CONTROLLED SYNCHRONIZATION IN TWO DYNAMICAL SYSTEMS WITH SECTOR BOUNDED NONLINEARITIES

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Abstract
We study synchronization in two delay-coupled FitzHugh-Nagumo systems with slow-varying delays, which are the simplest model of neural network. Further we generalize FitzHugh-Nagumo model and consider synchronization in the more common case. We show that external stimulus can be used to control synchronization. We develop one algorithm for synchronization of FitzHugh-Nagumo systems and the other one to control synchrony in the generalized neural model, and find the conditions of their applicability. The first algorithm provides synchronization in the case, when all parameters are known and measurable, while the second one is an adaptive controller and is used in the case with uncertain parameters of the system.

Key words
Synchronization, Neural networks, Adaptive control

1 Introduction

The ability to control nonlinear dynamical systems has brought up a wide interdisciplinary area of research that has evolved rapidly in the past decades [Schöll and Schuster, 2008]. This field has various aspects comprising stabilization of unstable fixed points (steady states), control of oscillation in systems with several degrees of freedom [Plotnikov and Andrievsky, 2013], or control of network dynamics [Selivanov et al., 2012; Lehnert et al., 2014]. One of the important areas to consider is the synchronization in neural networks.

As any other kind of physical, chemical, or biological oscillators, such neurons could synchronize and exhibit collective behavior that is not intrinsic to any individual neuron. For example, partial synchrony in cortical systems is believed to generate various brain oscillations, such as the alpha and gamma EEG rhythms. Increased synchrony may result in pathological types of activity, such as epilepsy. Coordinated synchrony is needed for locomotion and swim pattern generation in fish [Izhikevich, 2005]. On the one hand, synchronization can be good, while, on the other hand, it can be harmful. Therefore, it is important to study synchronization in neural networks. The control of synchronization has so far focused on networks of identical nodes with constant parameters [Zhou, Lu and Lu, 2008; Lu and Qin, 2009; Lu et al., 2012; Selivanov et al., 2012; Guzenko, Lehnert and Schöll, 2013; Lehnert et al., 2014]. However, in realistic networks the nodes are always characterized by some diversity meaning that the parameters of the different nodes are not identical and may vary over the time. These variations and heterogeneities in the nodes can hinder or prevent synchronization.

In order to grasp the complicated interaction of neurons in large neural networks, those are often lumped into groups of neural populations each of which can be represented as en effective excitable element that is mutually coupled to other elements [Rosenblum and Pikovsky, 2004; Popovych, Hauptmann and Tass, 2004]. In this sense the simplest model which may reveal features of interacting neurons consists of two coupled neural oscillators. Each of this can be represented by a simplifiled FitzHugh-Nagumo system [FitzHugh, 1961; Nagumo, Arimoto and Yoshizawa, 1962].

We propose the algorithm to control synchronization in two delay-coupled systems with slowly-varying delays and show that it can be used to counteract the variations in connections time between the neurons. Further we generalize FitzHugh-Nagumo model and consider synchronization in the more common case with sector bounded nonlinearities and unknown parameters. The adaptive controller, which is based on SG method [Fradkov, 1979; Fradkov, 2007], allows to estimate unknown parameters of the system and provide the synchronization.

The paper is organized as follows. After this introduction we describe the system model, define the synchronization problem for two FHN-systems and develop the control algorithm in Sec. 2. Section 3 describes the control algorithm for two generalized systems with uknown parameters. Section 4 shows the results of the numerical simulation of developed algorithm performance. Finally, we conclude with Sec. 5.

2 Synchronization of two FHN-systems

We consider two delay-coupled FitzHugh-Nagumo (FHN) systems [FitzHugh, 1961; Nagumo, Arimoto and Yoshizawa, 1962], which are the simplest model of neural network. The FHN model is paradigmatic for excitable dynamics close to a Hopf bifurcation [Lindner et al., 2004], which is not only characteristic for neurons but also occurs in the context of other systems ranging from electronic circuits [Heinrich et al., 2010] to cardiovascular tissues and the climate system [Murray, 1993; Izhikevich, 2000]. The plant is described as follows:

$$\begin{aligned} \varepsilon \dot{u_1} &= u_1 - \frac{u_1^3}{3} - v_1 + C[u_2(t-\tau) - u_1(t)] + I, \\ \dot{v_1} &= u_1 + a, \\ \varepsilon \dot{u_2} &= u_2 - \frac{u_2^3}{3} - v_2 + C[u_1(t-\tau) - u_2(t)], \\ \dot{v_2} &= u_2 + a, \end{aligned} \tag{1}$$

