PREDICTABILITY THROUGH FINITE TIME LYAPUNOV EXPONENTS IN TWO COUPLED SYSTEMS

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Abstract
We analyse the predictability information of two coupled Rössler systems through the study of finite time lyapunov exponents distributions. Using these techniques one can derive the system shadowing properties, and characterise the possible nonhyperbolic nature of the system and the goodnes of the computed orbit against the real one. Our work focuses in how these results may depend on the considered finite time intervals. By using arbitrarily selected initial deviation directions, we aim to correlate the selection of the intervals lengths with the flow physical timescales.

Key words
Finite Lyapunov Exponents, Shadowing, Unstable Dimension Variability, Hyperbolicity

1 Introduction

In the last century, the numerical approach to solving physical problems has gained in relevance with the increase of computational facilities. Methods derived from chaos theory and nonlinear dynamics techniques are quite useful in solving real problems where chaos is present and a strong dependence on initial conditions is a key issue. When dealing with numerical explorations, it is important how to consider both its predictability and instability.

Predictability refers to the assessment of the likely errors in a forecast, and refers directly to the (in)stability of the true orbit. But is also refers to its reliability, understood as the coincidence of a calculated orbit with the real one. This is tightly related to the shadowing phenomenon.

Given a model with strong sensitivity to initial conditions, and since all numerical methods introduce errors, it is very likely that a numerically computed orbit will diverge from a real one. A shadow is an exact solution in a given model that remains close to a numerical solution for a given amount of time. In the so-called pseudo-deterministic systems, when nonhyperbolicity is sourced to Unstable Dimension Variability (UDV), the shadowing maz be only valid during very short times. So the estimation of those scales in which the calculated orbit is reliable, i.e. is followed by a real orbit, is of importance when modelling these systems. Through the study of the finite time Lyapunov exponents distributions, we will analyse the predictability information of a given system and the proper timescales to consider.

2 Description of the model

We have studied a model formed by two identical, symmetrically diffusively coupled continous Rössler systems which may undergo a chaos-hyperchaos transition,

$$\left.\begin{array}{l} \dot{x_1} = -y_1 - z_1 \\ \dot{y_1} = x_1 + a y_1 \\ \dot{z_1} = b + z_1(x_1 - c) + d(z_2 - z_1) \\ \dot{x_2} = -y_2 - z_2 \\ \dot{y_2} = x_2 + a y_2 \\ \dot{z_2} = b + z_2(x_2 - c) + d(z_1 - z_2) \end{array}\right\} \quad (1)$$

We have selected it because its prototypical behaviour in which concerns the hyperchaos transition and presence of UDV phenomenon. A very similar system was studied in [Yanchuck, 2001], where a chaos-hyperchaos transition and the presence of UDV was described in detail. The parameters $b = 2.0$ and $c = 4.0$ have been fixed. The parameter a will control every Rössler system behavior, meanwhile the parameter d will be the coupling parameter. Figure. 1 shows the evolution of oscillator (x_1, y_1, z_1) and the oscillator (x_2, y_2, z_2), until time $T = 10000$, showing the different initial evolution as the control parameter is varied in three typical cases.

