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# MATHEMATICAL MODEL OF THE MOTION OF A BODY THROUGH A BORDER OF MULTIPHASE MEDIA

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**Abstract**
Investigations of the physical objects and phenomena is an interesting and complex task. This is especially true in regard to the cybernation of scientific research. This paper deals with a model of a cylindrical body displacements through border of two viscous media. The model enables one to set up the problem of displacements of a cylindrical body for optimum energy consumption, the time and distance of the displacement being given. The problem has a number of special features. First, it is irregular [Krasovskii, 1968], because the Euler–Lagrange equations do not contain controls in an explicit form, and, hence, the optimal controls cannot be determined in terms of the state and adjoint variables. Second, as it was found out, there are impulse components in the control forces and momentums optimum programs. Therefore, the classical variational techniques cannot be directly applied to find these programs. The third feature follows from the second one and consists of calculating the energy consumption. So we consider a new problem from the viewpoint of the theory of singular or degenerate [Gurman, 1985] solutions of dynamic optimization problems.

**Key words**
Control problem, nonlinear systems, fluid dynamics, modeling.

## 1 Introduction

Recently, control problems of moving objects in various media are relevant and interesting. Various works produced in coastal shelves such as a lining of pipelines, search of mineral deposits, service works are at the bottom of the given researches [Beletskii, 1973]. Construction adequate 3D models considering all physical nuances is a difficult task. Therefore we will be limited to 2D consideration in absence of forces of a superficial tension. Models of this kind were considered in the book [Zavalishchins and Zavalishchin, 2002]. We also will consider only a movement through border of media. The created model can be used for design of perspective new machines.

## 2 Mathematical Model

Here we are attempt to construct a model in plane $Oxy$ of the body moving through a viscous media border (see Fig.1). The medium of smaller density is located above axis $Ox$. More dense one is located below axis $Ox$. Body movement only through the border of viscous media is considered. In the initial state $l_t = l$ and in final state $l_t = 0$. Here $l$ is the length of the body. Thus the length of a tail $l_t$ in the first medium changes from $l$ to $0$. Length of the body part being in the second one is equal $l - l_t$. The location of the body inertia center $l_c$ depends on size of body immersing in the second medium. It should be noted that the inertia center doesn't coincide with the center of mass.

### 2.1 Forces and Moments Acting on the Body

It is obvious that the state of the body is described by generalized coordinates $x$, $y$ and $\varphi$. Let $V$ be the vector of centroid velocity $\mathbf{V} = (\dot{x}; \dot{y})^T$, $F$ be the force acting along a body axis $\mathbf{F} = (F\sin\varphi; F\cos\varphi)^T$, $U$ be the angular moment, $E$ be the unit vector $\mathbf{E} = (\sin\varphi; \cos\varphi)^T$, $D$ and $D^{\perp}$ are the drag force and lift force respectively. It is necessary to note that because of presence of two viscous media drag forces and lift forces will be different. Let the body moves from the first medium 1 to the second medium 2. Forces acting in different media will create the moment. The drag force and lift force acting in $i$-th medium are equivalent to

$$\mathbf{D}_i = (-D_i \sin(\varphi - \alpha); -D_i \cos(\varphi - \alpha))^T, \quad \mathbf{D}_i^{\perp} = (D_i^{\perp} \cos(\varphi - \alpha); -D_i^{\perp} \sin(\varphi - \alpha))^T, \tag{1}$$

their resultant force acts at a point defined by $l_c$

$$l_c = \frac{(D_1 + D_2)l_t + D_2 l}{2(D_1 + D_2)}. \tag{2}$$

### 2.2 Description of the Hydrodynamic Forces, Coefficients and Limitations

Let a body of bounded size with sufficiently smooth boundary $S$ moves in fluid. One of the fluid mechanics axioms is the sticking condition: at the body surface points the velocity vector of fluid particle is equal to the velocity vector of the corresponding body point. This condition implies that in the case of translational motion of the body the following equality is fulfilled at its surface (see [Slezkin, 1955])

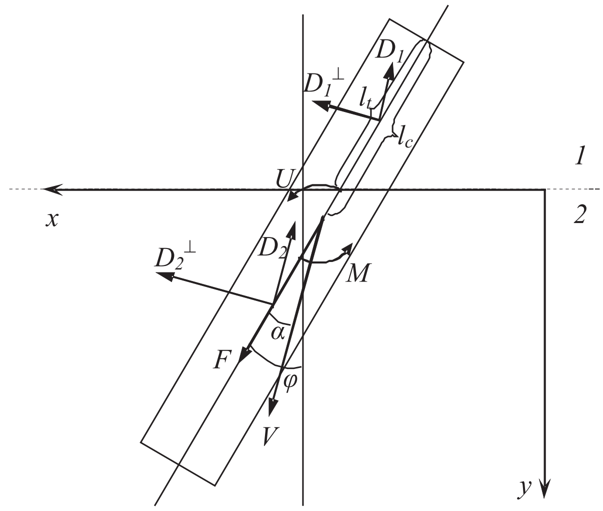


Figure 1. Forces and moments acting on a body.