where u_i and v_i denote the membrane potential and recovery variable of the nodes $i = 1, 2$ respectively, ε is a time-scale parameter and typically small, meaning that u_i is a fast variable, while v_i changes slowly. τ is the delay, i.e., the time the signal needs to propagate between two nodes. Let assume that the delay τ is a differentiable function with $\dot{\tau} \leqslant d < 1$ (this is tha case of *slowly-varying delays*). I is external stimulus and will be considered as a control. The coupling strength is given by C. In the uncoupled system ($C = 0$), a is the threshold parameter: for $a > 1$ the system is excitable while for $a < 1$ it exhibits self-sustained periodic firing. This is due to a supercritical Hopf bifurcation at $a = 1$ with a locally stable fixed point for $a > 1$ and a stable limit cycle for $a < 1$.

Let state the problem of variable value synchronization in two coupled FHN systems. We subtract the third equation from the first one, and the fourth one from the second one (1) making the following substitution

$$\delta_1 = u_1 - u_2, \qquad \delta_2 = v_1 - v_2, \tag{2}$$

and get

$$\begin{aligned} \varepsilon \dot{\delta}_1(t) &= (1 - C - \phi(t))\delta_1(t) \\ &\quad - C\delta_1(t - \tau(t)) - \delta_2(t) + I(t), \\ \dot{\delta}_2(t) &= \delta_1(t), \end{aligned} \tag{3}$$

$\phi = 1/3(u_1^2 + u_1 u_2 + u_2^2)$, $\phi(t) \geqslant 0\ \forall t$ is nonnegative function. Then the control goal can be described as follows

$$\delta_1(t) \to 0, \quad \delta_2(t) \to 0, \quad \text{while } t \to \infty. \tag{4}$$

We want to find the control $I(t)$ to ensure the control goal (4). For this purpose let introduce the following Lyapunov function

$$V(t, \mathbf{\Delta}(t)) = \varepsilon\delta_1^2 + \delta_2^2 + \theta_0 \int\limits_{t-\tau(t)}^{t} \delta_1^2(s)\mathrm{d}s, \tag{5}$$

where $\mathbf{\Delta} = (\delta_1, \delta_2)$, while $\theta_0 > 0$ is some positive parameter. Find its derivative according to the system (1)

$$\begin{aligned} \dot{V}(t, \mathbf{\Delta}(t)) &= (2 - 2C - 2\phi(t) + \theta_0)\delta_1^2(t) \\ &\quad - 2C\delta_1(t)\delta_1(t-\tau) \\ &\quad - \theta_0(1 - \dot{\tau})\delta_1^2(t-\tau) + 2\delta_1(t)I(t). \end{aligned} \tag{6}$$

Let choose the control $I(t)$ in form

$$I(t) = -\theta_1\delta_1(t) + \theta_2\delta_1(t-\tau), \tag{7}$$

where $\theta_1 \geqslant 0$, θ_2 are control parameters. We substitute the chosen control to the expression (6). We should choose control parameters such that to make the Lyapunov function derivative negative for all $\delta_1(t)$, $\delta_1(t-\tau)$ except zero, i.e., the following inequality should be fulfilled

$$\begin{aligned} &(2 - 2C - 2\phi(t) + \theta_0 - 2\theta_1)\delta_1^2(t) - 2(C - \theta_2) \\ &\quad \times \delta_1(t)\delta_1(t-\tau) - \theta_0(1 - \dot{\tau})\delta_1^2(t-\tau) < 0, \end{aligned} \tag{8}$$

that can be presented in form

$$\begin{bmatrix}\delta_1(t) & \delta_1(t-\tau)\end{bmatrix} W \begin{bmatrix}\delta_1(t) \\ \delta_1(t-\tau)\end{bmatrix} < 0, \tag{9}$$

where

$$W = \begin{bmatrix} 2(1 - C - \phi - \theta_1) + \theta_0 & -C + \theta_2 \\ -C + \theta_2 & -\theta_0(1 - \dot{\tau}) \end{bmatrix} < 0. \tag{10}$$

Thus, we should make matrix W be negative-definite (10) by tuning the control parameters θ_0, θ_1, θ_2. We use Sylvester's criterion to determine whether the matrix W is negative-definite. Since $\theta > 0$ and $\dot{\tau} \leqslant d < 1$ then the lower principal minor of matrix W is negative. The matrix W is negative-definite if and only if its determinant is positive. Since $\phi(t) \geqslant 0$ and $\dot{\tau} \leqslant d < 1$, then control parameters must satisfy the inequality

$$(2\theta_1 + 2C - \theta_0 - 2)\theta_0(1 - d) - (\theta_2 - C)^2 > 0, \tag{11}$$

that can be rewritten as

$$-(1-d)\theta_0^2 + 2(\theta_1 + C - 1)\theta_0 - (\theta_2 - C)^2 > 0. \tag{12}$$

