Atomic Probability Amplitude Stabilization with Feedback Control Optical Field

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***Abstract*— We demonstrate the possibility to stabilize the probability amplitude of the upper level for a single quantum two-level atom in a classical optical field with feedback control scheme.**

I. INTRODUCTION

The methods of feedback control are widely used in the modern physics, but still they are not very popular in quantum optics. What is sufficient that very often this "cybernetical" approach does not demand to involve a very complicated physical devices and can be arranged in a trivial nonlinear system [1].

We apply this technique to control the energy of two-level atom in the optical external field $E(t)$ in the frame of the so-called "semi classical model" of the atom–field interaction that describes a single quantum two-level atomic system (all other levels are neglected) with classical electromagnetic field.

Recently other authors studied the control of two-level atoms in the frame of open loop-ideology when the controlling field was known *a priori*. It allowed to get the different forms of atomic energy spectra, producing π- and $\frac{\pi}{2}$- pulses [2], taking special non-constant shapes of external field [3] etc.

The main feature of the model proposed here is that it is based on the closed-loop approach. It means that we do not define initially the dependency of the field on time, but restore this function for every moment from the current values of the probability amplitudes of the atomic ground and excited levels.

We use the standard notations following [4], but in our model the optical field plays the role of a control signal $u(t)$ for closed-loop or feedback control scheme in the form of speed-gradient (SG) method [5].

For this purpose we use the real positive goal function Q, measuring how far at the moment we are from the desired state of the atom. As a result we calculate the control signal $u(t)$, i.e. we restore the shape of the electromagnetic filed $E(t)$ to keep atom at the upper level.

In the second section of this work we construct the feedback control model for the single two-level atom in external controlling optical field. Then, in the third section, we apply feedback speed gradient scheme to the non-decay case.

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II. TWO-LEVEL ATOM IN CONTROL OPTICAL FIELD

Let's consider the interaction of an optical field $E(t)$ linearly polarized along the x-axis with a two-level atom.

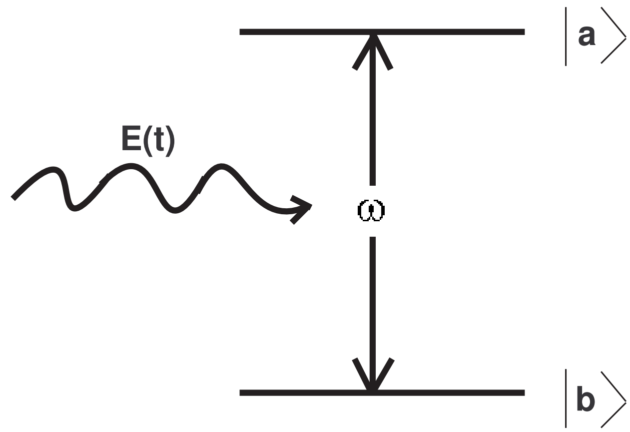

Fig. 1. Interaction of a single two-level atom with an optical field.

Let $|a\rangle$ and $|b\rangle$ represent the upper and lower level states of the atom, i.e., they are eigenstates of the unperturbed part of the Hamiltonian $\hat{H}_0$ with the eigenvalues: $\hat{H}_0|a\rangle = \hbar\omega_a|a\rangle$ and $\hat{H}_0|b\rangle = \hbar\omega_b|b\rangle$. The wave function of a two-level atom can be written in the form

$$|\psi(t)\rangle = C_a(t)|a\rangle + C_b(t)|b\rangle,$$

where C_a and C_b are the probability amplitudes of finding the atom in states $|a\rangle$ and $|b\rangle$, respectively. The corresponding Schrödinger equation is:

$$|\dot{\psi(t)}\rangle = -\frac{\iota}{\hbar}\hat{H}|\psi(t)\rangle,$$

with $\hat{H} = \hat{H}_0 + \hat{H}_1$, where $\hat{H}_0$ and $\hat{H}_1$ represent the unperturbed and interaction parts of the Hamiltonian, respectively [4]:

$$\begin{aligned}\hat{H}_0 &= \hbar\omega_a|a\rangle\langle a| + \hbar\omega_b|b\rangle\langle b| \ ; \\ \hat{H}_1 &= -\Big(\wp_{ab}|a\rangle\langle b| + \wp_{ba}|b\rangle\langle a|\Big)E(t),\end{aligned}$$

where $\wp_{ab} = \wp_{ba}^* = e\langle a|x|b\rangle$ is the matrix element of the electric dipole moment. We neglected the decay of the levels. We express the electric field as

$$E(t) = E_0 u(t),$$

where E_0 is the amplitude and $u(t)$ is the dimensionless control signal. The equations of motion for the amplitudes C_a and C_b may be written as

$$\dot{C_a} = -\iota\omega_a C_a + \iota\Omega_R u(t) e^{-\iota\phi} C_b,$$
$$\dot{C_b} = -\iota\omega_b C_b + \iota\Omega_R u(t) e^{\iota\phi} C_a,$$

where the "Rabi frequency" is defined as $\Omega_R = \frac{|\wp_{ba}|E_0}{\hbar}$, and ϕ is the phase of the dipole matrix element $\wp_{ba} = |\wp_{ba}|e^{\iota\phi}$.