$$\left(\frac{\partial \mathbf{v}}{\partial x}\right)^* \mathbf{n} = 0, \tag{3}$$

where $\mathbf{n}$ is the unit vector of the outward normal to the surface $S$ at the point $x$. The stress on an element $dS$ of the body surface is calculated by the formula $\mathbf{p}_n = P\mathbf{n}$, where $\mathbf{n}$ is the unit vector of the outward normal to $dS$. This equality and (3) yield the formula for the principal vector of the forces acting from fluid upon the body surface (hydrodynamic forces)

$$\mathbf{R} = \iint_S \left(-pE + \mu \frac{\partial \mathbf{v}}{\partial x}\right) \mathbf{n}\, dS. \tag{4}$$

Here $p$ is average normal stress at each point, $\mu$ is the dynamic viscosity.

We need further the so-called moving coordinate system $O_c y_1 y_2 y_3$ with the body inertia center as the origin and the axes rigidly connected with the body.

To find the principal vector and momentum, one has to calculate on the body surface the pressure and the Frechet derivative of the fluid velocity vector. To do this, one has to solve a certain boundary-value problem for the vector-valued Navier–Stokes equation. This equation is written out below in the moving system $O_c y_1 y_2 y_3$ with axes parallel to the corresponding axes of the system $Ox_1 x_2 x_3$ (the body is assumed to move translationally). Let $\mathbf{V}$ be the velocity vector of the body, and $x_c(t)$ be the radius vector of its inertia center. In the moving coordinate system, denote the absolute velocity vector of fluid and the pressure as follows: $\hat{\mathbf{v}}(t, y) = \mathbf{v}(t, x_c(t) + y)$, $\hat{p}(t, y) = p(t, x_c(t) + y)$. Then the Navier–Stokes equation is of the form

$$\frac{\partial \hat{\mathbf{v}}}{\partial t} = -\frac{\partial \hat{\mathbf{v}}}{\partial y}(\hat{\mathbf{v}} - \mathbf{V}) - \frac{1}{\rho}\left(\frac{\partial \hat{p}}{\partial y}\right)^* + \nu \operatorname{div} \frac{\partial \hat{\mathbf{v}}}{\partial y} + \mathbf{F}, \tag{5}$$

where $\mathbf{F}$ is the strength of the gravity field, $\rho$ is the fluid density, $\nu = \mu/\rho$ is the kinematic viscosity coefficient.

Now, the above-mentioned boundary-value problem is reduced to finding the solution of a system of partial differential equations, namely, equation (5) plus the equation of continuity $\operatorname{div}\hat{\mathbf{v}} = 0$. This solution must satisfy the sticking condition $\hat{\mathbf{v}}(t, y)\Big|_S = \mathbf{V}$ and the natural condition $\lim\limits_{y\to\infty} \hat{\mathbf{v}}(t, y) = 0$. A flow is called steady-state or stationary if the field of its absolute velocity vectors in the moving coordinate system does not change in time. Obviously, if the body moves translationally, the necessary condition for the flow to be stationary is $\mathbf{V} = \mathbf{V}_0 = \text{const}$. Suppose that the body has a symmetry axis. If the body moves in such a manner that this axis remains in a given plane (for example, in the plane $Oxy$), then, according to the statics theorems for an absolutely solid body, the totality of forces acting from fluid upon the body can be reduced to the resultant one called the hydrodynamic force. As usual the point of intersection of the symmetry axis and the line of the hydrodynamic force action is referred to as center of pressure. The hydrodynamic force is resolved into components parallel to the velocity vector $\mathbf{V}$ of the body inertia center and perpendicular to $\mathbf{V}$. It should be noted that $x$ and $y$ are the coordinates of body inertia center. The first component $\mathbf{D}$ is known as the drag force, and the second one $\mathbf{D}^{\perp}$ is called the lift force.

Let $\mathbf{i}, \mathbf{j}$ be the unit vectors in the directions $Ox$ and $Oy$ respectively. We need further a mapping that puts a vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ into correspondence to $\mathbf{a}^{\perp} = -a_2\mathbf{i} + a_1\mathbf{j}$. Let $V$ be the magnitude of $\mathbf{V}$, $D$ be that of the drag force, and $D^{\perp}$ be that of the lift force. For needs of forthcoming references, it is convenient to formulate the following assertion as lemma.

