

Forecasting transitions in systems with high dimensional stochastic complex dynamics

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We describe a new procedure to monitor and forecast the onset of transitions in high dimensional stochastic complex systems (see Phys.Rev. Lett. **113**, 264102 (2014)). We illustrate the methodology by an application to the Tangled Nature model of evolutionary ecology but expect the method to be of general applicability. The quasi-stable configurations of the full stochastic dynamics are taken as input for a stability analysis by means of the deterministic mean field equations. Numerical analysis of the high dimensional stability matrix allows us to identify unstable directions associated with eigenvalues with positive real part. The overlap of the instantaneous configuration vector of the full stochastic system with the eigenvectors of the unstable directions of the deterministic mean field approximation is found to be a good early-warning of the transitions occurring intermittently.

Introduction - Many complex high dimensional systems are characterised by intermittent dynamics, where relatively long quiescent periods are interrupted by sudden and quick bursts of activity during which the system suffers hectic rearrangements. These rearrangements can be seen as transitions between metastable states. Examples of abrupt transitions have been identified in a broad range of systems [1]: in biological ecosystems [2, 3] transitions from a flourishing to a wild state can occur, in financial markets [4] endogenous crisis can destabilize an existing balance, in the human brain [5] epileptic seizures signals a switch from a regular to an irregular condition, climate [6] can exhibit sudden changes both overall or in one of its subsystems, like when a bloom of harmful algae suddenly forms in the sea [7].

We focus on the Tangled Nature (TaNa) model [8–10] of evolutionary ecology. The initial aim of the model was to establish a sound and simple mathematical framework for "punctuated equilibrium", i.e. the observed intermittent mode of macro-evolution. The TaNa model is an individual based stochastic model of coevolution. The model's phenomenology is in good agreement with biological observations [11]. At the microscopic level of individuals the dynamics is unfolding at a smooth constant pace: agents reproduce, mutate and die at essentially constant rates. On the contrary, at the systemic level the generated ecological network structures jump from one metastable configuration to another (denoted quasi-Evolutionary Stable Strategies or qESS). We investigate these macroscopic instabilities by performing a Linear Stability Analysis (LSA) of the mean field representation of the dynamics about the actual configurations produced by the full stochastic dynamics. LSA is obviously a standard procedure to analyze the nature of fixed points for deterministic autonomous equations of motion. Here we develop the method to allow applications to high dimensional *stochastic* dynamics.

When we apply this procedure to very high dimensional situations like the TaNa model it is not possible to solve directly the fixed point equation: $\mathbb{F}(\mathbf{n}^*) = 0$. Instead, we can use the observed qESS configurations generated by the full stochastic dynamics to approximate \mathbf{n}^* and perform a LSA of the mean field dynamics about these configurations. To our knowledge this procedure for applying LSA to high dimensional stochastic dynamics has hardly been attempted before. Only recently, LSA of agent-based models have been studied: in [12] the stability properties of the attractors of a generalized Sznajd model are derived from its mean-field formulation, whereas in [13] a similar analysis has been done for a network of pulse-coupled neurons. Neither of these systems, however, exhibits intermittent behavior as we observe in the TaNa. The intermittent dynamics allows us to define a new mean-field based early-warning measure for the occurrence of abrupt transitions.

How effective an unstable eigendirection is in destabilising the configuration $\mathbf{n}(t)$ will depend on the overlap between the deviation from $\bar{\mathbf{n}}_{stoc}$: $\delta\mathbf{n}(t) = \mathbf{n}(t) - \bar{\mathbf{n}}_{stoc}$ and the unstable directions. During the qESS we therefore introduce the following instability indicator:

$$Q(t) = \max_{\lambda \in Sp^+(\mathbb{M}[\bar{\mathbf{n}}_{stoc}])} |e^\lambda \langle (\mathbf{n}(t) - \bar{\mathbf{n}}_{stoc}), \mathbf{e}_\lambda \rangle| \quad (1)$$

where the eigenvalues λ and the correspondent eigenvectors \mathbf{e}_λ of $\mathbb{M}[\bar{\mathbf{n}}_{stoc}]$ can be computed numerically for high dimensions (in our case with the Intel DGEEV routine). $Sp^+(\mathbb{M}[\bar{\mathbf{n}}_{stoc}])$ refers to the eigenvalues with positive real part and the brackets denotes the scalar product

In Fig. 1 we show Q as a function of the microscopic time steps (blue curve). We observe that $\|\delta\mathbf{n}(t)\|$ fluctuates during the qESS. In contrast Q only grows when a transition is about to occur. Typically Q starts to increase several generations prior to the transition corresponding, in this particular case, to thousands of single update events.

