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STATE ESTIMATION FOR BILINEAR IMPULSIVE CONTROL SYSTEMS UNDER UNCERTAINTIES

Oxana G. Matviychuk
Department of Optimal Control
Krasovskii Institute of Mathematics and Mechanics,
Russian Academy of Sciences,
Ekaterinburg, Russia
vog@imm.uran.ru

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Abstract
The state estimation problem for uncertain impulsive control systems with a special structure is considered. The initial states are taken to be unknown but bounded with given bounds. We assume here that the coefficients of the matrix included in the differential equations are not exactly known, but belong to the given compact set in the corresponding space. We present here algorithms that allow to find the external ellipsoidal estimates of reachable sets for such bilinear impulsive uncertain systems.

Key words
Bilinear control systems, impulsive control, ellipsoidal calculus, trajectory tubes, estimation.

1 Introduction

The paper deals with an impulsive control systems with unknown but bounded uncertainties related to the case of a set-membership description of uncertainty [Kurzhanski and Valyi, 1997; Schweppe, 1973; Walter and Pronzato, 1997; Boyd, El Ghaoui, Feron and Balakrishnan, 1994]. Systems with such uncertainties may be found in many applied areas such as engineering problems in physics and cybernetics [Ceccarelli and etc., 2006], economics, biological and ecological modeling when it occurs that a stochastic nature of errors is questionable. A special type of nonlinear system is considered in the paper. The matrix of the system is uncertain and only the bounds of the admissible values of these matrix coefficients are known. For such bilinear systems the reachable sets are star-shaped sets and, in particular, can be non-convex.

Such systems can simulate various types of systems whose parameters are unknown, but can vary within certain limits, when the stochastic nature of errors is questionable due to limited data or because of the complexity of the model [August, Lu and Koeppl, 2012; Boscain, Chambrion and Sigalotti, 2013; Boussaïd, Caponigro and Chambrion, 2013; Ceccarelli and etc., 2006; Gough, 2008; Nihtila, 2010]. For instance one can indicate mechanical systems in which the stiffness or friction coefficients are given inaccurately. Electrical systems where the resistance, capacitance, inductance, or feedback coefficients are known with a certain accuracy can also be described within the framework of this model.

Related results connected with a so-called bounded-error characterization, also called set-membership approach, has been proposed and intensively developed. Among them we mention here [Kurzhanski and Valyi, 1997; Kurzhanski and Varaiya, 2014; Mazurenko, 2012; Schweppe, 1973; Walter and Pronzato, 1997]. For models with linear dynamics under such set-membership uncertainty there are several constructive approaches which allow to find effective estimates of reachable sets of control systems under uncertainty [Gusev, 2017; Chernousko, 1996; Filippova, 2010; Kurzhanski and Valyi, 1997; Kurzhanski and Varaiya, 2014; Polyak and etc., 2004].

The paper deals with the guaranteed state estimation problem and uses ellipsoidal calculus [Chernousko, 1994; Kurzhanski and Valyi, 1997] to construct external reachable sets estimates for such systems. Here we develop the set-membership approach based on ellipsoidal calculus for the considered system. Also we generalize earlier results [Filippova and Matviychuk, 2015; Filippova, 2016; Matviychuk, 2017a], in particular we consider more complicated model of the control system than in [Matviychuk, 2017b]. In this paper the control function of studied bilinear impulsive control system is a pair of a classical (measurable) control and an impulsive control function. It is assumed that a classical control should belongs to a given finite-dimensional ellipsoid and an impulsive control function is the scalar function of bounded variation. The algorithms of constructing external ellipsoidal estimates for studied systems are given.

2 Basic Notations

Let $\mathbb{R}^n$ be the n–dimensional vector space, $\mathrm{comp}\,\mathbb{R}^n$ be the set of all compact subsets of $\mathbb{R}^n$, $\mathrm{conv}\,\mathbb{R}^n$ be the set of all convex and compact subsets of $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$ stands for the set of all real $n\times n$–matrices and $x'y = (x,y) = \sum_{i=1}^{n} x_i y_i$ be the usual inner product of $x, y \in \mathbb{R}^n$ with prime as a transpose, $\|x\| = (x'x)^{1/2}$. Let $I \in \mathbb{R}^{n\times n}$ be the identity matrix, $\mathrm{Tr}(A)$ be the trace of $n\times n$-matrix A (the sum of its diagonal elements), $\mathrm{diag}\, b = \mathrm{diag}\{b_i\}$ be the diagonal matrix A with $a_{ii} = b_i$ where b_i are components of the vector b. For a set $A \subset \mathbb{R}^n$ we denote its closed convex hull as $\overline{\mathrm{co}}\, A$.

