OPTIMIZATION OF BEAM DYNAMICS IN THE AXIALLY SYMMETRIC POTENTIAL FIELD

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Abstract
The problem of optimization of charged particle beam dynamics in an axially symmetric electric field is considered. The complex potential is represented with use Cauchy integral of a function defined on the boundary of the region and considered as the control function. Analytical representation for the variation of the optimized functional and the necessary optimality conditions are found. On the basis of the expression for the variation of the functional can be built directed methods of optimization. Various practical implementations of fields obtained in the optimization process are possible.

Key words
Beam Dynamics Optimization, Axially Symmetric Field.

1 Introduction

Many works [Svistunov, Ovsyannikov, (2010); Ovsyannikov, Ovsyannikov, Antropov, Kozynchenko, (2005); Svistunov, Durkin, Ovsyannikov, (2012); Ovsyannikov, Ovsyannikov, (2010)] are devoted to the problems of optimization of the dynamics of charged particles in electromagnetic fields. In particular, in work [Kozynchenko, Ovsyannikov, (2009)] electrostatic injectors for linear accelerator were investigated.

In this paper, the problem of optimization of beam dynamics of charged particles in the axial-symmetric electric field is considered. New analytical representation of variation of functional is found and the optimality conditions are formulated.

In some simply connected bounded three-dimensional area, which has axial symmetry, let us consider dynamics of charged particles described by a system of ordinary differential equations in cylindrical coordinates:

$$\ddot{r} = E_r(r, z, \varphi), \quad (1)$$

$$\ddot{z} = E_z(r, z, \varphi). \quad (2)$$

Note that the external electric field in equations (1), (2) is defined by specifying a control function φ on the curve L - the border of complex area G, which is a meridional cross section of investigated region in three-dimensional space. Curve L assumed to be smooth closed curve, and the function φ is defined and continuous on the curve L, and satisfies the Hölder condition:

$$|\varphi(\eta_1) - \varphi(\eta_2)| \le M|\eta_1 - \eta_2|^{\nu}, \nu > 0, M > 0. \quad (3)$$

In this case, the complex potential H of the field is represented in the following form [Molokovskiy, Sushkov, (1972)]:

$$H(z, r, \varphi) = \frac{1}{2\pi}\int_0^{2\pi} F(z + ir\cos\alpha, \varphi)\,d\alpha, \quad (4)$$

where F is a Cauchy integral:

$$F(\xi, \varphi) = \frac{1}{2\pi i}\int_L \frac{\varphi(\eta)}{\eta - \xi}\,d\eta. \quad (5)$$

Here $\xi = z + i\cdot r\cdot\cos(\alpha)$, $x = r\cdot\cos\theta$, $y = r\cdot\sin\theta$ $\xi \in G$, $\eta \in L$. Here and further the meridional plane of cylindrical coordinates system in R^3 will be identified with the complex plane C.

The complex potential is an analytic function defined in a domain G, and its real and imaginary parts are harmonic functions of real variables r, z and θ. Let us consider the function $U = \mathrm{Re}\,H$ as defining an axially symmetric potential electric field. Then the electric field intensity in cylindrical coordinates is given by

$$E_z = -\frac{1}{2\pi}\int_0^{2\pi}\left(\mathrm{Re}\frac{\partial F(z + ir\cos\alpha, \varphi)}{\partial\xi}\right)d\alpha, \quad (6)$$

$$E_r = \frac{1}{2\pi}\int_0^{2\pi}\left(\mathrm{Im}\frac{\partial F(z + ir\cos\alpha, \varphi)}{\partial\xi}\cos\alpha\right)d\alpha. \quad (7)$$

Note that the dynamic equation (1)-(2) may be converted to the form:

$$\frac{d\mathbf{a}}{dt} = \mathbf{f}(t, \mathbf{a}, \varphi). \quad (8)$$

Also consider changing of the density distribution of the particles along the trajectories

$$\frac{d\rho}{dt} = -\rho(t,\mathbf{a}) \cdot \mathrm{div}_{\mathbf{a}}\, \mathbf{f}(t,\mathbf{a},\varphi). \qquad (9)$$