This inequality is fulfilled for some positive θ_0, when the following quadratic equation for θ_0 has real roots

$$-(1-d)\theta_0^2+2(1-d)(\theta_1+C-1)\theta_0-(\theta_2-C)^2=0, \quad (13)$$

and the following inequality is fulfilled by Vieta's formulas

$$\theta_1+C-1>0. \quad (14)$$

Thus, the discriminant of the equation (13) must be positive

$$4(1-d)^2(\theta_1+C-1)^2-4(1-d)(\theta_2-C)^2>0, \quad (15)$$

that, considering inequality (14), can be presented as

$$\theta_1>\frac{|\theta_2-C|}{\sqrt{1-d}}-C+1. \quad (16)$$

Thus, the following theorem takes place

Theorem 1. *Let the delay τ be slowly-varying differential function in the plant* (1)*, i.e., $\dot{\tau}\leqslant d<1$. Then the control $I(t)$ in form* (7)*, where parameters $\theta_1\geqslant 0$ and θ_2 satisfy the inequality* (16)*, ensures the control goal* (4)*, meaning the substitutions* (2)*.*

3 Synchronization of two generalized systems

Now consider the case of two neural systems with uncertain parameters, which is described by the following equations

$$\begin{aligned}\varepsilon\dot{u_1}&=u_1-f_1(u_1)-v_1+C[u_2(t-\tau)-u_1(t)]+I,\\ \dot{v_1}&=u_1+a,\\ \varepsilon\dot{u_2}&=u_2-f_2(u_2)-v_2+C[u_1(t-\tau)-u_2(t)],\\ \dot{v_2}&=u_2+a,\end{aligned} \quad (17)$$

where f_1, f_2 are some functions, which satisfy $f_1(u_1)-f_2(u_2)=(u_1-u_2)g(u_1,u_2)$ with nonnegative function $g(t)\geqslant 0\;\forall t$. The coupling strength C and threshold a are uncertain parameters, and the delay τ is a measurable function.

Now we want to state the synchronization problem for the system (17). Let choose the substitutions (2) and the control goal (4). We subtract the third equation from the first one, and the fourth one from the second one (17) and get

$$\begin{aligned}\varepsilon\dot{\delta}_1(t)&=(1-C-g(t))\delta_1(t)\\ &\quad-C\delta_1(t-\tau(t))-\delta_2(t)+I(t),\\ \delta_2(t)&=\delta_1(t).\end{aligned} \quad (18)$$

Now let introduce the following Lyapunov function

$$V(t,\mathbf{\Delta}(t))=\varepsilon\delta_1^2+\delta_2^2+\frac{1}{\gamma_0}(\theta(t)-C)^2, \quad (19)$$

where $\mathbf{\Delta}=(\delta_1,\delta_2)$, while θ and $\gamma_0>0$ are tunable parameters which will be defined later. Find its derivative according to the system (17)

$$\begin{aligned}\dot{V}(t,\mathbf{\Delta}(t))&=2(1-C-g(t))\delta_1^2(t)+2\delta_1(t)I(t)\\ &\quad-2C\delta_1(t)\delta_1(t-\tau)+\frac{2}{\gamma_0}(\theta(t)-C)\dot{\theta}(t).\end{aligned} \quad (20)$$

Let choose the control $I(t)$ in form

$$I(t)=\theta(\delta_1(t)+\delta_1(t-\tau))-\gamma\delta_1(t), \quad (21)$$

where $\gamma\geqslant 1$ is a gain. The tuning of parameter θ can be performed by SG-algorithm [Fradkov, 2007]

$$\dot{\theta}(t)=-\gamma_0\delta_1(t)(\delta_1(t)+\delta_1(t-\tau)). \quad (22)$$

Let substitute the control $I(t)$ in form (21), (22) to the Lyapunov function derivative (20), then

$$\dot{V}(t,\mathbf{\Delta}(t))=-2(g(t)+\gamma-1)\delta_1^2(t), \quad (23)$$

which is negative for all δ_1 except zero.