We denote by $B(a, r) = \{x \in \mathbb{R}^n : \|x - a\| \le r\}$ the ball in $\mathbb{R}^n$ with a center $a \in \mathbb{R}^n$ and a radius $r > 0$ and by $E(a, Q) = \{x \in \mathbb{R}^n : (Q^{-1}(x-a), (x-a)) \le 1\}$ the *ellipsoid* in $\mathbb{R}^n$ with a center $a \in \mathbb{R}^n$ and a symmetric positive definite $n\times n$-matrix Q. Denote by $h(A, B)$ the Hausdorff distance between sets $A, B \in \mathbb{R}^n$.

3 Problem Formulation

Consider the following bilinear impulsive control system

$$\begin{gathered} dx = \big(A(t)x(t) + u(t)\big)dt + B(t)dv(t), \\ x(t_0 - 0) = x_0, \quad t \in [t_0, T], \end{gathered} \tag{1}$$

here $x \in \mathbb{R}^n$, vector-function $B(\cdot) \in \mathbb{R}^n$ is continuous on $[t_0, T]$. The initial condition $x(t_0 - 0) = x_0$ to the system (1) is assumed to be unknown but bounded

$$x_0 \in \mathcal{X}_0 = E(a_0, Q_0). \tag{2}$$

Let us assume that the control function $u(t)$ in (1) is Lebesgue measurable on $[t_0, T]$ and satisfies the constraint

$$u(t) \in \mathcal{U} = E(\hat{a}, \hat{Q}), \quad \text{for a.e. } t \in [t_0, T], \tag{3}$$

where $\hat{a} \in \mathbb{R}^n$, $\hat{Q} \in \mathbb{R}^{n\times n}$. The impulsive control function $v(\cdot) \in \mathbb{R}^n$ is a scalar function of bounded variation, monotonically increasing and right-continuous for $t \in [t_0, T]$. Also it is assumed that for some given $\mu > 0$ we have

$$\operatorname*{Var}_{t\in[t_0,T]} v(t) = \sup_{\{t_i\}} \sum_{i=1}^{k} |v(t_i) - v(t_{i-1})| \le \mu, \tag{4}$$

where supremum is taken over all $\{t_i\}$ such that $t_0 \le t_1 \le \ldots \le t_k = T$. Denote by $\mathcal{V}$ the class of all admissible controls $v(\cdot)$ for which (4) holds.

The $n\times n$–matrix function $A(t)$ in (1) has the special form

$$A(t) = A^0 + A^1(t) + A^2(t), \quad t \in [t_0, T],$$

where $A^0 \in \mathbb{R}^{n\times n}$ is given and the measurable, $A^1(t), A^2(t) \in \mathbb{R}^{n\times n}$ are unknown but bounded

$$A(t) \in \mathcal{A} = A^0 + \mathcal{A}^1 + \mathcal{A}^2, \quad t \in [t_0, T], \tag{5}$$

$$\begin{gathered} A^1(t) \in \mathcal{A}^1 = \big\{A = \{a_{ij}\} \in \mathbb{R}^{n\times n} : \\ |a_{ij}| \le c_{ij}, \ \ i, j{=}1, \ldots, n\big\}, \end{gathered} \tag{6}$$

$$\begin{gathered} A^2(t) \in \mathcal{A}^2 = \{A \in \mathbb{R}^{n\times n} \colon A = \mathrm{diag}\, a, \\ a = (a_1, \ldots, a_n) \in \mathbf{A}_0\}, \end{gathered} \tag{7}$$

$$\mathbf{A}_0 = \{a \in \mathbb{R}^n : \sum_{i=1}^{n} |a_i|^2 \le 1\},$$

where $c_{ij} \ge 0$ $(i, j = 1, \ldots n)$ are given.

Let the function $x(\cdot) = x(\cdot; t_0, x_0, A(\cdot), u(\cdot), v(\cdot))$ be a *solution* of the system (1)–(5) with initial state $x_0 \in \mathcal{X}_0$, with controls $u \in \mathcal{U}$, $v \in \mathcal{V}$ and with a matrix $A(\cdot) \in \mathcal{A}(\cdot)$.

