Manipulating phases of alternately excited self-oscillators: A way to organize hyperbolic chaos and some other phenomena of complex dynamics

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***Abstract*— The report is devoted to realization of some models and phenomena of nonlinear dynamics (Bernoulli map, Arnold's cat map, hyperbolic attractor of Smale-Williams type, robust strange nonchaotic attractor, Mandelbrot and Julia sets, hyperchaos). The idea is based on a use of a special class of systems composed of two or more coupled oscillators with periodically modulated parameters. The subsystems become active alternately and transfer the excitation each other. Manipulating with phases of the transferred excitation (due to a proper selection of the coupling terms in the equations) allows implementation of the named models and phenomena. The proposed systems may be designed e.g. as electronic devices.**

I. Introduction

Some important concepts and phenomena in the nonlinear science are not attributed yet with good examples of real-world systems, at least on a level of familiar to physicists differential equations. This situation relates to some classic objects like Bernoulli map, hyperbolic toral maps, strange hyperbolic attractors, phenomena of complex analytic dynamics associated with Mandelbrot and Julia sets [1-4]. The present research is devoted to implementation of these concepts in a special class of systems, which allow realization e.g. in electronics, nonlinear optics and laser physics. These systems are composed of oscillators excited and decaying alternately, in which transfer of the excitation is accompanied with appropriate transformation of the phases of the oscillations [5-9].

II. Chaos governed by the Bernoulli map

Let us consider a system of two van der Pol oscillators

$$\begin{aligned}&\ddot{x} - [A\cos(2\pi t/T) - x^2]\dot{x} + \omega_0^2 x = \varepsilon y \cos\omega_0(n-1)t,\\ &\ddot{y} - [-A\cos(2\pi t/T) - y^2]\dot{y} + n^2\omega_0^2 y = \varepsilon x^n,\end{aligned} \tag{1}$$

where an integer $n \geq 2$, and $T = 2\pi N/\omega_0$. Let the first oscillator have some phase φ on a stage of activity: $x \propto \cos(\omega_0 t + \varphi)$. The term x^n contains the n-th harmonic: $\cos(n\omega_0 t + n\varphi)$, and its phase is $n\varphi$. As the half-period comes to the end, the second oscillator inherits the phase $n\varphi$ because of its excitation in presence of that term. The backward transfer of the excitation takes place due to the term in the first equation with the spectral component on a difference frequency. Hence, on the next stage of excitation the first oscillator accepts the phase $n\varphi$. In successive epochs of excitation the phases of the first oscillator follow approximately the Bernoulli map

$$\varphi' = n\varphi \ \ (\mathrm{mod} 2\pi). \tag{2}$$

(Hereafter we ignore constant additive terms arising normally in the course of the phase transfer.) Fig.1 shows an iteration diagram for the phases obtained for a concrete regime of the model.

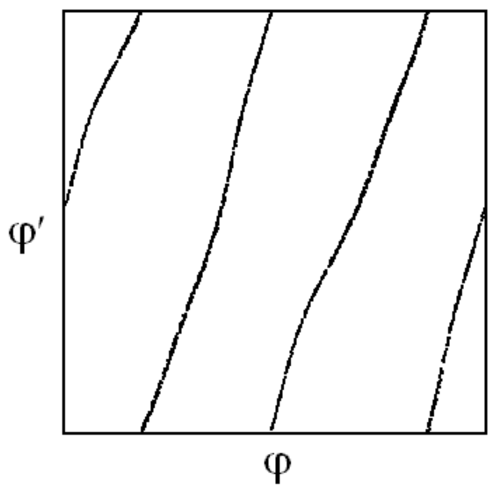

Fig. 1. The iteration diagram for phases in the model (1): n=3, A=3, T=10, ε=0.5, ω_0=2π. Observe correspondence of the topological nature of the empirical map to that of the map (2) with n=3

To say more accurately, the four-dimensional Poincaré map corresponding to evolution over a period T possesses a hyperbolic chaotic attractor of Smale – Williams type. We can define a toroidal absorbing domain, which is mapped inside itself been compressed longitudinally and stretched transversally with n turns around "the hole of the doughnut". The model (1) is a realistic example of a system with hyperbolic strange attractor. A similar system was studied recently in an experiment [6].

