PHYSCON 2015, Istanbul, Turkey, 19–22 August, 2015

ENTROPY OF WEIGHTED RECURRENCE PLOTS

Thomas Peron
São Carlos Institute of Physics
Universidade de So Paulo
13566-590 São Carlos, São Paulo, Brazil
thomas.peron@usp.br

Jürgen Kurths
Potsdam Institute for Climate Impact Research
14473 Potsdam, Germany

Department of Physics
Humboldt University
12489 Berlin, Germany
kurths@pik-potsdam.de

Abstract
The Shannon entropy of a time series is a standard measure to assess the complexity of a dynamical process and can be used to quantify transitions between different dynamical regimes. An alternative way of quantifying complexity is based on state recurrences, such as those available in recurrence quantification analysis. Although varying definitions for recurrence-based entropies have been suggested so far, for some cases they reveal inconsistent results. Here we suggest a method based on weighted recurrence plots and show that the associated Shannon entropy is positively correlated with the largest Lyapunov exponent. We demonstrate the potential on a prototypical example as well as on experimental data of a chemical experiment.

Key words
Recurrence Plots. Nonlinear dynamics. Entropy.

1 Introduction

First conceived to visualize the time-dependent behavior of complex dynamical systems, recurrence plots (RPs) have been shown to be a powerful technique to uncover statistically many characteristic properties of such systems [Eckmann et al., 1987; Marwan et al., 2007]. A crucial issue in the study of time series originating from complex systems is the detection of dynamical transitions, a task that RPs have been accomplishing due to a set of RP-based measures of complexity. Examples of their successful application in real-world systems can be found in neuroscience, earth science, astrophysics , and other areas of research [Marwan et al., 2007]. The so-called recurrence quantification analysis (RQA) provides measurements based on the density and the length of diagonal and vertical line patterns in RPs, which turn out to be an alternative way to quantify the complexity of physical systems. The time-dependent behavior of nonlinear time series can then be uncovered by setting sliding time windows in order to identify dynamical transitions, such as periodic to chaos transitions [Trulla et al., 1996] and even chaos-chaos transitions [Marwan et al., 2002]. The great merit of this approach resides in the fact that the calculation of Lyapunov exponents is often impracticable when the equations of motion are unknown. In this way, many measurements based on RQA have been proposed in order to better quantify the properties of dynamical systems. For instance, one of the most employed quantifiers able to detect bifurcation points are entropy-based measurements, e.g., the normalized entropy of recurrence times [Baptista et al., 2010; Little et al., 2007] or the Shannon entropy of the distribution of length of diagonal line segments [Trulla et al., 1996]. However, the entropy of the diagonal line segments is known to present in some cases a counterintuitive anticorrelation with Lyapunov exponents, leading to high values for periodic dynamics and low values in chaotic regimes [Trulla et al., 1996]. In this work we tackle this problem of the apparent contradiction by proposing an alternative RP-based entropy of weighted Recurrence Plots (wRPs). In other words, instead of considering binary RPs, we allow them to have weights proportional to the euclidean distances between the points in the phase space [Eroglu et al., 2014].

2 Recurrence Plots and Entropy

Given a trajectory $\mathbf{x}_i$ ($i = 1, ..., N$, $\mathbf{x} \in \mathbb{R}$) embedded in a m-dimensional phase space we define the weighted RP as [Eroglu et al., 2014]

$$\widetilde{W}_{ij} = e^{-||\mathbf{x}_i - \mathbf{x}_j||}, \qquad (1)$$

where $|| \cdot ||$ is the Euclidean norm. The definition in Eq. 1 scales the distances to be bounded in between $[0, 1]$, where $\widetilde{W}_{ij} \to 1$ for recurrent points and