The *trajectory tube* $\mathcal{X}(\cdot) = \mathcal{X}(\cdot; \mathcal{X}_0, \mathcal{A}, \mathcal{U}, \mathcal{V})$ of the system (1)–(5) is defined as the following set (see also [Filippova and Matviychuk, 2011])

$$\begin{gathered} \mathcal{X}(\cdot) = \bigcup \{x(\cdot) = x(\cdot;\, t_0, x_0, A(\cdot), u(\cdot), v(\cdot)) : \\ x_0{\in}\mathcal{X}_0, \ A(\cdot){\in}\mathcal{A}(\cdot), \ u{\in}\mathcal{U}, \ v{\in}\mathcal{V}\} \end{gathered} \tag{8}$$

and the *reachable set* of the system (1) at the time t is the cross-sections $\mathcal{X}(t)$ of the tube $\mathcal{X}(\cdot)$ (8) at the instant t ($t \in [t_0, T]$).

The main problem considered in this paper is to find the external ellipsoidal estimates for reachable sets $\mathcal{X}(t)$ of the dynamic control systems (1)–(5) with uncertain matrix of the system and uncertain initial state basing on the special structure of the data $\mathcal{A}, \mathcal{U}, \mathcal{V}$ and $\mathcal{X}_0$.

4 Preliminary Results

Consider first some auxiliary results.

4.1 Bilinear System

Consider first the following bilinear system

$$\begin{aligned} \dot{x} = A(t)\,x, \quad t_0 \le t \le T, \\ x_0 \in \mathcal{X}_0 = E(a_0, Q_0), \quad A(t) \in \mathcal{A}, \end{aligned} \tag{9}$$

where $x \in \mathbb{R}^n$, the set $\mathcal{A}$ is defined in (5).

The *reachable set* $\mathcal{X}(t) = \mathcal{X}(t; \mathcal{X}_0, \mathcal{A})$ at the time t ($t_0 < t \le T$) of the system (9) is defined as the following set

$$\begin{aligned} \mathcal{X}(t) = \bigcup \big\{ x(t) = x(t;\, t_0, x_0, A(t)) : \\ x_0 \in \mathcal{X}_0,\ A(t) \in \mathcal{A} \big\}. \end{aligned} \tag{10}$$

Note that the reachable sets $\mathcal{X}(t)$ for the bilinear system (9) are star-shaped sets.

A set $Z \subseteq \mathbb{R}^n$ is called *star-shaped* (with center c) if $c + \lambda(Z - c) \subseteq Z$ for all $\lambda \in [0, 1]$.

The set of all star-shaped compact subsets $Z \subseteq \mathbb{R}^n$ with center c will be denoted as $\mathrm{St}(c, \mathbb{R}^n)$, $\mathrm{St}\,\mathbb{R}^n = \mathrm{St}(0, \mathbb{R}^n)$.

Assumption 1. *For every $t \in [t_0, T]$ the inclusions $0 \in \mathcal{U}$ and $0 \in \mathcal{X}_0$ are true.*

We will assume further that Assumption 1 is satisfied.

Theorem 1. [Kurzhanski and Filippova, 1993] *Under Assumption 1 the reachable sets $\mathcal{X}(t)$ are star-shaped and compact for all $t \in [t_0, T]$ ($\mathcal{X}(t) \in \mathrm{St}\,\mathbb{R}^n$).*

Let $\rho(l|C)$ be the *support function* of a convex compact set $C \in \mathrm{conv}\,\mathbb{R}^n$, i.e.,

$$\rho(l|C) = \max\{l'c : \ c \in C\}, \quad l \in \mathbb{R}^n.$$

We will denote the *Minkowski function* of a set $M \in \mathrm{St}\,\mathbb{R}^n$ by

$$h_M(z) = \inf\{t > 0 : \ z \in tM,\ z \in \mathbb{R}^n\}.$$

We need the following notation

$$\mathcal{M} * X = \{z \in \mathbb{R}^n : \ z = Mx,\ M \in \mathcal{M},\ x \in X\},$$

where $\mathcal{M} \in \mathrm{conv}\,\mathbb{R}^{n \times n}$, $X \in \mathrm{conv}\,\mathbb{R}^n$.

Then the evolution equation known as the *integral funnel equation* [Kurzhanski and Filippova, 1993; Kurzhanski and Valyi, 1997] that describes the dynamics of star-shaped trajectory tubes is given in the following theorem.