Thus, the following theorem takes place

Theorem 2. *Suppose that in the plant* (17) *f_1, f_2 are some functions, which satisfy $f_1(u_1)-f_2(u_2)=-(u_1-u_2)g(u_1,u_2)$ with nonnegative function $g(t)\geqslant 0\;\forall t$. Then the control $I(t)$ in form* (21)*,* (22) *with $\gamma\geqslant 1$, $\gamma_0>0$ ensures the control goal* (4)*, meaning the substitutions* (2)*.*

4 Simulation

The simulation was carried out in Matlab R2009b.

Firstly, we consider the case of the system (1) behavior without control. The system parameters: $a=0.7$, $C=1$, $\varepsilon=0.1$, $\tau(t)=3+1/2\cos(t)$. The initial conditions: $u_1(t)=\cos(t)$, $u_2(t)=-\cos(t)$, $v_1(t)=\sin(t)$, $v_2(t)=-\sin(t)$ for $t\in[-\tau,0]$. The two coupled FHN-systems do not synchronize: Figure 1 shows in (a) and (b) the time series of the membrane potentials and the recovery variables, respectively, in (c) the synchronization errors of the membrane potentials, in (d) the phase portrait.

Now we use the control according to Eq. (7) with $\theta_1=5$, $\theta_2=1$ in order to synchronize the two systems. Figure 2 shows the results. After a transient time of approximately 20 units of time the two systems

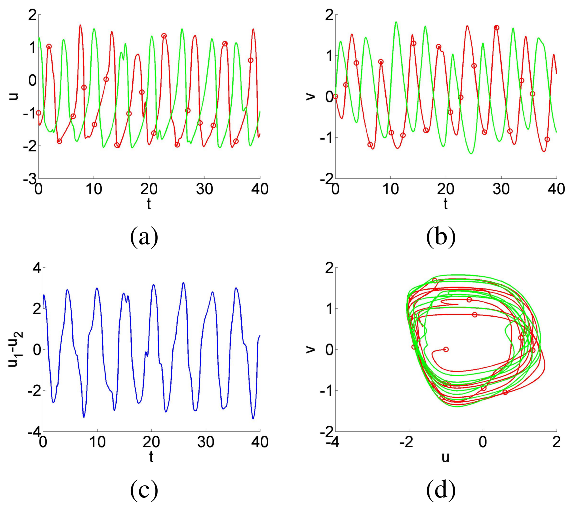

Figure 1. Dynamics of two coupled FitzHugh-Nagumo systems according to Eq. (1) without control. Green solid line marks node one, red line with circles marks node two. (a) and (b): time series of the membrane potential and the recovery variable, respectively; (c): time series of synchronization errors of the membrane potentials; and (d): phase space. The system parameters: $a = 0.7$, $C = 1, \varepsilon = 0.1, \tau(t) = 3 + 1/2\cos(t)$. The initial conditions: $u_1(t) = \cos(t), u_2(t) = -\cos(t), v_1(t) = \sin(t)$, $v_2(t) = -\sin(t)$ for $t \in [-\tau, 0]$.

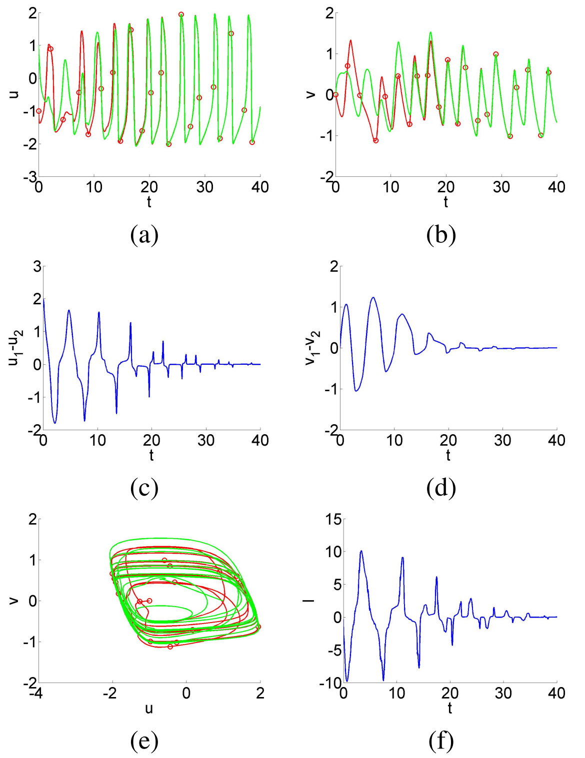

Figure 2. Control of synchronization of two coupled FitzHugh-Nagumo systems (Eq. (1)) with control algorithm in form (7). (a) and (b): time series of the membrane potential and the recovery variable, respectively; (c) and (d): time series of synchronization errors of the membrane potential and the recovery variable, respectively; (e): phase space; and (f): time series of the external stimulus adapted according to Eq. (7). $\theta_1 = 5, \theta_2 = 1$. Other parameters and initial conditions as in Fig 1.

