CYBERNETICS AND PHYSICS, VOL. 3, NO. 3, 2014 , 124–129

EXACT PENALTY FUNCTIONS METHOD FOR SOLVING PROBLEMS OF NONDIFFERENTIABLE OPTIMIZATION

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Abstract
In the article a method of exact penalty functions for minimizing a quasidifferentiable function under quasidifferentiable constraints is discusses. A regularity condition for the function which defines a constraint is introduced and prove that when it is running, there is an exact penalty parameter. The case when the constraint is convex is studied in detail.

Key words
Constrained optimization, quasidifferentiable functions, regularity condition, exact penalty functions.

1 Introduction

In nonlinear programming methods of penalty functions both external and internal have widespread in the development of algorithms for the constrained optimization. The basic idea of these techniques is to replace the conditional optimization problem to sequence of auxiliary unconstrained optimization problems. An auxiliary function in the methods of penalty functions is chosen so that it is coincided with the objective function for a given set and is increased outside this set.

The methods of exact penalty functions are received much attention at the present time. Penalty functions have always taken an important role in solving many constrained optimization problems in the fields such as industry design and management science.

In this regard, we note an interesting class of problems arising in the solution of various optimization problems of dynamics of charged particle beams and plasma [Zavadsky, Ovsyannikov, and Chung, 2009; Ovsyannikov *et al.*, 2009; Ovsyannikov, 1990; Ovsyannikov, 2009]. Investigated here problems of parametric optimization and minimax estimation often leads to the optimization of function of many variables subject to various constraints.

The existence of an exact penalty parameter for problems of the convex programming was seen by I.I.Ereminym [Eremin, 1967], a little later - U.I.Zangvilom [Zangwill, 1967]. Subsequently, this problem was the subject of many works [Demyanov, Giannessi, and Karelin, 1998; Demyanov *et al.*, 1996; Fletcher, 1973; Han and Mangasarian, 1979; Polyakova, 2001a].

2 Method of Penalty Function

Formulate some statements that hold for the penalty function method. Consider the problem of the constrained optimization

$$\inf_{x\in X} f(x), \tag{1}$$

where a function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous,

$$X = \{\, x \in \mathbb{R}^n \;\big|\; \varphi(x) = 0\}, \tag{2}$$

a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is also continuous and

$$\varphi(x) > 0 \quad \forall x \notin X.$$

Then the set X is the closed set of the points of global minimizers of φ on $\mathbb{R}^n$. For solving the problem (1) by this method we introduce penalty functions

$$F(c, x) = f(x) + c\varphi(x),$$

where c is a nonnegative number, called *a penalty parameter*. Then consider the unconstrained minimization problem

$$\inf_{x\in\mathbb{R}^n} F(c, x). \tag{3}$$

Let infimum in (3) be attained for each nonnegative number c. Denote by

$$f^* = \min_{x\in X} f(x), \quad F^*(c) = \min_{x\in\mathbb{R}^n} F(c,x),$$

$$x(c) = \arg\min_{x\in\mathbb{R}^n} F(x,c). \quad x^{**} = \arg\min_{x\in\mathbb{R}^n} f(x),$$

$$f^{**} = f(x^{**}).$$

Let $f^{**} > -\infty$. Notice that $f^* \geq F^*(c)$ for each positive c.

Choose a monotonically increasing sequence $\{c_k\}(k = 1, 2, \dots),\ c_k > 0,\ c_k \to +\infty$.

Theorem 1. *If* $\{x(c_k)\}$ *is a sequence of solutions of the auxiliary problem* (3) *then*

$$1) \qquad F^*(c_k) = F(c_k, x(c_k)) \leq$$

$$\leq F(c_{k+1}, x(c_{k+1})) = F^*(c_{k+1});$$

$$2) \qquad f(x(c_k)) \leq f(x(c_{k+1}));$$

$$3) \qquad f(x(c_k)) \leq F(c_k, x(c_k)) \leq f^*.$$

Lemma 1. *Under the above assumptions the inequality*

$$\varphi(x(c_{k+1})) \leq \varphi(x(c_k)) \quad \forall k > 0, \tag{4}$$

is true.

Denote by

$$\mathcal{L}(\varphi, x^{**}) = \{x \in \mathbb{R}^n \mid \varphi(x) \leq \varphi(x^{**})\}.$$

From the inequality (4) it follows that all points of the sequence $\{x(c_k)\}$ are in the set $\mathcal{L}(x^{**})$.

Lemma 2. *For any sequence of positive numbers* $\{c_k\},\ c_k \to +\infty$, *the formula*

$$\varphi(x(c_k)) \underset{k\to+\infty}{\longrightarrow} 0.$$

is correct.