Theorem 2. [Filippova and Lisin, 2000] *The trajectory tube $\mathcal{X}(t)$ of the bilinear differential system* (9) *with constraints* (2), (5) *is the unique solution to the evolution equation*

$$\lim_{\sigma \to +0} \sigma^{-1} h\big(\mathcal{X}(t+\sigma), (I + \sigma\mathcal{A}) * \mathcal{X}(t)\big) = 0, \tag{11}$$

with initial condition $\mathcal{X}(t_0) = \mathcal{X}_0,\ t \in [t_0, T]$.

From Theorem 2 we have

$$\mathcal{X}(t_0 + \sigma) \subseteq (I + \sigma\mathcal{A}) * \mathcal{X}_0 + o(\sigma)B(0, 1),$$

where $\sigma^{-1}o(\sigma) \to 0$ for $\sigma \to +0$. Taking into account (5), we note that

$$\begin{aligned} (I + \sigma\mathcal{A}) * \mathcal{X}_0 = \\ = (I + \sigma(A^0 + \mathcal{A}^1)) * \mathcal{X}_0 + \sigma\mathcal{A}^2 * \mathcal{X}_0, \end{aligned} \tag{12}$$

where sets $\mathcal{A}^1$ and $\mathcal{A}^2$ are defined in (6) and (7) respectively.

Consider the auxiliary bilinear system

$$\begin{aligned} \dot{x} = A(t)\,x, \quad t \in [t_0, T], \\ x_0 \in \mathcal{X}_0 = E(a_0, Q_0), \quad A(t) \in A^0 + \mathcal{A}^1. \end{aligned} \tag{13}$$

The external ellipsoidal estimate of set $(I + \sigma(A^0 + \mathcal{A}^1)) * \mathcal{X}_0$ may be found by applying the following theorem.

Theorem 3. [Chernousko, 1996] *Let $a^*(t)$ and $Q^*(t)$ be the solutions of the following system of nonlinear differential equations*

$$\dot{a}^* = A^0 a^*, \quad a_1^+(t_0) = a_0, \tag{14}$$

$$\dot{Q}^* = A^0 Q^* + Q^* A^{0\prime} + qQ^* + q^{-1}G, \tag{15}$$

$$Q^*(t_0) = Q_0, \quad t_0 \le t \le T,$$

$$q = \big(n^{-1}\,\mathrm{Tr}\,((Q^*)^{-1}G)\big)^{1/2},$$

$$G = \mathrm{diag}\Big\{(n - v)\Big[\sum_{i=1}^{n} c_{ji}|a_i^*|$$

$$+\big(\max_{\sigma = \{\sigma_{ij}\}} \sum_{p,q=1}^{n} Q_{pq}^* c_{jp} c_{jq} \sigma_{jp} \sigma_{jq}\big)^{1/2}\Big]^2\Big\}.$$

Here the maximum is taken over all $\sigma_{ij} = \pm 1$, $i, j = 1, \ldots, n$, such that $c_{ij} \neq 0$ and v is a number of such indices i for which we have: $c_{ij} = 0$ for all $j = 1, \ldots, n$. Then the following external estimate for the reachable set $\mathcal{X}(t)$ of the system (13) *is true*

$$\mathcal{X}(t) \subseteq E(a^*(t), Q^*(t)), \quad t_0 \le t \le T. \tag{16}$$

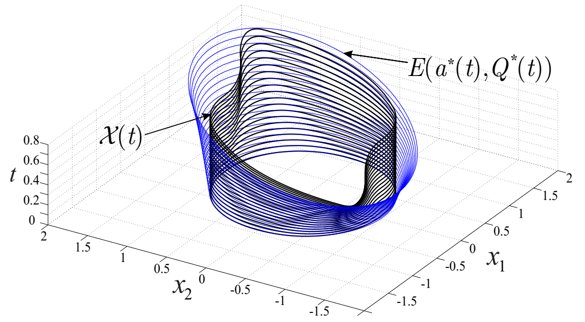

Figure 1. Trajectory tube $\mathcal{X}(t)$ and its ellipsoidal estimating tube $E(a^*(t), Q^*(t))$ for the bilinear system with uncertain initial states.