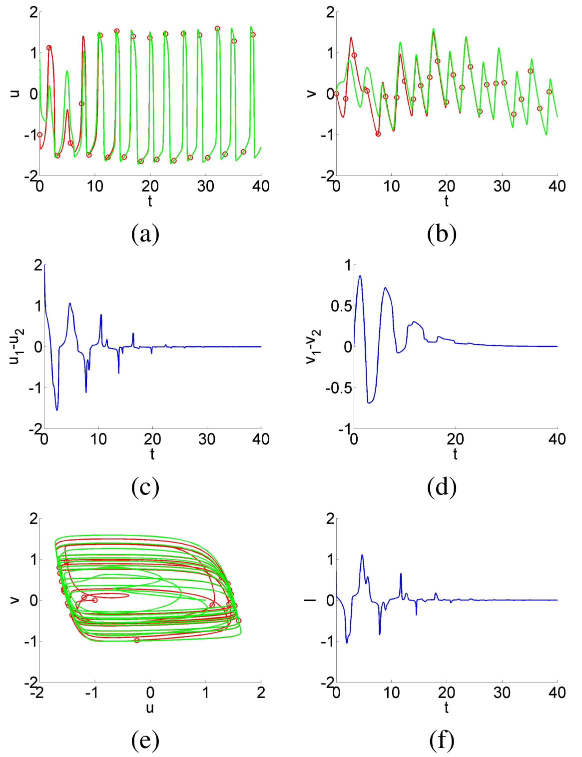

Figure 3. Control of synchronization of two coupled neural systems (Eq. (24)) with adaptive controller in form (21), (22). (a) and (b): time series of the membrane potential and the recovery variable, respectively; (c) and (d): time series of synchronization errors of the membrane potential and the recovery variable, respectively; (e): phase space; and (f): time series of the external stimulus adapted according to Eq. (21), (22). $\gamma = 0, \gamma_0 = 0.1$. Other parameters and initial conditions as in Fig 1.

reach the desired synchronized state (see the time series of the membrane potential and the recovery variable in Fig. 2(a), (b) and their synchronization errors in Fig. 2(c), (d).) Thus, the control is successful. Note that the control I is bounded and tends to zero while $t \to \infty$ (see Fig. 2(f)).

Finally, we consider the following system

$$
\begin{aligned}
\varepsilon \dot{u_1} &= u_1 - \frac{u_1^5}{5} - v_1 + C[u_2(t-\tau) - u_1(t)] + I,\\
\dot{v_1} &= u_1 + a,\\
\varepsilon \dot{u_2} &= u_2 - \frac{u_2^5}{5} - v_2 + C[u_1(t-\tau) - u_2(t)],\\
\dot{v_2} &= u_2 + a,
\end{aligned}
\tag{24}
$$

with the same parameters and initial conditions as in the previous cases. Suppose that parameters a, C and ε are uknown. Therefore, we use the adaptive controller in form (21), (22) with $\gamma = 1$, $\gamma_0 = 0.1$ in order to synchronize the two systems. Figure 3 shows the results. After a transient time of approximately 20 units of time the two systems reach the desired synchronized state (see the time series of the membrane potential and the recovery variable in Fig. 3(a), (b) and their synchronization errors in Fig. 3(c), (d).) Thus, the control is successful. As in the previous case, the control I is bounded and tends to zero while $t \to \infty$ (see Fig. 3(f)).

5 Conclusion

We have proposed two methods for controlling synchrony in two delay-coupled neural systems with time-varying delays. The first method is used to control of two FitzHugh-Nagumo systems, a neural model which is considered to be generic for excitable systems close to a Hopf bifurcation. It deals with the case, when all parameters of the system are measurable. We have posed the synchronization problem and introduced the Lyapunov function to find control and prove the synchronization problem. Based on this function we have derived a controller, which makes the synchrony stable despite the time-varying delay.

Then, we have considered the case of generalized system with sector bounded nonlinearities and uncertain parameters. For the estimation of uncertain parameters we have used the speed-gradient algorithm. Based on SG-algorithm we have derived an adaptive controller, which makes the synchrony stable despite the time-varying delay and uncertain parameters.

We have found the conditions of applicability of two proposed methods and have formulated the theorems of control goal achievement. The simulation has shown that these methods ensure the control goal. Given the paradigmatic nature of the FitzHugh-Nagumo system we expect our method to be extended to the case with several nodes of the system and be applicable in a wide range of neural system models.

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