**Lemma.** The drag and lift forces are calculated by the formulae

$$\begin{aligned} \mathbf{D} &= \operatorname{sign}(\mathbf{V},\mathbf{D}) D V^{-1}\mathbf{V}, \\ \mathbf{D}^{\perp} &= \operatorname{sign}(\mathbf{V},\mathbf{D}) s D^{\perp} V^{-1}\mathbf{V}^{\perp}, \\ s &= \operatorname{sign}((\mathbf{V},\mathbf{e})(\mathbf{V},\mathbf{e}^{\perp})), \end{aligned} \tag{6}$$

where $\mathbf{e}$ is the directing vector of the body symmetry axis.

In the framework of the listed constraints, the coefficient $C_D$ is a function of the body shape, Reynolds number and, probably, the angle of attack between the velocity vector of the body inertia center and the symmetry axis, i.e., $C_D = C_D(\text{shape,Re},\alpha)$ [Daily and Harleman, 1966]. To determine the angle of attack one can use the formula

$$\alpha = -s \arccos|(\mathbf{e}, \mathbf{V}/V)|. \tag{7}$$

The nonstationarity of the flow can be partially taken into account by means of introducing the apparent additional mass [Daily and Harleman, 1966].

### 2.3 Derivation of the Equations of Motion

Kinetic energy is equal to

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}\frac{ml^2}{3}\dot{\varphi}^2. \tag{8}$$

Using the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \tag{9}$$

one can obtain body movement equations

$$\begin{aligned} m\ddot{x} &= Q_x \\ m\ddot{y} &= Q_y \\ \tfrac{1}{3}ml^2\ddot{\varphi} &= Q_\varphi \end{aligned} \tag{10}$$

The generalized forces corresponding to the generalized coordinates will be the following

$$\begin{aligned} Q_x &= -D\sin(\varphi-\alpha) + D^{\perp}\cos(\varphi-\alpha) + F\sin(\varphi) \\ Q_y &= -D\cos(\varphi-\alpha) - D^{\perp}\sin(\varphi-\alpha) + F\cos(\varphi) - mg \\ Q_\varphi &= U + M \end{aligned} \tag{11}$$

The expression for the power of the control forces and momentums is of the form

$$\dot{W} = (\dot{x}\sin\varphi + \dot{y}\cos\varphi)F + \omega U\,. \tag{12}$$

The system of equations (10) and (11) describes body movement.

## 3 Optimization Problem Statements

Now the optimization problem can be formulated.

**Problem 1.** It is required to find controls $F^0(t)$ $U^0(t)$, $0 \leqslant t \leqslant t_k$, moving with the minimum power expenses, $W(t_k) \to \min$, a body for given time $t_k$ for the set distance.

Such problem is nonregular. Euler–Lagrange equations do not contain controls and do not allow to define their optimum values in terms of the phase and interfaced variables.

The problem reduction is proved by that body movement occurs in a potential gravity field. And the changeable part of work of control forces is used for change of body kinetic energy. Therefore the varied part of work will be equivalent to power expenses for overcoming of hydrodynamic forces of resistance and will be equal to scalar product $(\mathbf{D}^T, \mathbf{V})$

$$N = -D\sin(\varphi - \alpha) - D^{\perp}\cos(\varphi - \alpha) - M\varphi \tag{13}$$

Power of hydrodynamic forces is equal to

$$\dot{N} = D(\dot{\varphi} - \dot{\alpha})(-\cos(\varphi-\alpha) + \sin(\varphi-\alpha)) - M\dot{\varphi}\,. \tag{14}$$

Now it is possible not to consider dynamics of a body, having assigned function of control to derivatives of the generalized coordinates. Thus the initial problem is equivalent to following problem.

**Problem 2.** It is required to find functions $\mathbf{V}(t) = (V_x(t), V_y(t))^T$ $\omega(t)$, minimizing terminal functional $N(t_k)$ at dynamical relations (12) and restrictions

$$\begin{aligned} &x(t_k) = x_k\,, \quad y(t_k) = y_k\,, \quad \varphi(t_k) = \varphi_k\,, \\ &\cos\alpha = \dot{x}\cos\varphi + \dot{y}\cos\varphi\,. \end{aligned} \tag{15}$$