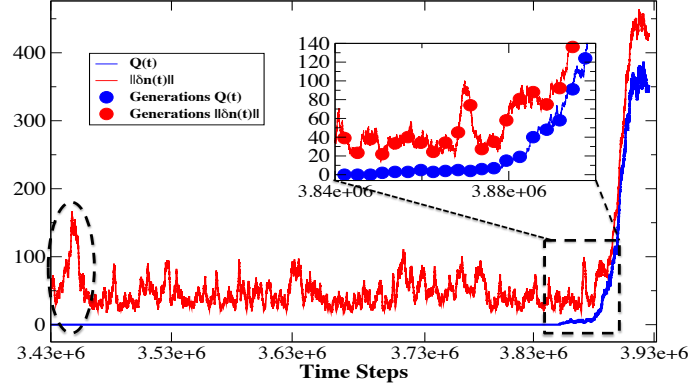


Figure 1. Typical behavior of $Q(t)$ and $\|\delta\mathbf{n}(t)\|$ in a single run of the TaNa in time steps. Clearly $Q(t) \simeq 0$ even for more rare strong fluctuations (dashed circle) inside the qESSs, whereas it starts to increase rapidly before the actual transition. In the inset, we zoom on the transition and indicate with markers the points observed at the coarse-grained level of generations. Notice that between two generations many time steps (events) are present.