To solve for C_a and C_b, let's write the equations of motion for the slowly varying amplitudes:

$$c_a = C_a e^{\iota\omega_a t} \;\; ; \;\; c_b = C_b e^{\iota\omega_b t},$$

then

$$\dot{c_a} = \iota\Omega_R u(t) e^{-\iota\phi} c_b e^{\iota\omega t} \; ;$$
$$\dot{c_b} = \iota\Omega_R u(t) e^{\iota\phi} c_a e^{-\iota\omega t} \; ,$$

where $\omega = \omega_a - \omega_b$ is the atomic transition frequency. The phase ϕ can be excluded from the system. Really, if we put $\widetilde{c_b} = c_b e^{-\iota\phi}$:

$$\dot{\widetilde{c_b}} = \iota\Omega_R u(t) e^{-\iota w t} c_a$$

Later for simplicity we will denote $\widetilde{c_b}$ with c_b, then finally:

$$\dot{c_a} = \iota\Omega_R u(t) e^{\iota w t} c_b \tag{1}$$

$$\dot{c_b} = \iota\Omega_R u(t) e^{-\iota w t} c_a \tag{2}$$

Now let's suppose that we have the initial conditions:

$$c_a(0) = 0 \;\; ; \;\; c_b(0) = 1 \tag{3}$$

and our goal is to stabilize the atomic system at the upper level: $|c_a|^2 = 1$.

III. Speed gradient method for probability amplitude control

Still we did not specify the time-dependent function $u(t)$. To find it, we apply the speed gradient (SG) method [5] to control the system behavior.

In this approach the control action is chosen in the maximum descent direction for a scalar goal function.

The goal in the control process is a smooth scalar function Q with the limit relation

$$\lim_{t\to\infty} Q(x(t), t) \to 0.$$

The purpose of the SG method is to minimize the goal function

$$Q = \frac{1}{2}\Big(|c_a|^2 - 1\Big)^2, \tag{4}$$

where $|c_a|^2 = c_a c_a^*$.

SG represents the control signal u with the time derivative of the goal function Q.

In the case of proportional feedback with some positive coefficients $\Gamma_0, \; \Gamma_1$, it is defined in the form:

$$u = \Gamma_0 - \Gamma_1 \frac{\partial \dot{Q}}{\partial u} \tag{5}$$

Thus

$$u(t) = \Gamma_0 + \iota\Gamma_1\Omega_R\Big(|c_a|^2 - 1\Big)\Big(e^{-\iota w t} c_a c_b^* - e^{\iota w t} c_b c_a^*\Big) \tag{6}$$

Putting value of $u(t)$ from Eq.(6) in Eqs.(1) and (2), we have the following system of equations:

$$\begin{aligned}
\dot{c_a} &= \iota\Omega_R\Gamma_0 e^{\iota w t} c_b + \Gamma_1\Omega_R^2\Big(|c_a|^2 - 1\Big) \\
&\quad \Big(e^{2\iota w t} c_a^* c_b^2 - c_a|c_b|^2\Big) \; ; \\
\dot{c_b} &= \iota\Omega_R\Gamma_0 e^{-\iota w t} c_a + \Gamma_1\Omega_R^2\Big(|c_a|^2 - 1\Big) \\
&\quad \Big(c_b|c_a|^2 - e^{-2\iota w t} c_a^2 c_b^*\Big).
\end{aligned}$$

Now suppose that

$$\rho_a = c_a c_a^* = |c_a|^2 \;\; ; \;\; \rho_b = c_b c_b^* = |c_b|^2 \;\; ;$$
$$\iota\rho_- = e^{-\iota w t} c_a c_b^* - e^{\iota w t} c_a^* c_b \;\; ;$$
$$\rho_+ = e^{-\iota w t} c_a c_b^* + e^{\iota w t} c_a^* c_b.$$