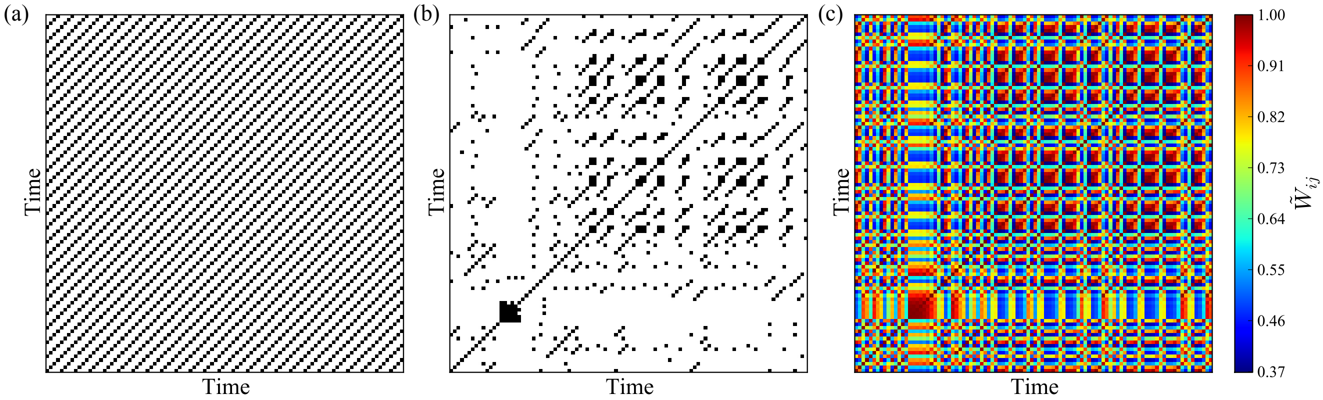

Figure 1. Recurrence plot of logistic map for (a) periodic regime and (b) chaotic regime and (c) weighted recurrence plot of logistic map for chaotic regime.

$\widetilde{W}_{ij} \rightarrow 0$ for distant states. This definition presents the benefit that it does not require selection of a recurrence threshold ε, for which there is no general method available yet [Marwan et al., 2007] .

Figures 1(a) and (b) show typical RPs of the logistic map in the periodic and chaotic regime, respectively; while Fig. 1 shows the corresponding wRP for the chaotic regime.

Having the weighted recurrence matrix $\widetilde{\mathbf{W}}$ we then define the strength of point i in the time series as [Eroglu et al., 2014]

$$s_i = \sum_{j=1}^{N} \widetilde{W}_{ij}. \tag{2}$$

Thus, by the computing the heterogeneity of the strength distribution $p(s)$ we are able to quantify the amount of statistical disorder in the system. Therefore, such task can be accomplished by the calculation of the Shannon entropy associated to the distribution of strengths, i.e., [Eroglu et al., 2014]

$$S_{\mathrm{wRP}} = -\sum_{\{s\}} p(s) \ln p(s). \tag{3}$$

Figure 2 shows the comparison between our approach, the traditional RP-based entropy measurement based on length of diagonal lines and the Lyapunov applied to to the logistic map. As we can see, although S_{RP} predicts the dynamical transitions, it is strongly anticorrelated with the Lyapunov exponent for some intervals of the bifurcation parameter a. In contrast, the entropy associated to wRPs is, in general, positively correlated with $\lambda_{\max}$ throughout almost the entire range of parameter a. In [Eroglu et al., 2014] we further show that similar results are obtained with time series originated by the Rössler system and also data obtained from real experiments with coupled chemical oscillators.