Corollary 1. *Under conditions of the Theorem 3 the following inclusion holds*

$$(I + \sigma(A^0 + \mathcal{A}^1)) * \mathcal{X}_0 \subseteq E(a^*(t_0 + \sigma), Q^*(t_0 + \sigma)) + o(\sigma)B(0, 1), \tag{17}$$

where $\sigma^{-1}o(\sigma) \to 0$ *for* $\sigma \to +0$.

The following example illustrates the Theorem 3.

Example 1. Consider the following bilinear system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = c(t)\, x_1, \end{cases} \quad 0 \le t \le 0.8,$$

where $x_0 \in \mathcal{X}_0 = B(0,1)$, the uncertain but bounded measurable function $c(t)$ satisfies the inequality $|c(t)| \le 1$ $(0 \le t \le 0.8)$.

The trajectory tube $\mathcal{X}(t)$ and its external ellipsoidal estimating tube $E(a^*(t), Q^*(t))$ calculated by the Theorem 3 are given in the Figure 1.

The following theorem is hold.

Theorem 4. [Filippova and Lisin, 2000] *For every* $z \in \mathbb{R}^n$ *such that* $z_i \neq 0$ $(i = 1, \dots, n)$ *the following formula is true*

$$h_{\mathcal{A}^2 * \mathcal{X}_0}(z) = \min \Big\{ \max_{l \neq 0} \frac{1}{\rho(l|\mathcal{X}_0)} \sum_{i=1}^{n} l_i z_i a_i^{-1} : a \in \mathbf{A}_0,\ a_i \neq 0,\ i = 1, \dots, n \Big\}.$$

Remark 1. [Filippova and Lisin, 2000] Let the set $\mathcal{A}^2$ is defined in (7) and $\mathcal{X}_0 = E(0, Q_0)$, then the following formula is true

$$h_{\mathcal{A}^2 * E(0,Q_0)}(z) = \|Q_0^{-\frac{1}{2}} z\|_{l_1}.$$

The external ellipsoidal estimate of set $\sigma\mathcal{A}^2 * \mathcal{X}_0$ may be found by applying the following theorem.

Theorem 5. [Matviychuk, 2016] *For* $\mathcal{X}_0 = E(a_0, Q_0)$ *and all* $\sigma > 0$ *the following external estimate is true*

$$\sigma\mathcal{A}^2 * \mathcal{X}_0 \subseteq E(a_0, \tilde{Q}(\sigma)) + o(\sigma)B(0, 1), \tag{18}$$

where $\sigma^{-1}o(\sigma) \to 0$ *for* $\sigma \to +0$,

$$\tilde{Q}(\sigma) = \mathrm{diag}\{(p^{-1} + 1)\sigma^2 (a_i^0)^2 + (p + 1) r^2(\sigma)\},$$

$$a_0 = \{a_i^0\}, \quad r(\sigma) = \sigma \max_z \|z\| \left(\|Q_0^{-\frac{1}{2}} z\|_{l_1}\right)^{-1}.$$

Here p *is the unique positive root of the equation* $\sum_{i=1}^n 1/p + \alpha_i = n/p(p+1)$, *where* $\alpha_i \ge 0$ $(i = 1, ..., n)$ *being the roots of the following equation* $\prod_{i=1}^n \left(\sigma^2 (a_i^0)^2 - \alpha r^2(\sigma)\right) = 0$.

Then an external ellipsoidal estimate of the trajectory tube $\mathcal{X}(t)$ of the system (9) may be found by applying the following new result.

Theorem 6. [Matviychuk, 2017b] *For the trajectory tube* $\mathcal{X}(t)$ *of the system (9) and for all* $\sigma > 0$ *the following inclusion holds*

$$\mathcal{X}(t_0 + \sigma) \subseteq E(a^+(\sigma), Q^+(\sigma)) + o(\sigma)B(0, 1), \tag{19}$$

where $\sigma^{-1}o(\sigma) \to 0$ *for* $\sigma \to +0$,

$$a^+(\sigma) = a^*(t_0 + \sigma),$$
$$Q^+(\sigma) = (p^{-1} + 1)\tilde{Q}(\sigma) + (p + 1)Q^*(t_0 + \sigma).$$

Here $a^*(t_0 + \sigma)$, $Q^*(t_0 + \sigma)$, $\tilde{Q}(\sigma)$ *are defined in Theorem 3 and Theorem 5 and* p *is the unique positive root of the equation* $\sum_{i=1}^n 1/p + \alpha_i = n/p(p+1)$, *where* $\alpha_i \ge 0$ $(i = 1, ..., n)$ *being the roots of the following equation* $|\tilde{Q}(\sigma) - \alpha Q^*(t_0 + \sigma)| = 0$.

