Open Loop Control of Lyapunov Exponents in Fixed Points of Nonlinear Oscillator

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Abstract

We investigate the Lyapunove exponents in the fixed points of one dimensional nonlinear oscillation driven by rapidly changing external force. We present phase portraits of the systems in the neighbourhood of the fixed points and demonstrate the changing of the Lyapunov spectrum under the application of different forms of feed forward control.

1 Introduction

Here we discuss the case of Kapitza pendulum driven by sin- or cos-rapidly oscillating periodical force [1]. We use the averaging procedure with respect to the rapidly changing movement and start from the effective potential energy of the pendulum [2]. Our purpose is to investigate how the open-loop control scheme influences on the structure of the Lyapunov spectrum in the fixed points of the dynamical system. We investigate the phase portraits of the systems in the neighbourhood of the fixed points and demonstrate the changing of the Lyapunov spectrum under the application of different forms of feedforward control.

2 Lyapunov exponents of the pendulum driven by external periodical force: Horizontal Modulation

Consider the motion of a pendulum of mass m whose point of support oscillates horizontally with a high frequency $\gamma >> \sqrt{\frac{g}{l}}$,where g is the gravitational acceleration and l is the length of pendulum.Now the differential equation of such a dynamical system is

$$ml\ddot{\varphi} = -\frac{1}{l}\frac{dU_{eff}}{d\varphi} \tag{1}$$

where $\varphi = \varphi(t)$ is the angular displacement of the pendulum and

$$U_{eff} = mgl[-\cos\varphi + (\frac{a^2\gamma^2}{4gl})\cos^2\varphi]$$

is the effective potential energy, where a is the amplitude of the oscillation. Therefore our dynamical system for horizontal modulation becomes

$$ml^2\ddot{\varphi} + \frac{1}{2}ma^2\gamma^2\sin\varphi\cos\varphi = mgl\sin\varphi \tag{2}$$

Now equation (2) in Cauchy Form can be written as

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -\frac{g}{l}\sin x_1(1 - \frac{a^2\gamma^2}{2gl}\cos x_1) \end{cases} \tag{3}$$

By solving simultaneous equations

$$\dot{x_1} = 0 \quad and \quad \dot{x_2} = 0$$

The fixed points are

$$(k\pi, 0), k \in Z \quad and \quad (\arccos(\frac{2gl}{a^2\gamma^2}), 0)$$

Now we check the behaviour of the system by determining the stability of the fixed points

At fixed point (0,0) the Lyapunov exponents are

$$\lambda_1^+ = +\frac{a\gamma}{\sqrt{2}l}\sqrt{1-\frac{2gl}{a^2\gamma^2}} \quad and \quad \lambda_2^- = -\frac{a\gamma}{\sqrt{2}l}\sqrt{1-\frac{2gl}{a^2\gamma^2}}$$

The Lyapunov exponents are real and distinct

i.e$\lambda_1^+ > 0, \lambda_2^- < 0$ if $\frac{2gl}{a^2\gamma^2} < 1$

The Lyapunov exponents are real and coincide

i.e$\lambda_1^+ = 0, \lambda_2^- = 0$ if $\frac{2gl}{a^2\gamma^2} = 1$

Lyapunov exponents are distinct and pure imaginary

if $\frac{2gl}{a^2\gamma^2} > 1$

At fixed point $(\pi, 0)$ the Lyapunov exponents are

$$\lambda_1^+ = +\frac{a\gamma}{\sqrt{2}l}\sqrt{1+\frac{2gl}{a^2\gamma^2}} \quad and \quad \lambda_2^- = -\frac{a\gamma}{\sqrt{2}l}\sqrt{1+\frac{2gl}{a^2\gamma^2}}$$

The Lyapunov exponents are real and distinct
i.e$\lambda_1^+ > 0, \lambda_2^- < 0$ if $\frac{2gl}{a^2\gamma^2} < 1, \frac{2gl}{a^2\gamma^2} = 1, \frac{2gl}{a^2\gamma^2} > 1$
At fixed point $(\arccos(\frac{2gl}{a^2\gamma^2}), 0)$ the Lyapunov exponents are

$$\lambda_1^+ = +\frac{a\gamma}{\sqrt{2}l}\sqrt{\Big(\frac{2gl}{a^2\gamma^2}\Big)^2-1} \quad and \quad \lambda_2^- = -\frac{a\gamma}{\sqrt{2}l}\sqrt{\Big(\frac{2gl}{a^2\gamma^2}\Big)^2-1}$$

For $x_1 = \arccos(\frac{2gl}{a^2\gamma^2}), \quad \cos x_1 \leq 1$
Therefore Lyapunov exponents are distinct and pure imaginary
if $\frac{2gl}{a^2\gamma^2} < 1$
The Lyapunov exponents are real and coincide
i.e $\lambda_1^+ = 0, \lambda_2^- = 0$ if $\frac{2gl}{a^2\gamma^2} = 1$

3 Lyapunov exponents of the pendulum driven by external periodical force: Vertical Modulation

In this case

$$U_{eff} = mgl[-\cos\varphi + (\frac{a^2\gamma^2}{4gl})\sin^2\varphi] \tag{4}$$

is the effective potential energy, where a is the amplitude of the oscillation. Therefore our dynamical system for vertical modulation becomes

$$ml^2\ddot{\varphi} + mgl\sin\varphi = \frac{1}{2}ma^2\gamma^2\sin\varphi\cos\varphi \tag{5}$$

