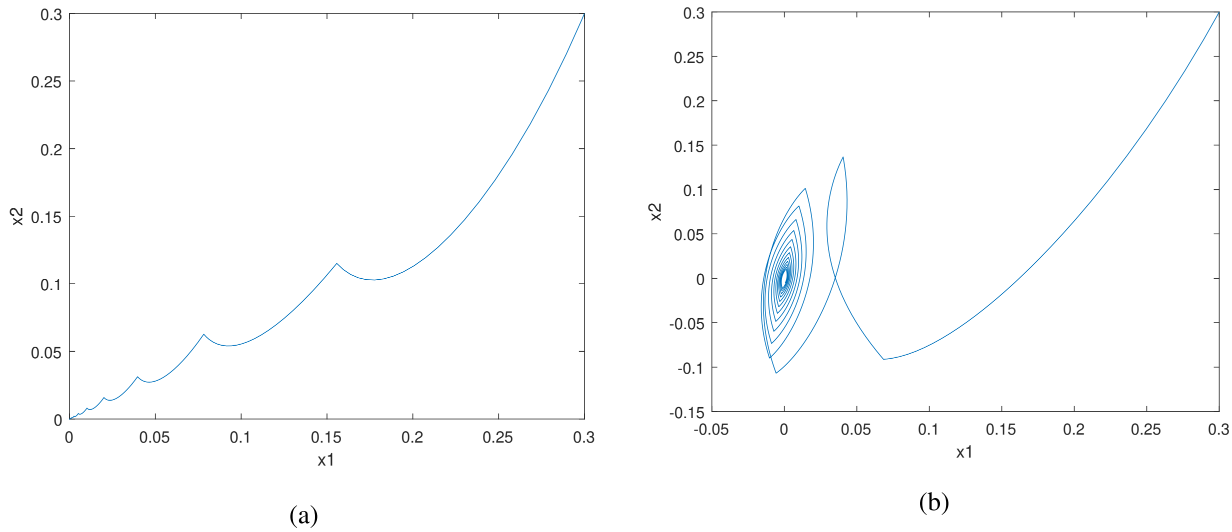

Figure 2. A trajectory enters zero with the increase of time. (a) Non-oscillative behavior, $k = 5, T = 0.5$ (b) Oscillative behavior, $k = 5, T = 0.75$

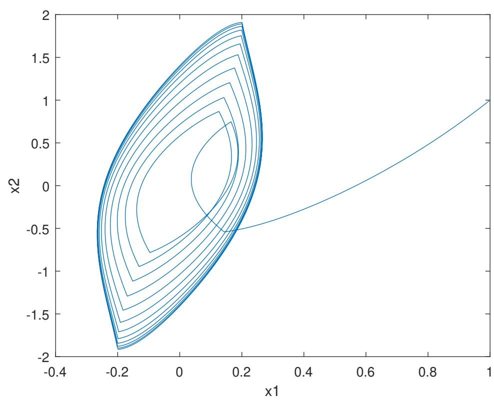

Figure 3. Stabilization fails, the trajectory tends to infinity (oscillative behavior), $k = 5, T = 0.8$

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