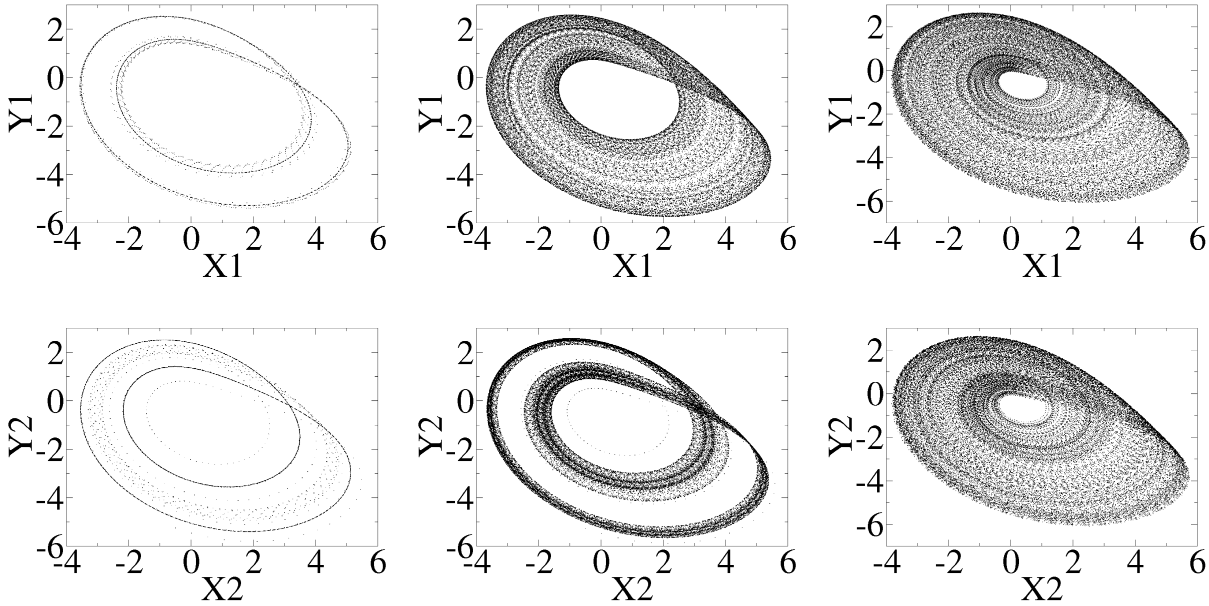

Figure 1. Different initial evolution of the two coupled Rössler oscillators, as the control parameter a is varied. The oscillators are displayed just up time $T = 10000$. From left to right, $a = 0.342$, $a = 0.365$, $a = 0.389$. Fixed coupling strength parameter $d = 0.25$. Upper row is oscillator 1, and bottom row is oscillator 2. These three cases are indicated in Fig. 2 as A,B and C.

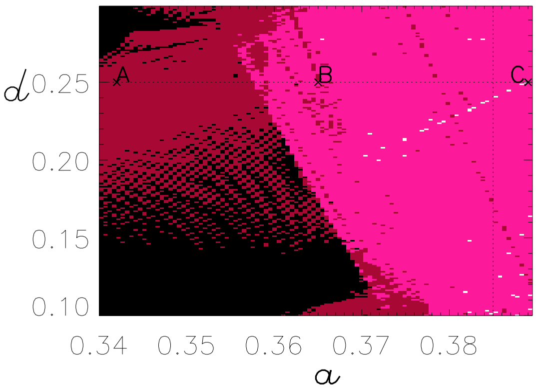

Figure 2. Hyperchaocity chart. The number of positive asymptotic Lyapunov exponents varies with the Rössler parameter a and the coupling strength parameter d. Black means 0 positive exponents. Dark red means only 1. Pink means 2, and White means 3. Points A, B, and C are the three plots of Fig. 1.

3 Finite Lyapunov exponents distributions

The standard, global or asymptotic Lyapunov exponent gives a measure for the total predictability of a system through globally averaged growth rates in the limit of large time and small uncertainty. By computing all relevant exponents, we get insight on the different regimes of the system. We can see the areas with no chaos (i.e. no positive exponents), chaos (one positive exponent) and hyperchaos (more than one). Figure 2 shows how the hyperchaos is born in a complex waz depending on the individual nature of the oscillator a, and the coupling strenght d.

The definition of global Lyapunov exponent requires an infinite integration time. In practice, however, all numerically computed exponents are calculated using finite time integration intervals. When this is the case, they are generically named as finite Lyapunov exponents and are given by,

$$\chi(t) = \frac{1}{t} \log_e \frac{\delta z(t)}{\delta z(0)}. \tag{2}$$

When computed during finite times, the values of the exponents are generally different and may change in sign along one orbit. These generic exponents reflect the growth rate of the axes of one ellipse centered in the initial position as the system evolves up given time t. Conversely to the local Lyapunov Exponent, which does not depend on the initial arbitrary perturbation, as it is defined when the perturbation has converged to the so called backward Lyapunov vector, generic finite Lyapunov exponents depend by definition on the finite integration time interval $t = \Delta t$ and the initial direction of the chosen deviation direction (or initial vectors used as ellipse axes). This is because for large Δt, there is a large enough number of Benettin steps for allowing the initial axes converge towards a given direction, or even to the asymptotic final direction. But being Δt small enough, such a convergence could not be reached.