Here $\mathbf{a}$ the vector of phase variables $\mathbf{a} = (r, \dot{r}, z, \dot{z})^T$, and vector function $\mathbf{f}$ in equation (8) has the following form

$$\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} a_2 \\ E_r(a_1, a_3, \varphi) \\ a_4 \\ E_z(a_1, a_3, \varphi) \end{pmatrix}. \qquad (10)$$

Equations (8) and (9) we will consider with the initial conditions

$$\mathbf{a}(0) = \mathbf{a}_0 = (r_0, v_{r0}, z_0, v_{z0})^T \in M_0 \subset R^4, \quad (11)$$

$$\rho(0,\mathbf{a}(0)) = \rho_0(\mathbf{a}_0), \quad \mathbf{a}_0 \in M_0. \qquad (12)$$

where M_0 a compact set such that for any point $\mathbf{a}(0) \in M_0$ satisfies $(r_0 + i \cdot z_0) \in G$; $\rho_0(\mathbf{a}_0)$ — some non-negative continuous function. Vector $\mathbf{a}_t = \mathbf{a}(t, \mathbf{a}_0, \varphi)$ is the vector of phase variables of system (8) at time t corresponding to initial condition (11) and given control function φ.

Function φ will be referred to hereafter as boundary control or simply control. The class of admissible controls D is the set of continuous functions φ on a curve L satisfying the Hölder condition (3) and such that $\varphi(\eta) \in \Phi$ when $\eta \in L$, where Φ is a convex compact set in the complex plane.

We assume further that the solutions of system (8), (9) are defined and are unique to some fixed interval $[0,T]$, for all initial conditions (11) and for all admissible controls.

On the trajectories of the system (1), (2), we introduce the functional of quality of the form

$$I(\varphi) = \int_0^T \int_{M_{t,\varphi}} p(t,\mathbf{a}_t)\rho(t,\mathbf{a}_t)d\mathbf{a}_t dt + \int_{M_{T,\varphi}} q(\mathbf{a}_T)\rho(T,\mathbf{a}_T)d\mathbf{a}_T. \qquad (13)$$

Here p and q are given non-negative, continuously differentiable functions. Set $M_{t,\varphi}$ is a section of the beam of trajectories of the system (8) coming from the initial set M_0 at the time t with the given control function φ.

Let us consider further the minimization of the functional on the admissible class of controls. Let φ is an admissible control. Variation of the control $\Delta\varphi$ is admissible if the control $\tilde{\varphi} = \varphi + \Delta\varphi$ is also admissible control.

2 Variation of functional

Variation functional (13) with admissible variation of the control function φ such that $\|\Delta\varphi\| = \max_{\eta \in L} |\Delta\varphi(\eta)| \to 0$, can be represented as follows:

$$\delta I(\Delta\varphi) = = \mathrm{Re} \int_0^T \int_{M_{t,\varphi}} \int_0^{2\pi} \frac{\lambda_\alpha(t,\mathbf{a}_t)}{4\pi^2 i} \int_L \frac{\Delta\varphi(\eta) d\eta d\alpha}{(\eta - \xi_\alpha(t,\mathbf{a}_t))^2} d\mathbf{a}_t dt. \qquad (14)$$

Complex functions λ_α and ξ_α have the form

$$\lambda_\alpha = \mu_4 + i \cdot \mu_2 \cos\alpha, \qquad (15)$$

$$\xi_\alpha(t,\mathbf{a}_t) = z(t,\mathbf{a}_t) + i \cdot r(t,\mathbf{a}_t)\cos\alpha. \qquad (16)$$