The following algorithm is based on Theorem 6 and may be used to produce the external ellipsoidal estimates for the reachable sets of the system (9).

Algorithm 1. The time segment $[t_0, T]$ is subdivided into subsegments $[t_i, t_{i+1}]$ where $t_i = t_0 + i\sigma$ $(i = 1, \dots, m)$, $\sigma = (T - t_0)/m$, $t_m = T$.

- For the given $\mathcal{X}_0 = E(a_0, Q_0)$ we find the external estimate $E(a^+(\sigma), Q^+(\sigma))$ by Theorem 6 such that

$$\mathcal{X}(t_1) = \mathcal{X}(t_0 + \sigma) \subseteq E(a^+(\sigma), Q^+(\sigma)).$$

- Consider the system on the next subsegment $[t_1, t_2]$ with $E(a^+(\sigma), Q^+(\sigma))$ as the initial ellipsoid at instant t_1.
- The next step repeats the previous iteration beginning with new initial data.

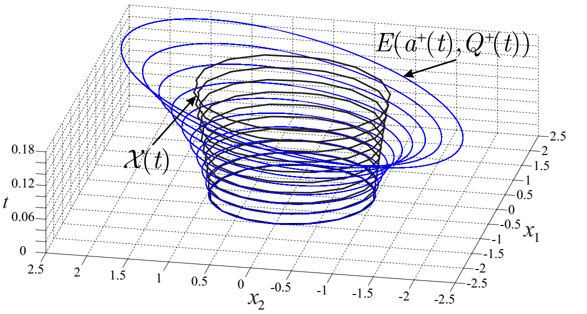

Figure 2. Trajectory tube $\mathcal{X}(t)$ and its ellipsoidal estimating tube $E(a^+(t),Q^+(t))$ for the bilinear control system with uncertain initial states.

At the end of the process we will get the external estimate of the tube $\mathcal{X}(\cdot)$ of the system (9) with accuracy tending to zero when $m \to \infty$.

The following example illustrates the Algorithm 1.

Example 2. Consider the following system

$$\begin{cases} \dot{x}_1 = a_1 x_1 + x_2, \\ \dot{x}_2 = a_2 x_2 + c(t)\, x_1, \end{cases} \quad 0 \le t \le 0.18,$$

where $x_0 \in \mathcal{X}_0 = B(0,1)$, $c(t)$ is an unknown but bounded measurable function with $|c(t)| \le 1$, the uncertain bounded matrix function $A^2(t) \in \mathcal{A}^2$ where

$$\mathcal{A}^2 = \Big\{ A^2(t) : A^2(t) = \mathrm{diag}\{a_1, a_2\}, \\ a_1^2 + a_2^2 \le 1, \;\; t \in [0, 0.18] \Big\}.$$

The trajectory tube $\mathcal{X}(t)$ and its external ellipsoidal estimate $E(a^+(t),Q^+(t))$ calculated by Algorithm 1 are given in the Figure 2.

5 Main Result

Consider the bilinear impulsive control system (1) with restrictions (2)–(5)

$$dx = \big(A(t)x(t) + u(t)\big)dt + B(t)dv(t),$$
$$x(t_0 - 0) = x_0 \in \mathcal{X}_0 = E(a_0, Q_0), \quad t \in [t_0, T],$$
$$A(t) \in \mathcal{A}, \quad u \in \mathcal{U} = E(\hat{a}, \hat{Q}), \quad v \in \mathcal{V}.$$

Let us introduce a new time variable [Rishel, 1965] $\eta = \eta(t)$ and a new state coordinate $\tau = \tau(\eta)$

$$\eta(t) = t + \int_{t_0}^{t} dv(s), \quad \tau(\eta) = \inf\{t \,:\, \eta(t) \ge \eta\}.$$

Consider the following differential inclusion

$$\frac{d}{d\eta}\begin{pmatrix} z \\ \tau \end{pmatrix} \in H(\tau, z), \tag{20}$$
$$z(t_0) = z_0 \in \mathcal{X}_0 = E(a_0, Q_0),$$
$$\tau(t_0) = t_0, \quad t_0 \le \eta \le T + \mu,$$
$$H(\tau, z) = \bigcup_{0 \le \nu \le 1} \Big\{ (1-\nu)\begin{pmatrix} A(\tau)z + E(\hat{a}, \hat{Q}) \\ 1 \end{pmatrix} + \nu \begin{pmatrix} B(\tau) \\ 0 \end{pmatrix} \Big\}.$$

Denote by $w = \{z, \tau\}$ the extended state vector of the differential inclusion (20) and by $W(\eta) = W(\eta; t_0, \mathcal{X}_0 \times \{t_0\}, \mathcal{A})$ $(t_0 \le \eta \le T + \mu)$ the reachable set of the system (20).