Now equation (5) in Cauchy Form can be written as

$$\begin{cases} \dot{x_1} = x_2 \\ \\ \dot{x_2} = -\frac{g}{l}\sin x_1(1 + \frac{a^2\gamma^2}{2gl}\cos x_1) \end{cases} \tag{6}$$

By solving simultaneous equations

$$\dot{x_1} = 0 \quad and \quad \dot{x_2} = 0$$

The fixed points are

$$(k\pi, 0), k \in Z \quad and \quad (\arccos(\frac{-2gl}{a^2\gamma^2}), 0)$$

Now we check the behaviour of the system by determining the stability of the fixed points
At fixed point (0,0) the Lyapunov exponents are

$$\lambda_1^+ = +\frac{a\gamma}{\sqrt{2}l}\sqrt{1 + \frac{2gl}{a^2\gamma^2}}\ i \quad and \quad \lambda_2^- = -\frac{a\gamma}{\sqrt{2}l}\sqrt{1 + \frac{2gl}{a^2\gamma^2}}\ i$$

The Lyapunov exponents are distinct and pure imaginary if

$\frac{2gl}{a^2\gamma^2} > 1, \frac{2gl}{a^2\gamma^2} = 1, \frac{2gl}{a^2\gamma^2} < 1$

At fixed point $(\pi, 0)$ the Lyapunov exponents are

$$\lambda_1^+ = +\frac{a\gamma}{\sqrt{2}l}\sqrt{\frac{2gl}{a^2\gamma^2} - 1} \quad and \quad \lambda_2^- = -\frac{a\gamma}{\sqrt{2}l}\sqrt{\frac{2gl}{a^2\gamma^2} - 1}$$

The Lyapunov exponents are distinct and pure imaginary if$\frac{2gl}{a^2\gamma^2} < 1$
The Lyapunov exponents are real and coincide

i.e$\lambda_1^+ = 0, \lambda_2^- = 0$ if$\frac{2gl}{a^2\gamma^2} = 1$

The Lyapunov exponents are distinct and real

i.e$\lambda_1^+ > 0, \lambda_2^- < 0$ if$\frac{2gl}{a^2\gamma^2} > 1$

At fixed point $(\arccos(\frac{-2gl}{a^2\gamma^2}), 0)$ the Lyapunov exponents are

$$\lambda_1^+ = +\frac{a\gamma}{\sqrt{2}l}\sqrt{\left(\frac{2gl}{a^2\gamma^2}\right)^2 + 1}\ i \quad and \quad \lambda_2^- = -\frac{a\gamma}{\sqrt{2}l}\sqrt{\left(\frac{2gl}{a^2\gamma^2}\right)^2 + 1}\ i$$

For $x_1 = \arccos(\frac{-2gl}{a^2\gamma^2}), \quad \cos x_1 \geq -1$

Therefore the Lyapunov exponents are distinct and pure imaginary

if $\frac{2gl}{a^2\gamma^2} < 1$ and $\frac{2gl}{a^2\gamma^2} = 1$

4 Conclusions

We present our results in the form of Table 1.

	$\frac{2gl}{a^2\gamma^2} < 1$	$\frac{2gl}{a^2\gamma^2} = 1$	$\frac{2gl}{a^2\gamma^2} > 1$
(a)	$(\arccos(\frac{2gl}{a^2\gamma^2}), 0)$ $\lambda_1$andλ_2 are Pure Imaginary	$(\arccos(\frac{2gl}{a^2\gamma^2}), 0)$ λ_1 and λ_2 Coincides	
Horizontal oscillation	(0,0) $\lambda_1 > 0$ and $\lambda_2 < 0$	(0,0) λ_1 and λ_2 Coincides	(0,0) λ_1 and λ_2 are Pure Imaginary
	$(\pi, 0)$ $\lambda_1 > 0$ and $\lambda_2 < 0$	$(\pi, 0)$ $\lambda_1 > 0$ and $\lambda_2 < 0$	$(\pi, 0)$ $\lambda_1 > 0$ and $\lambda_2 < 0$
(b)	$(\arccos(-\frac{2gl}{a^2\gamma^2}), 0)$ λ_1 and λ_2 are Pure Imaginary	$(\arccos(-\frac{2gl}{a^2\gamma^2}), 0)$ λ_1 and λ_2 are Pure Imaginary	
Vertical oscillation	(0,0) λ_1 and λ_2 are Pure Imaginary	(0,0) λ_1 and λ_2 are Pure Imaginary	(0,0) λ_1 and λ_2 are Pure Imaginary
	$(\pi, 0)$ λ_1 and λ_2 are Pure Imaginary	$(\pi, 0)$ λ_1 and λ_2 Coincides	$(\pi, 0)$ $\lambda_1 > 0$ and $\lambda_2 < 0$

From the table we can see that the fixed points of the system are limited in their types; and the Lyapunov spectrum in their neighbourhod can be controlled with the open-loop scheme.

References

[1] P. L. Kapitza, Dynamic stability of a pendulum with an oscillating point of suspension, Journal of Experimental and Theoretical Physics 21 (1951) 588.

[2] L. D. Landau, E. M. Lifshitz, Mecanics, Pergamon Press, Oxford, 1960.