If we make a partition of the whole integration time along one orbit into a series of time intervals of size Δt, then it is possible to compute the finite time Lyapunov exponent $\chi(\Delta t)$ for every interval. Depending on the nature of the system, the distribution will depend on a third factor, the total integration time along the finite exponents are recollected [Vallejo, 2003]. It may happen that several transients appear and the final nature of the orbit is only obtained but after a given integration time.

When the distribution of $\chi(\Delta t)$ along the integrated orbit is normalized, dividing it by the total number of intervals, we obtain a density function $P(\chi)$ that gives the probability of getting a given value χ between $[\chi, \chi + d\chi]$. From this density, we can get information about the degree of instability of the orbit. The variation of the distributions with Δt provides an indication on the threshold for passing from the local dynamics scales to the timescale when the initial directions are allowed to evolve [Vallejo 2008].

We hace calculated the finite exponents distributions for a range of parameters $a - d$, and Δt intervals. From those distributions we will get insight into the flow predictability indicators. All the exponents were calculated by selecting as initial directions a randomly chosen set of independent vectors. If allowed to evolve time enough, the Benettin algorithm returns the asymptotic values ordered from the largest to the smallest. For small Δt intervals, the values have been not evolved towards the asymptotic ones and such an ordering is not yet achieved. We remark here we have used for the initial ellipse an arbitrary set of independent axes not aligned with any privilegued direction. In this way, the trending time will be tightly related with the flow typical timescales.

4 Hyperbolicity and UDV

Hyperbolicity is different from stability. An orbit can be unstable but be hyperbolic. Hyperbolicity means good shadowing behaviour. In hyperbolic systems, the angle between the stable and unstable manifolds is away from zero and phase space is locally spanned by a fixed number of distinct stable and unstable directions [Viana, 2000]

The nonhyperbolic behaviour can arise from tangencies (homoclinic tangencies) between stable and unstable manifold, from UDV, or from both. In the case of just tangencies, there is a higher, but still moderate obstacle to shadowing. But in the so called pseudo-deterministic systems, the shadowing in only valid during trajectories of reasonable (short) length, due to the UDV, as larger times are improbable because of diffusive processes.

A finger print of UDV is the fluctuating behavior around zero of the finite time exponent closest to zero. Even when these oscillations are not always associated to UDV, they are still a good indication for non hyperbolic behavior.

We have analysed the fluctuations around zero of the finite time exponent closest to zero [Davidchack, 2000]. Those oscillations will be detected by calculating the F_+ index, or probability of getting a positive $\chi(\Delta t)$. It can be defined as,