Thus, the sequence of the points $\{x(c_k)\}$ is a minimizing sequence for the function φ.

3 Quasidifferential Functions

Let a function f be defined on $\mathbb{R}^n$ and be directionally differentiable at a point $x \in \mathbb{R}^n$ and its directional derivative $f'(x,g)$ can be represented in the form [Demyanov and Rubinov, 1995]

$$f'(x,g) = \lim_{\lambda\downarrow 0} \frac{f(x+\lambda g) - f(x)}{\lambda} =$$

$$= \max_{v\in\underline{\partial} f(x)} \langle v, g\rangle + \min_{w\in\overline{\partial} f(x)} \langle w, g\rangle,$$

where $\underline{\partial} f(x) \subset \mathbb{R}^n,\ \overline{\partial} f(x) \subset \mathbb{R}^n$ are convex compact sets in $\mathbb{R}^n$. The function f is called *a quasidifferential* at a point $x \in \mathbb{R}^n$. A pair of sets $\mathcal{D}f(x) = [\underline{\partial} f(x), \overline{\partial} f(x)]$ is called *a quasidifferentiable* of quasidifferentiable function f at x. The set $\underline{\partial} f(x) \subset \mathbb{R}^n$ is called a subdifferential of f at x, the set $\overline{\partial} f(x) \subset \mathbb{R}^n$ is called a superdifferential of f at x. Differentiable, convex, concave functions, the maximum functions are quasidifferentiable functions.

4 Exact Penalty Functions

Exact penalty functions are the penalty functions for which there exists a parameter $c^* > 0$ that for any $c \geq c^*$ the set of minimum points of $F(c,x)$ on $\mathbb{R}^n$ coincides with the set of solutions of (1). A parameter c^* is called *an exact penalty parameter* for the family of functions $F(c,x)$. Therefore, any number greater then an exact penalty parameter is also an exact penalty parameter. Obviously, the implementation of the method of exact penalty functions is primarily dependent from the properties of the function φ.

In practice it would be useful to find conditions which guarantee that there exists the exact penalty parameter $c^* \geq 0$ such that the set

$$\{x \in \mathbb{R}^n \mid x = \arg\min_{x\in\mathbb{R}^n} F(c,x)\}$$

coincides with the set

$$\{x \in \mathbb{R}^n \mid x = \arg\min_{x\in X} f(x)\}$$

The implementation of exact penalty functions methods first of all depends on the properties of the function φ. Therefore various conditions are imposed on φ to make it possible to solve our problem.

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz quasidifferentiable function given the set X of the form (2).

Regularity condition. [Polyakova, 2001b]

We say, that a regularity condition is satisfied for the function φ if for each boundary point $x^* \in \text{bd } X$ there exist positive real numbers $\varepsilon(x^*)$ and $\beta(x^*)$, such that

$$\frac{o(\alpha, \bar{x}, g)}{\alpha} > -\varphi'(\bar{x}, g) + \beta(x^*), \tag{5}$$

$$\forall \alpha \in (0, \varepsilon(x)],\ \forall \bar{x} \in \mathcal{A}(X) \cap S_{\varepsilon(x)}(x),$$

$$\forall g \in N(X, \bar{x}),\ ||g|| = 1.$$

where

$$\mathcal{A}(X) = \{x \in \text{bd}(X) \mid \exists z \notin X, x = \text{pr}(z, X)\},$$

and $N(X, x)$ is the normal cone to the set X at the point $x \in X$, an expression $x = \text{pr}(z, X)$ means that the point $x \in X$ is the orthogonal projection of the point $z \notin X$ on the set X.

The regularity condition is the condition about the behavior of the function φ only at the boundary points of the set X.

We can prove that in order to satisfy the regularity condition it is necessary that the function φ is nondifferentiable at boundary points of the set X.

Theorem 2. *If the set X is compact and the regularity condition is satisfied for $F(c_k, x)$ then there exists an exact penalty parameter c^*, and*

$$x(c_k) \in X \quad \forall c_k \geq c^*.$$

Theorem 3. *If for the function φ at each boundary point $x^* \in X$ the regularity condition is hold, then*

$$\Gamma(X, x^*) = \gamma_0(X, x^*) \tag{6}$$

where

$$\Gamma(X, x^*) = \text{cl}\Big\{g \in \mathbb{R}^n \mid x_k \in X,\ x_k \neq x^*,$$
$$x_k \to x^*,\ \frac{x_k - x^*}{||x_k - x^*||} \to \frac{g}{||g||}\Big\}.$$

$$\gamma_0(X, x^*) = \{g \in \mathbb{R}^n \mid \langle \varphi'(x^*, g), g\rangle = 0\}.$$