Hence we have the following four equations:

$$\begin{aligned}
\dot{\rho_a} &= \Omega_R\Gamma_0\rho_- + 2\Gamma_1\Omega_R^2\Big(\rho_a - 1\Big) \\
&\quad \Big[\Big(\frac{\rho_+^2 - \rho_-^2}{4}\Big) - \rho_a\rho_b\Big] \; ; \\
\dot{\rho_b} &= -\Omega_R\Gamma_0\rho_- + 2\Gamma_1\Omega_R^2\Big(\rho_a - 1\Big) \\
&\quad \Big[\rho_a\rho_b - \Big(\frac{\rho_+^2 - \rho_-^2}{4}\Big)\Big] \; ; \\
\dot{\rho_+} &= w\rho_- \; ; \\
\dot{\rho_-} &= 2\Omega_R\Big(\rho_b - \rho_a\Big)\Big[\Gamma_0 - \Gamma_1\Omega_R\Big(\rho_a - 1\Big)\rho_-\Big] \\
&\quad -w\rho_+ \; .
\end{aligned} \tag{7}$$

Also from Eq.(6) the control signal $u(t)$ becomes

$$u(t) = \Gamma_0 - \Gamma_1\Omega_R\Big(\rho_a - 1\Big)\rho_- \tag{8}$$

With initial conditions $\rho_a(0) = 0, \; \rho_b(0) = 1$ we have

$$\dot{\rho_a} + \dot{\rho_b} = 0 \; ,$$

that means in fact:

$$|c_a(t)|^2 + |c_b(t)|^2 = 1,$$

which is the simple statement that the probability to find the atom in one of its states $|a\rangle$ or $|b\rangle$ is 1.

Thus, we can simplify the system (7), putting $\rho_b = 1 - \rho_a$.

The system (7) has two equilibrium(fixed) points:

$$(\rho_a, \rho_+, \rho_-) = \Big(\frac{1}{2}\Big[1 \pm \frac{\omega}{\sqrt{4\Gamma_0^2\Omega_R^2 + \omega^2}}\Big], -\frac{2\Gamma_0\Omega_R}{\omega}, 0\Big)$$

From above we can easily get

$$\rho_b = \frac{1}{2}\Big[1 \pm \frac{\omega}{\sqrt{4\Gamma_0^2\Omega_R^2 + \omega^2}}\Big] \; .$$

Thus, it is necessary to include the constant part of the signal, Γ_0 into the SG scheme (5) not to start the control procedure from the equilibrium(fixed) point.

On the Fig.2,3 we demonstrate the result of our control procedure for: $\Gamma_0 = 0.01$, $\Gamma_1 = 0.1 \ sec$, $\Omega_R = 10^2 \ sec^{-1}$ and $\omega = 10^3 \ sec^{-1}$.

On Fig.2 we show the solution of Eq.(7a).

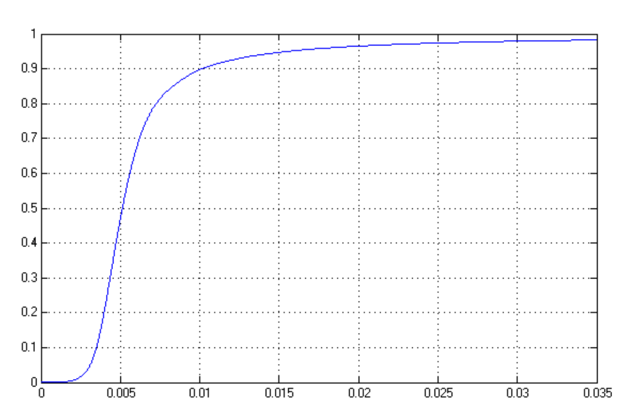

Fig. 2. The density matrix element $\rho_a(t)$ for the control procedure (4)-(5)

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On Fig.3 we show the control signal(8).

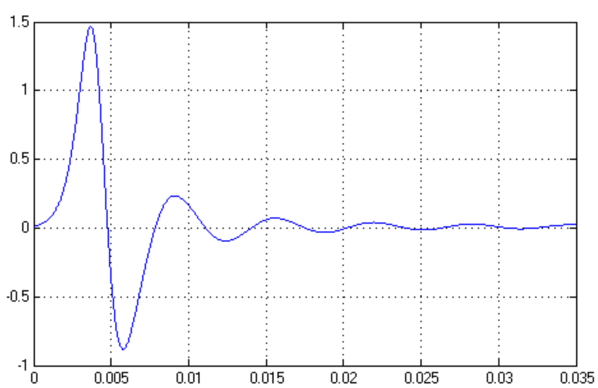

Fig. 3. The control signal $u(t)$ for the system (7).

IV. Conclusions

The SG algorithm can be easy applied to establish feedback control for the probability amplitudes of two-level atom.

This scheme, nevertheless, should be sufficiently modified if we take into consideration the decay of the atom levels, because in this case the goal (4) is not achievable for SG algorithm in principle.

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