Theorem 7. *For any $\sigma > 0$ following inclusion holds*

$$W(t_0 + \sigma) \subseteq W(t_0, \sigma) + o(\sigma)B^{n+1}(0,1), \tag{21}$$
$$\lim_{\sigma \to +0} \sigma^{-1} o(\sigma) = 0,$$
$$W(t_0, \sigma) = \bigcup_{0 \le \nu \le 1} W(t_0, \sigma, \nu),$$
$$W(t_0, \sigma, \nu) = \begin{pmatrix} E\big(a^+(t_0, \sigma, \nu), Q^+(t_0, \sigma, \nu)\big) \\ t_0 + \sigma(1-\nu) \end{pmatrix},$$
$$a^+(t_0, \sigma, \nu) = \tilde{a}^*(\sigma, \nu) + \sigma(1-\nu)\hat{a} + \sigma\nu B(t_0),$$
$$Q^+(t_0, \sigma, \nu) = (p^{-1} + 1)\sigma^2(1-\nu)^2 \bar{Q}(\sigma) + (p+1)\tilde{Q}^*(\sigma, \nu),$$
$$\bar{Q}(\sigma) = (q^{-1} + 1)\hat{Q} + (q+1)\tilde{Q}(\sigma),$$

where $\tilde{Q}(\sigma)$ is defined in the Theorem 5 and functions $\tilde{a}^*(\sigma, \nu)$, $\tilde{Q}^*(\sigma, \nu)$ are calculated as $a^*(t)$, $Q^*(t)$ in the Theorem 3 but when we replace matrix A^0 in (14), (15) *by $\tilde{A}^0 = (1-\nu)A^0$. Here $p = p(\sigma, \nu)$ is the unique positive root of the equation $\sum_{i=1}^{n} 1/p + \lambda_i = n/p(p+1)$, $\lambda_i = \lambda_i(\sigma, \nu) \ge 0$ being the roots of the equation $|\sigma^2(1-\nu)^2\bar{Q}(\sigma) - \lambda\tilde{Q}^*(\sigma, \nu)| = 0$, and $q = q(\sigma, \nu)$ is the unique positive root of the equation*

$$\sum_{i=1}^{n} 1/q + \alpha_i = n/q(q+1),$$

where $\alpha_i = \alpha_i(\sigma, \nu) \ge 0$ satisfy the equation $|\bar{Q}(\sigma) - \alpha\tilde{Q}(\sigma)| = 0$.

Proof. The proof of this theorem uses the procedure of external ellipsoidal estimating a sum of two ellipsoids [Chernousko, 1994; Kurzhanski and Valyi, 1997]. Applying the scheme from [Filippova and Matviychuk, 2011; Filippova and Matviychuk, 2015] and using results of the Theorem 6 we can find the upper estimatesfor reachable sets $W(t_0 + \sigma)$ of the differential inclusion (20). □

Remark 2. To find the estimate of the reachable set $W(t_0 + \sigma)$ we introduce a small parameter $\varepsilon > 0$ and embed the degenerate ellipsoid $W(t_0, \sigma, \nu)$ in the nondegenerate ellipsoid $E\big(w_\varepsilon(t_0, \sigma, \nu), O_\varepsilon(t_0, \sigma, \nu)\big)$:

$$W(t_0, \sigma, \nu) \subseteq E\big(w_\varepsilon(t_0, \sigma, \nu), O_\varepsilon(t_0, \sigma, \nu)\big),$$
$$w_\varepsilon(t_0, \sigma, \nu) = \begin{pmatrix} a^+(t_0, \sigma, \nu) \\ t_0 + \sigma(1-\nu) \end{pmatrix},$$
$$O_\varepsilon(t_0, \sigma, \nu) = \begin{pmatrix} Q^+(t_0, \sigma, \nu) & 0 \\ 0 & \varepsilon^2 \end{pmatrix}.$$

Thus, for all small $\varepsilon > 0$ we get

$$W(t_0 + \sigma) \subseteq W(t_0, \sigma) \subseteq W_\varepsilon(t_0, \sigma),$$
$$W_\varepsilon(t_0, \sigma) = \bigcup_{0 \le \nu \le 1} E\big(w_\varepsilon(t_0, \sigma, \nu), O_\varepsilon(t_0, \sigma, \nu)\big),$$

where $\lim\limits_{\varepsilon \to +0} h(W(t_0, \sigma), W_\varepsilon(t_0, \sigma)) = 0$.