The cone $\Gamma(X, x^*)$ is called *the cone of feasible directions at a point x^* to the set X.*

Lemma 3. *If for the function φ defining the set X* of the form (2) *at $x^* \in \mathcal{A}(X)$ the regularity condition 1 is hold, then*

$$\varphi'(x^*, g) \geq \beta(x^*) \quad \forall g \in N(X, x^*), \quad ||g|| = 1.$$

Theorem 4. If there is a positive number $\beta(X)$, that the inequality

$$\inf_{x \in \text{bd}\,(X)} \min_{\substack{\|g\| = 1,\\ g \in N(X, x)}} \varphi'(x, g) \geq \beta(X), \tag{7}$$

holds, then for the family of penalty functions $\{F(c_k, x)\}$ there exist the exact penalty parameter.

Note that the condition (7) is more constructive than the regularity condition (5), and it can also be used as a condition of regularity in the problem of the existence of the exact penalty parameter.

Example. Let

$$\varphi(x) = \max\{0, f_1(x), f_2(x)\}, \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

where

$$f_1(x) = x_1^2 + x_2^2 - 1, \quad f_2(x) = -(x_1 - 1)^2 - x_2^2 + 1.$$

The set $X = \{x \in \mathbb{R}^2 \mid \varphi(x) = 0\}$ is not convex. Consider boundary points of X. Let at the point $x^1 \in X$

$$f_1(x^1) = 0,\ f_2(x^1) < 0,$$

$x^1 \in \mathcal{A}(X)$ and the regularity condition 1 be hold for the set X then the condition (6) is hold and

$$\underline{\partial}\varphi(x^1) = \text{co}\,\{(0, 0), f_1'(x^1)\},\ \overline{\partial}\varphi(x^1) = (0, 0),$$

$$N(X, x^1) = \{g \in \mathbb{R}^2 \mid g = \lambda f_1'(x^1), \quad \lambda \geq 0\}.$$

Then

$$\min_{\substack{\|g\| = 1,\\ g \in N(X, x^1)}} \varphi'(x^1, g) = 2.$$

If $x^1 \in \mathbb{R}^2$ and

$$f_1(x^1) < 0, \quad f_2(x^1) = 0,$$

then $x^1 \in \mathcal{A}(X)$ and the condition (6) is hold and

$$\underline{\partial}\varphi(x^1) = \text{co}\,\{(0, 0), f_2'(x^1)\},\ \overline{\partial}\varphi(x^1) = (0, 0),$$

$$N(X, x^1) = \{g \in \mathbb{R}^2 \mid g = \lambda f_2'(x^1), \lambda \geq 0\}.$$

Then

$$\min_{\substack{\|g\| = 1,\\ g \in N(X, x^1)}} \varphi'(x^1, g) = 2.$$

Let

$$x^1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right),\ x^2 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right).$$

Then

$$f_1(x^1) = 0,\ f_2(x^1) = 0,\ \ f_1(x^2) = 0,\ f_2(x^2) = 0,$$

$x^1, x^2 \in \mathcal{A}(X)$ and the condition (6) is hold. Then

$$\underline{\partial}\varphi(x^1) = \text{co}\,\{(0,0), f_1'(x^1), f_2'(x^1)\} =$$

$$= \text{co}\,\{(0,0), (1, \sqrt{3}), (1, -\sqrt{3})\},\ \overline{\partial}\varphi(x^1) = (0,0),$$

$$\underline{\partial}\varphi(x^2) = \text{co}\,\{(0,0), f_1'(x^2), f_2'(x^2)\} =$$

$$= \text{co}\,\{(0,0), (1, -\sqrt{3}), (1, \sqrt{3})\},\ \overline{\partial}\varphi(x^1) = (0,0),$$

and

$$\min_{\substack{\|g\| = 1, \\ g \in N(X, x^1)}} \varphi'(x^1, g) = 1,$$

$$\min_{\substack{\|g\| = 1, \\ g \in N(X, x^2)}} \varphi'(x^2, g) = 1.$$

Hence, $\beta(X)$ can be equal 1.

5 Convex Constraints

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a convex function. In this case at a point $x \in X$ the equality

$$\gamma_0(X, x) = -\left(\text{cone}\ (\partial\varphi(x))\right)^*,$$

holds, where through cone (A) denotes the conical hull of a set A, $\partial\varphi(x)$ is the subdifferential of φ at $x \in$ bd X in the sense of convex analysis.