According to classical Euler–Lagrange procedure it is necessary to write out Hamiltonian $H = \lambda_0\dot{Y} + \lambda_1\dot{x} + \lambda_2\dot{y} + \lambda_3\dot{\varphi}$ and conjugated system with boundary conditions

$$\begin{aligned} -\dot{\lambda}_0 &= \frac{\partial H}{\partial N} = 0\,, & \lambda_0(t_k) &= \frac{\partial \Phi}{\partial N(t_k)} \\ -\dot{\lambda}_1 &= \frac{\partial H}{\partial x}\,, & \lambda_1(t_k) &= \frac{\partial \Phi}{\partial x(t_k)} \\ -\dot{\lambda}_2 &= \frac{\partial H}{\partial y}\,, & \lambda_2(t_k) &= \frac{\partial \Phi}{\partial y(t_k)} \\ -\dot{\lambda}_3 &= \frac{\partial H}{\partial \varphi}\,, & \lambda_3(t_k) &= \frac{\partial \Phi}{\partial \varphi(t_k)} \end{aligned} \tag{16}$$

Here $\Phi = N(t_k) + \nu_1(x(t_k) - x_k) + \nu_2(y(t_k) - y_k) + \nu_3(\varphi(t_k) - \varphi_k)$ is functional describing boundary conditions.

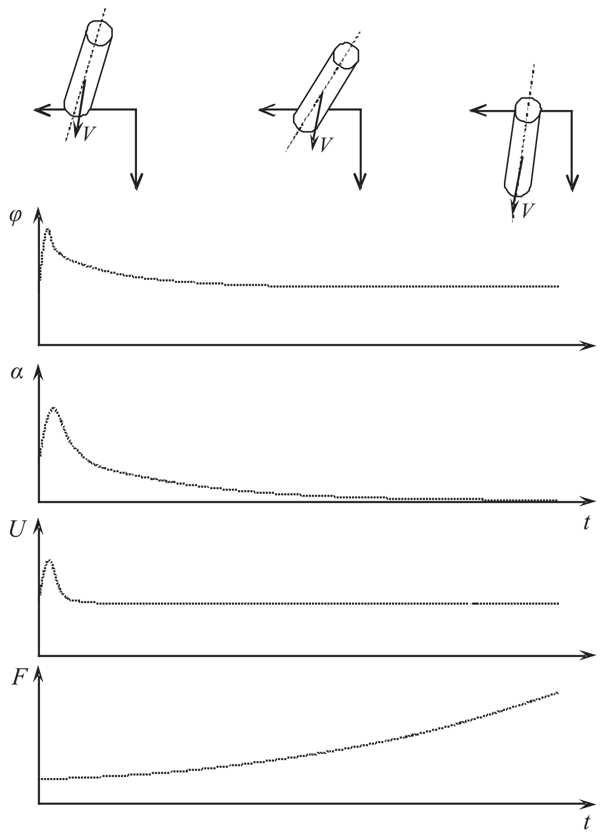


Figure 2. The angles of orientation and attack and control forces

Euler–Lagrange equations

$$\begin{aligned}\frac{\partial H}{\partial \dot{x}} &= \lambda_1 + \frac{\partial \dot{N}}{\partial \dot{x}} = 0\\ \frac{\partial H}{\partial \dot{y}} &= \lambda_2 + \frac{\partial \dot{N}}{\partial \dot{y}} = 0\\ \frac{\partial H}{\partial \dot{\varphi}} &= \lambda_3 + \frac{\partial \dot{N}}{\partial \dot{\varphi}} = 0\end{aligned} \qquad (17)$$

allow to calculate Lagrange multipliers and at having substituted them in the conjugated system (16) to write out the equations of optimal movement

$$\begin{aligned}\dot{x} &= V_x\,, \quad \frac{d}{dt}\left(\frac{\partial \dot{N}}{\partial V_x}\right) = \frac{\partial \dot{N}}{\partial x}\\ \dot{y} &= V_y\,, \quad \frac{d}{dt}\left(\frac{\partial \dot{N}}{\partial V_y}\right) = \frac{\partial \dot{N}}{\partial y}\\ \dot{\varphi} &= \omega\,, \quad \frac{d}{dt}\left(\frac{\partial \dot{N}}{\partial \omega}\right) = \frac{\partial \dot{N}}{\partial \varphi}\end{aligned} \qquad (18)$$

In Fig. 2 the basic modes of border overcoming are presented. It should be noted that zero moment corresponds to the time of entering into the boundary is difficult to control.

## 4 Conclusion

Thus the system of the differential equations describing a rigid body movement through the boundary of a viscous media is obtained. It allows to model various modes of such movement. The totality of the problems solved in the present paper can be used in both the applied theory of singular dynamic optimization problems and design of perspective samples of new machines.

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