The passage to the family of nondegenerate ellipsoids enables one to use the algorithms of [Vzdornova and Filippova, 2006] and construct an external estimate of the union of the ellipsoids

$$W_\varepsilon(t_0, \sigma) \subset E_\varepsilon(w^+(\sigma), O^+(\sigma)).$$

The following lemma explains the reason of construction of the auxiliary differential inclusion (20).

Lemma 1. [Filippova and Matviychuk, 2011] *The set $\mathcal{X}(T) = \mathcal{X}(T, t_0, \mathcal{X}_0)$ is the projection of $W(T + \mu)$ at the subspace of variables z $\mathcal{X}(T) = \pi_z W(T + \mu)$.*

The next iterative algorithm is based on Theorem 7 and allows to find the external ellipsoidal estimates of the reachable sets of the studied bilinear impulsive control system (1)–(5).

Algorithm 2. The time segment $[t_0, T + \mu]$ is subdivided into subsegments $[t_i, t_{i+1}]$ where $t_i = t_0 + i\sigma$ $(i = 1, \ldots, m)$, $\sigma = (T + \mu - t_0)/m$, $t_m = T + \mu$. Subdivide the segment $[0, 1]$ into subsegments $[\nu_j, \nu_{j+1}]$ where $\nu_j = jh_*$, $h_* = 1/k$, $\nu_0 = 0$, $\nu_k = 1$ $(j = 1, \ldots, k)$.

1. For the given $\mathcal{X}_0 = E(a_0, Q_0)$ define sets $W(t_0, \sigma, \nu_j)$, $j = 0, \ldots, k$ by Theorem 7.
2. Fix the small parameter $\varepsilon > 0$ and for sets $W(\sigma, \nu_j)$ $(j = 0, \ldots, k)$ find ellipsoids $E(w_\varepsilon(t_0, \sigma, \nu_j), O_\varepsilon(t_0, \sigma, \nu_j))$ by Remark 2.
3. Find ellipsoid $E_\varepsilon(w_1(\sigma), O_1(\sigma))$ in $\mathbb{R}^{n+1}$ such that

$$W_\varepsilon(t_0, \sigma) = \bigcup_{j=1}^{m} E\big(w_\varepsilon(t_0, \sigma, \nu_j), O_\varepsilon(t_0, \sigma, \nu_j)\big)$$
$$\subseteq E_\varepsilon(w_1(\sigma), O_1(\sigma)).$$

   At this step we find the ellipsoidal estimate for the union of a finite family of ellipsoids [Filippova and Matviychuk, 2011; Matviychuk, 2012].
4. Find the projection of $E_\varepsilon(w_1(\sigma), O_1(\sigma))$ at the subspace of variables z by Lemma 1: $E(a_1, Q_1) = \pi_z E_\varepsilon(w_1(\sigma), O_1(\sigma))$.
5. Consider the system on the next subsegment $[t_1, t_2]$ with $E(a_1, Q_1)$ as the initial ellipsoid at instant t_1.
6. The next step repeats the previous iteration beginning with new initial data.

At the end of the process we will get the external estimate $E(a^+(T), Q^+(T))$ of the reachable set $\mathcal{X}(T)$ of the impulsive control system (1)–(5) with uncertain matrix of the system and uncertain initial state basing on the special structure of the data $\mathcal{A}, \mathcal{U}$ and $\mathcal{X}_0$.

6 Conclusion

The problem of state estimation of the reachable sets for uncertain impulsive control systems for which we assume that the initial state is unknown but bounded with given constraints and the matrix in the linear part of state velocities is also unknown but bounded was considered in this paper.

The modified state estimation method which uses the special constraints on the controls and uncertainty and allows to construct the external ellipsoidal estimates of reachable sets is presented here. This method is based on results of ellipsoidal calculus developed earlier for some classes of uncertain systems.

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