$$F_+ = \int_0^{\infty} P(\chi) d\chi. \tag{3}$$

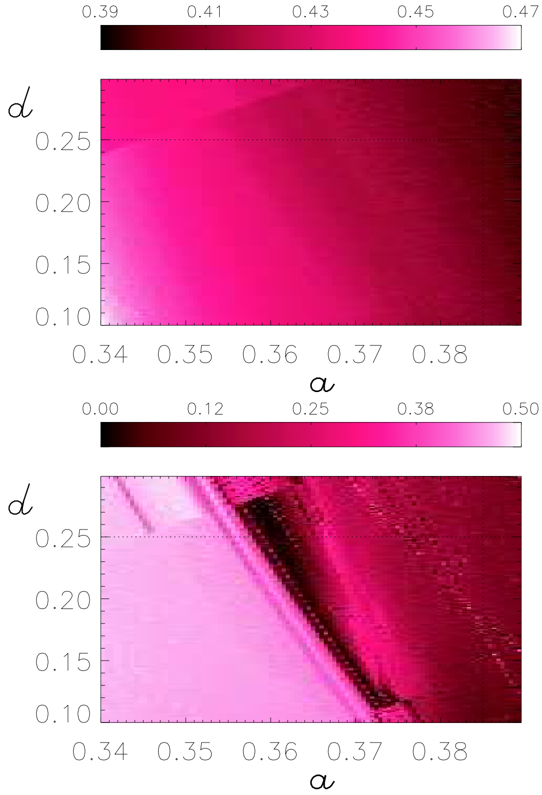

Figure 3. Probability of Positivity of the closest to zero exponent, for given Oscillator parameter a and Coupling Strength d. Scaled values give the distance to $P_+ = 0.5$. Darker areas are distances nearly to 0.0, meaning $P_+ \sim 0.5$. This means distributions centered around zero (stretched or shrinked). Brighter areas with larger values, are those farther from 0.5 in positive or negative direction. Top: $\Delta t = 1$ and $T = 10000$. Bottom: $\Delta t = 100$ and $T = 100000$.

The oscillations can be identified when F_+ is nearly 0.5. In Fig. 3 we can appreciate the oscillations around zero of the closest to zero exponents, which are a mark of nonhyperbolic nature of the local dynamics. This figure shows how the results depend on the selected interval lengths. In the top of Fig. 3, we see that when using $\Delta t = 1.0$, the flow has not evolved yet towards the most expanding/contracting direction, and no clear oscillation around zero is detected. When $\Delta t = 100.0$, the closest to zero exponent has centered around the asymptotic value, and the oscillations around zero are clearly seen. Note every point in the parametric space has their own timescale for evolving towards the final state, so with $\Delta t = 100.0$ some tails in the distributions leading to have a broad span for the positivity index can be seen. As consequence, both using the smaller and the larger interval lengths, the different flow behavior areas are identified. The upper leftmost corner, corresponding to the higher coupling strenght and smaller a control values, is clearly identified even at the smaller intervals. However, only with the larger ones, the fine structures of the oscillating nature of the exponents in the parametric space are fully resolved.

5 Shadowing and Predictability charts

One major goal of our work is the characterization of the predictability understood as the confidence (reliability) of a calculated orbit and how this characterisation depends on the selection of the interval size. The probability distributions for the shadowing can be justified from statistical properties of the finite-time exponents. When an oscillation around zero is present, the shadowing distance typically mimics a random walk behavior, swapping from exponential increases to decreases in the hyperbolic regions. It also can be described as a diffusion equation of a particle which may find different escape routes along its trajectory.

This diffusion approximation assumes independent and identically distributed innovations m and σ, with $\sigma = \lambda_T\sqrt{T}$, being T the time interval beyond the system decorrelation time. So when we use the closest to zero exponent and assume both m and σ to be very small, the shadowing time τ is given by [Sauer, 1997],

$$\tau \sim \delta^{-h} \qquad h = \frac{2\|m\|}{\sigma^2}, \tag{4}$$

where δ is the the round-off precision of the computer and h the so-called hyperbolicity index. When h is 0 or nearly 0, it is the worst case as there is no improvement in τ even increasing a lot δ. Reversely, larger the index, better the shadowing.

In Fig. 4 we have traced the different values of h as obtained using Eq.4 from the closest to zero exponent, for given Δt intervals and a parameter values, with fixed $d = 0.25$. At the smallest intervals, the exponents have not been allowed to evolve, and there is no clear distinction among the different regimes. At the larger intervals, such a distinction is clearly observed, and in the hyperchaotic regime the indexes are lower as expected. Indeed, there is clear separation of regimes before and after $a = 0.365$, where a transition takes place. So this diagram may serve for guessing the proper decorrelation time, or equivalently, the Δt that can be used for getting a reliable index value.

Keeping this in mind, Fig. 5 shows the h index as derived from the closest to zero exponent in the $a - d$ parameter space, for two different Δt values. These figures can be considered the main charts providing the overall predictability of the two coupled systems.

This figure allows to see how the information depends on the selected interval. In addition, we can also compare these predictability charts with the hyperchaos Fig. 2 and the oscillations around zero Fig. 3 figures, aiming to see the sources for the nonhyperbolic behavior. We can see here that the areas of low predictability can be associated largely to those areas with hyperchaos, and that the border of two regimes is mainly because the oscillation around zero phenomenon. However, additional structures are also visible, pointing to other sources for the predictability of the flow.