Using this formula write the analytic representation of the normal cone $N(X, x)$ to the set X at $x \in$ bd (X)

$$N(X, x) = \text{cl cone}\ (\partial\varphi(x)).$$

Then at $x \in$ bd (X) the formula

$$\min_{\substack{\|g\| = 1, \\ g \in N(X, x)}} \varphi'(x, g) = \beta(x) > 0.$$

holds. Thus, if

$$\inf_{x \in \text{bd}\,(X)} \beta(x) = \beta(X) > 0,$$

then the function φ can be used for constructing of the family of exact penalty functions.

Remark. From the condition (7) is followed that the existence of the exact penalty parameter depends on the behavior of the function φ at the boundary points of the set X, belonging to the set $\mathcal{A}(X)$.

Consider the optimization problem: find

$$\inf_{x \in X} f(x),$$

where

$$X = \{x \in \mathbb{R}^n \mid f_1(x) \le 0\} =$$

$$= \{x \in \mathbb{R}^n \mid \max\{0, f_1(x)\} \le 0\}, \qquad (8)$$

the function $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz, and $f_1 : \mathbb{R}^n \to \mathbb{R}$ is convex. Obviously, X is convex.

Assume that the set X is not an isolated point and the Slater's condition holds, i.e., there is a point $\hat{x} \in X$ for which $f_1(\hat{x}) < 0$. Consider the case where the minimizer of f does not belong to the set X. By virtue of the Slater's condition the set X is a convex body (set with non-empty interior). Let

$$\varphi(x) = \max\{f_1(x), 0\}.$$

If $x \in \text{bd}(X)$ then

$$\partial\varphi(x) = \text{co}\,\{\partial f_1(x), 0\},$$

If for any boundary point $x \in \text{bd}(X)$ a supporting hyperplane is unique, then the set X is called *a body with smooth boundary* or *smooth body*. For a smooth convex body the normal cone $N(X, x)$ at each boundary point $x \in$ bd (X) consists of a ray, spanned by the normal of the supporting hyperplane. Therefore, if the function f_1 is continuously differentiable, then the surface

$$X_1 = \{x \in \mathbb{R}^n \mid f_1(x) = 0\}$$

is a smooth manifold with $X_1 =$ bd (X) and

$$N(X, x) = \{g \in \mathbb{R}^n \mid g = \lambda f_1'(x),\ \lambda \ge 0\}.$$

5.1 Calculating of the Exact Penalty Parameter When the Function f_1 is Strongly Convex

Show that if the set X is given by using a strongly convex function f_1, then there exists the exact penalty parameter. Let f_1 be a strongly convex function on $\mathbb{R}^n$ and $m > 0$ is its a constant strong convexity. From convex analysis [Rockafellar, 1970] it is known that

$$\langle v(x) - v(y), x - y \rangle \geq 2m\|x - y\|^2$$

$$\forall v(x) \in \partial f_1(x),\ \forall v(y) \in \partial f_1(y),\ \forall x, y \in \mathbb{R}^n.$$

Let x_1^* be a minimizer of f_1 on $\mathbb{R}^n$. Then $x_1^* \in \text{int}\, X$ and

$$||v(x)|| \geq 2m||x - x_1^*||\ \forall x \in \mathbb{R}^n. \quad (9)$$

In addition, for each $x_0 \in \mathbb{R}^n,\ f(x_0) > f(x_1^*)$ and

$$L(f_1, x_0) = \{x \in \mathbb{R}^n \mid f_1(x) \leq f_1(x_0)\}$$

is compact.

Let $S(x_1^*, r)$ be a ball of the maximum radius centered at x_1^*, contained in the set X. Then from (9) we have

$$||v(x)|| \geq 2mr\ \forall v(x) \in \partial f_1(x),\ \forall x \notin S(x_1^*, r).$$

Thus, for every boundary point of the set X

$$||v(x)|| \geq 2mr \quad \forall x \in \text{ bd}X.$$

Theorem 5. If the function f_1 is strongly convex, the set X is given in the form (8), then for the function φ, defining a set at any boundary point of the set the regularity condition (5) holds and

$$\beta(X) = \min_{x \in \text{bd}\, X} \min_{\substack{\|g\| = 1, \\ g \in N(X,x)}} \varphi'(x, g) \geq 2md > 0.$$

In this case the exact penalty parameter c^* is equal to $c^* = \dfrac{L}{2md}$, where L is a constant Lipschitz of the function f on the set $\mathcal{L}(\varphi, x^{**})$.

Acknowledgements

This work is supported by RFFI in project of 12-01-00752. The authors acknowledge Saint Petersburg State University for a research grant 9.38.205.2014.

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