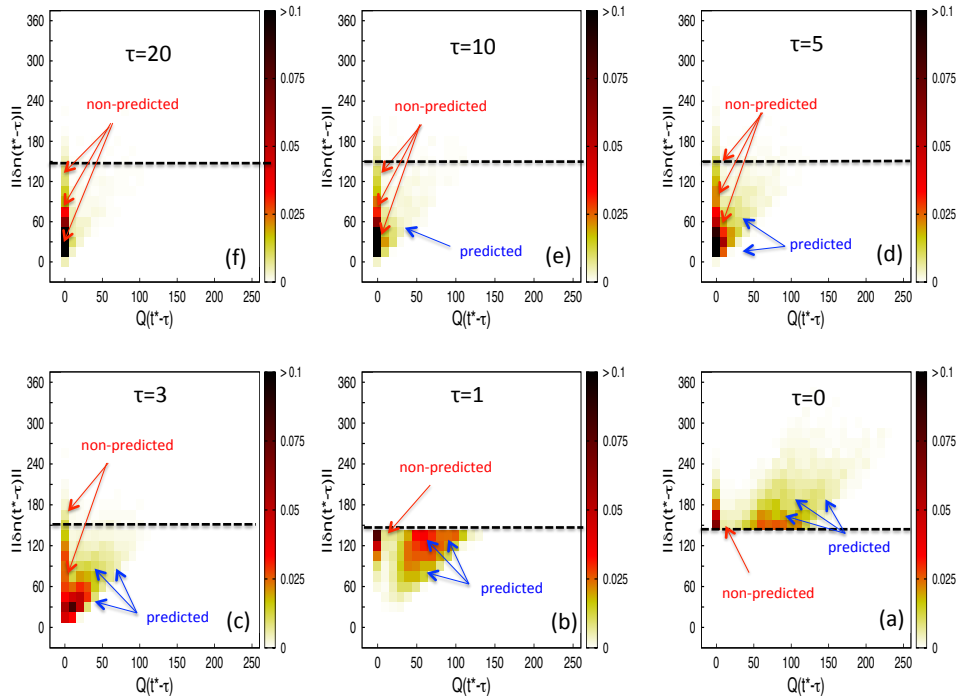


Figure 2. 2D distribution $P(\|\delta\mathbf{n}(t^* - \tau)\|, Q(t^* - \tau))$ averaged over 17000 transitions for different values of τ . The predictive power of Q is evident: typical fluctuations inside the qESSs are not signaled by Q (panels (e-f)), whereas dangerous perturbations leading to a transition are recognized by the increasing of Q away from zero (panels (a-d)). This is already seen for $\tau = 5$, which is still remarkably far from the transition. Examples of predicted/non predicted transitions are then shown with arrows in panels (d-a). The other plots can be interpreted in a similar way.

To understand the relation between $\|\delta\mathbf{n}\|$ and Q we show in Fig. 2 the joint probability density $P(\|\delta\mathbf{n}(t^* - \tau)\|, Q(t^* - \tau))$ for τ generations before the time t^* of the transition. We identify t^* as the time when the condition: $\|\delta\mathbf{n}(t^*)\| > d$ holds persistently for at least 10 generations for a fixed threshold $d = 150$, corresponding to the typical amplitude of the fluctuations inside the qESSs. From the way the region of largest support move in the $Q - \|\delta\mathbf{n}\|$ plane as

the transition is approached we see to what extent monitoring Q allows one to predict the transition. Note that a significant support for values of Q larger than about 10 starts to develop from around $\tau = 5$. At these times the deviation $\|\delta\mathbf{n}\|$ is still most often below the inherent qESS fluctuation level of 150. We may encounter situations where Q gives a false signal, by increasing significantly in correspondence to small amplitude perturbations of $\mathbf{n}(t)$. Remarkably, we can see that these events happen with low probability, thus not affecting significantly the performance of the Q measure.

Finally, our success rate in predicting transitions is approximately 84-86%, however non-predicted transitions do occur and are related to a non-vanishing probability that a direction which is weakly stable (negative eigenvalues close to zero) of the mean field can trigger a transition[14].

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- [1] M. Scheffer, *Critical transitions in nature and society* (Princeton University Press, 2009).
- [2] M. Scheffer, S. Carpenter, J. A. Foley, C. Folke, and B. Walker, *Nature* **413**, 591 (2001).
- [3] M. Scheffer and S. R. Carpenter, *Trends in Ecology & Evolution* **18**, 648 (2003).
- [4] R. M. May, S. A. Levin, and G. Sugihara, *Nature* **451**, 893 (2008).
- [5] B. Litt and J. Echaux, *The Lancet Neurology* **1**, 22 (2002).
- [6] T. M. Lenton, H. Held, E. Kriegler, J. W. Hall, W. Lucht, S. Rahmstorf, and H. J. Schellnhuber, *Proceedings of the National Academy of Sciences* **105**, 1786 (2008).
- [7] K. G. Sellner, G. J. Doucette, and G. J. Kirkpatrick, *Journal of Industrial Microbiology and Biotechnology* **30**, 383 (2003).
- [8] K. Christensen, S. A. di Collobiano, M. Hall, and H. J. Jensen, *J. Theor. Biol.* **216**, 73 (2002).
- [9] M. Hall, K. Christensen, S. A. di Collobiano, and H. J. Jensen, *Phys. Rev. E* **66** (2002).
- [10] E. P. Anderson and H. J. Jensen, *J. Theor. Biol.* **232**, 551 (2005).
- [11] S. Laird, D. Lawson, and H. J. Jensen, in *In Mathematical Modeling of Biological Systems. Vol. 2*, edited by A. Duetsch, L. Bruschi, H. Bryne, G. de Vries, and H. Herzel (Birkhauser, 2007), pp. 49–62.
- [12] A. M. Timpanaro and C. P. Prado, *Physical Review E* **86**, 046109 (2012).
- [13] S. Olmi, A. Politi, and A. Torcini, *The Journal of Mathematical Neuroscience (JMN)* **2**, 1 (2012).
- [14] A. Cairolo, D. Piovani, and H. J. Jensen, *Phys.Rev. Lett.* **113**, 264102 (2014).