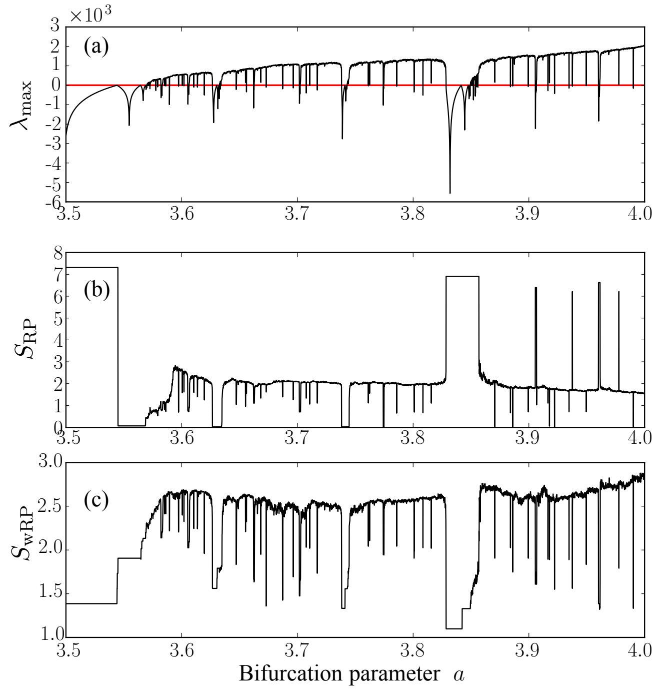

Figure 2. Comparison between the Lyapunov exponent $\lambda_{\max}$, (b) Entropy S_{RP} associated to the distribution of length of line segments in RPs and (c) entropy S_{wRP} of wRPs.

3 Conclusion

We have presented a recurrence-based matrix to quantify the dynamical properties of a given system. The Shannon entropy of the recurrence matrix has been defined as a complexity measure and compared with the Shannon entropy of other recurrence-based approaches. Although entropy is a well known measure of disorder, in recurrence plot terminology, entropy is determined as a heuristic measure, in order to detect the transitions between different regimes. The probability of occurrence of diagonal line segments of different lengths is not equal since a recurrence plot is a square matrix whose dimension is limited by the length of the time series. The Shannon entropy is computed from the diagonal line distribution in the RP approach. Hence, the commonly adopted entropic measures based on line segments can often yield counterintuitive results when quantifying the complexity of a given system. This was exemplified with the logistic map case in which the entropy of black and white dots was observed to be anticorrelated with the Lyapunov exponent. On the other hand, the entropy of weighted RPs presented here recovered the expected dependence as a function of the systems complexity, i.e., showing higher values within regions in which chaos is observed. Moreover, for the continuous systems such as the Rossler attractor and experimental time series of electrochemical oscillators, although black dots and weighted entropies are both positively correlated with the emergence of chaotic behavior, the latter definition was observed to have more stable values for voltage ranges that lead to periodic time series. The ideas presented here can be extended and applied to other complex systems with the potential to better identify dynamical transitions in time series originating from them.

Acknowledgements

T.K.D.M.P. would like to acknowledge D. Eroglu, N. Marwan, F. A. Rodrigues, L. da F. Costa, M. Sebek and I. Z. Kiss. This research was founded by FAPESP (Grant No. 2012/22160-7) and IRTG Grant No. 1740 (DFG and FAPESP).

References

Baptista, M., Ngamga, E., Pinto, P. R., Brito, M., and Kurths, J. (2010). Kolmogorov–sinai entropy from recurrence times. *Physics Letters A*, 374(9):1135–1140.

Eckmann, J.-P., Kamphorst, S. O., and Ruelle, D. (1987). Recurrence plots of dynamical systems. *EPL (Europhysics Letters)*, 4(9):973.

Eroglu, D., Peron, T. K. D., Marwan, N., Rodrigues, F. A., Costa, L. d. F., Sebek, M., Kiss, I. Z., and Kurths, J. (2014). Entropy of weighted recurrence plots. *Physical Review E*, 90(4):042919.

Little, M. A., McSharry, P. E., Roberts, S. J., Costello, D. A., Moroz, I. M., et al. (2007). Exploiting nonlinear recurrence and fractal scaling properties for voice disorder detection. *BioMedical Engineering OnLine*, 6(1):23.

Marwan, N., Carmen Romano, M., Thiel, M., and Kurths, J. (2007). Recurrence plots for the analysis of complex systems. *Physics Reports*, 438(5):237–329.

Marwan, N., Wessel, N., Meyerfeldt, U., Schirdewan, A., and Kurths, J. (2002). Recurrence-plot-based measures of complexity and their application to heart-rate-variability data. *Physical Review E*, 66:026702.

Trulla, L., Giuliani, A., Zbilut, J., and Webber, C. (1996). Recurrence quantification analysis of the logistic equation with transients. *Physics Letters A*, 223(4):255–260.