Here vector function $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3, \mu_4)^T = \boldsymbol{\mu}(t, \mathbf{a}_t)$ satisfies the following system defined on the trajectories of the system (8)

$$\frac{d\boldsymbol{\mu}}{dt} = -\left(\frac{\partial \mathbf{f}}{\partial \mathbf{a}} + \mathbf{E}\,\mathrm{div}_{\mathbf{a}}\, \mathbf{f}\right)^T \boldsymbol{\mu} + \left(\frac{\partial p(t,\mathbf{a}_t)}{\partial \mathbf{a}}\right)^T, \quad (17)$$

where $\mathbf{E}$ the identity matrix of dimension 4, and with terminal conditions

$$\boldsymbol{\mu}(T) = -\left(\frac{\partial q(\mathbf{a}_T)}{\partial \mathbf{a}}\right)^T. \qquad (18)$$

3 Necessary optimality conditions

Let the boundary L has the following parameterization:

$$\eta = \chi(s) = x(s) + i \cdot y(s), \quad s \in [0, S]. \qquad (19)$$

Then the integral over the complex circuit in the formula (14) may be replaced by definite integral. The result is

$$\delta I(\Delta\varphi) = \mathrm{Re} \int_0^T \int_{M_{t,\varphi}} \int_0^{2\pi} \frac{\lambda_\alpha(t,\mathbf{a}_t)}{4\pi^2 i} \int_0^S \frac{\Delta\varphi(\chi(s))\dot{\chi}(s) ds d\alpha}{(\chi(s) - \xi_\alpha(t,\mathbf{a}_t))^2} d\mathbf{a}_t dt. \quad (19)$$

By changing the order of integration in (19), we obtain:

$$\delta I(\Delta\varphi) = -\mathrm{Re} \int_0^S \Delta\varphi(\chi(s))\dot{\chi}(s)\omega(s) ds. \qquad (20)$$

Here

$$\omega(s) = -\frac{1}{4\pi^2 i} \int_0^T \int_{M_{t,\varphi}} \int_0^{2\pi} \frac{\lambda_\alpha(t,\mathbf{a}_t) d\alpha d\mathbf{a}_t dt}{(\chi(s) - \xi_\alpha(t,\mathbf{a}_t))^2}. \quad (21)$$

Theorem 1. Let φ_0 minimizes the functional (13). Then for any admissible variation of control function $\Delta\varphi$ the functional variation is non-negative

$$\delta I(\varphi_0, \Delta\varphi) = -\mathrm{Re} \int_0^S \Delta\varphi(\chi(s))\dot{\chi}(s)\omega(s) ds \geq 0. \quad (22)$$

Theorem 2. Let the φ_0 minimizes the functional (13). Then the condition of the maximum principle

$$\max_{c \in \Phi} \mathrm{Re}[c \cdot \dot{\chi}(s)\omega(s)] = \mathrm{Re}[\varphi_0(\chi(s)) \cdot \dot{\chi}(s)\omega(s)] \quad (24)$$

for all $s \in [0, S]$. Here c is a complex value of the admissible set Φ of values of the control function.

4 Conclusion

In this paper a new optimization model of charged particle beam dynamics in electrostatic axially symmetric field was proposed. Analytical representation for the variation of the optimized functional defined on the beam of trajectories of the investigated system (8) is found. The necessary optimality conditions are formulated. Various practical implementations of fields obtained in the optimization process are possible. Obtained results are similar to the conditions found in [Ovsyannikov, (2013)] for plane parallel field but unlike these studies one can consider more complicated model of the beam dynamics in axially symmetric field. On the basis of the obtained expression for the variation of the functional the various directed optimization methods can be built similar to [Ovsyannikov, Ovsyannikov, (2010); Ovsyannikov, Ovsyannikov, Antropov, Kozynchenko, (2005); Svistunov, Durkin, Ovsyannikov, (2012); Svistunov, Ovsyannikov, (2010)]. Also it is possible including by different ways to the model the charged particles interaction [Ovsyannikov (2014).] and the magnetic field.

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