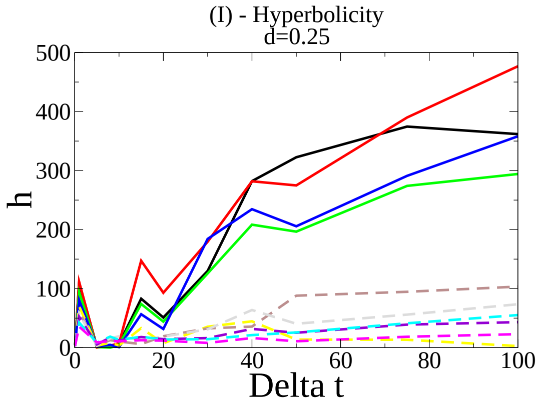

Figure 4. Predictability Indexes h calculated from the distributions of the closest to zero exponent, for different Δt intervals. Fixed coupling strength $d = 0.25$. Calculations start in $a = 0.34$, every line increases a in 0.05 units. Continous lines are $a < 0.365$. Dashed lines are those with $a > 0.365$. The regimes with low and high hyperbolicity are clearly identified, but only with a large enough $\Delta t \sim 25$ interval.

6 Conclusion

We have calculated the predictability index h of a system formed by two coupled Rössler oscillators, by using finite time exponents distributions. This h index allows to characterise the shadowing behavior. A shadow is an exact solution to a given model that remains close to a numerical solution for a given amount of time. So h index allows to estimate during how much time the calculated orbit is reliable, i.e. is followed by a real orbit.

The main result of our work is the analysis of the dependency of h index with the considered time interval Δt. We have performed numerical explorations in a broad range of the parametric space when transitions take place in a complex way.

Knowing the predictability and its time interval length dependencies in a given point is of interest, among other reasons, for chosing the most adequate integration scheme, as any error in the initial condition, or by what counts, the machine truncation error, may be amplified by the value of the hyperbolicity index h.

Independently of their convergence efficiency, the majority of commonly used chaoticity indicators are *global* or *averaged* indicators. They average along a given integration time, larger or shorter depending on the convergence rate. But when the shadowing times are short, averaged quantities should be handled with care, and the proper shadowing times taken into account. Our method does not use global averaged quantities during long intervals, unless strictly needed. So it can be used for open systems where transient chaos is found.

The identification of areas with low predictability are also of interest when applying control chaos methods as those based in the foundations made in [Ott, 1990], where by applying carefully chosen control impulses, it

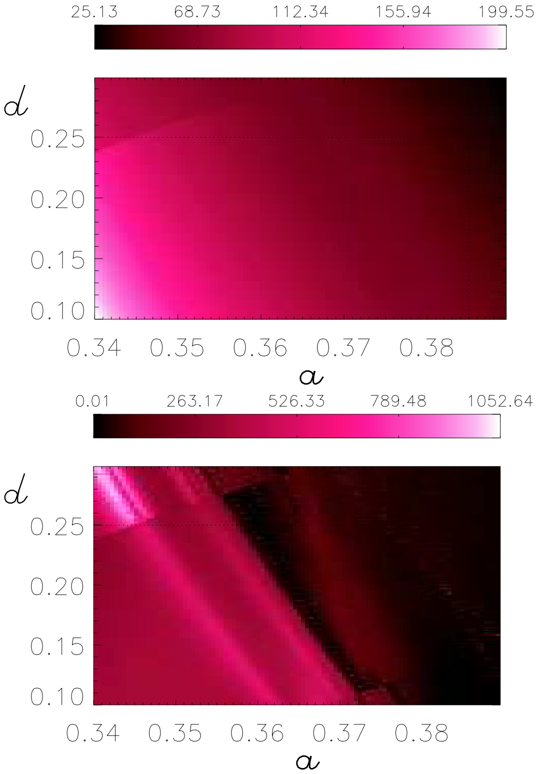

Figure 5. Predictability chart, or h index derived from the closest to zero exponent, for given a and d. Darker values reflects h lower and means poor predictability. Top: $\Delta t = 1$ and $T = 10000$. Bottom: $\Delta t = 100$ and $T = 100000$.

should be possible to carry the actual orbit towards the stable manifold. In the areas with tangencies, such an approach should be taken with care. The identification of these areas has been the subject of our work.

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