III. Robust strange nonchaotic attractor

The next construction we discuss is a system with quasi-periodic driving [9]:

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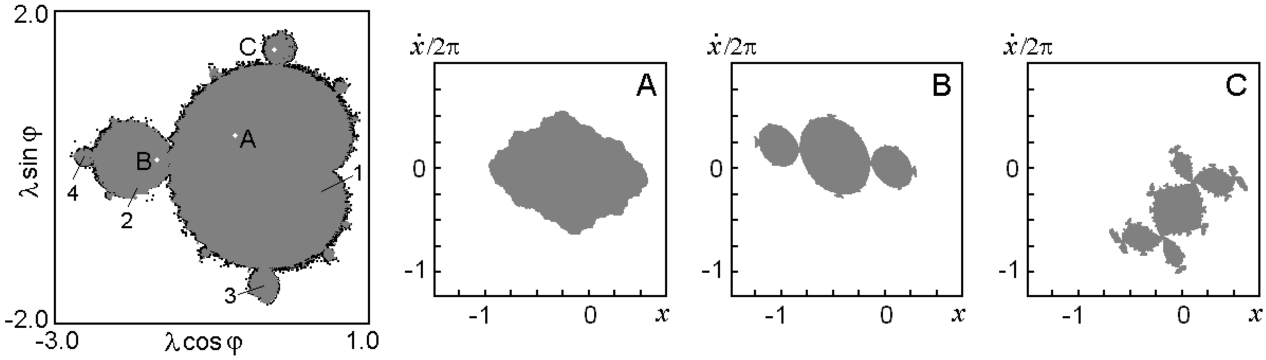

Fig. 2. Zone on the parameter plane $\lambda e^{i\varphi}$ corresponding to bounded dynamics in the model (5) and cross-sections of the basins of attractors at ω_0=2π, T=10, F=7, g=0.5, ε=1

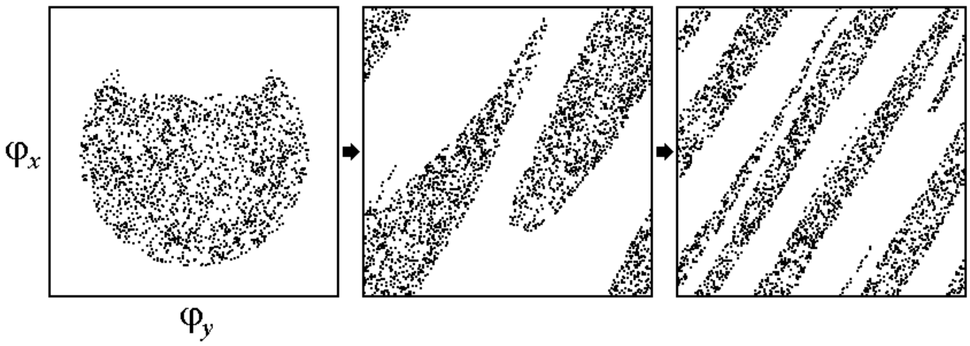

Fig. 3. Transformation of the area of the cat face in a the plane of phases of active oscillators in the model (6) for time intervals $2T$ and $4T$

$$\begin{aligned}&\ddot{x} - (A\sin(2\pi t/T) - x^2)\dot{x} + \omega_0^2 x = \varepsilon y \sin(\omega_0 t + \theta),\\ &\ddot{y} - (-A\sin(2\pi t/T) - y^2)\dot{y} + 4\omega_0^2 y = \varepsilon x \sin\omega_0 t,\\ &\dot{\theta} = \omega_0 w/T,\ \ w = (\sqrt{5}-1)/2.\end{aligned} \tag{3}$$

In this case, for the phase of the first oscillator and for that of the quasiperiodic force we get in a certain approximation

$$\varphi' = \varphi - \theta, \quad \theta' = \theta + 2\pi w \pmod{2\pi}. \tag{4}$$

This is the map used by Hunt and Ott in their construction of an example of robust strange nonchaotic attractor (SNA) [10]. The robust SNA appears as one adds small nonlinear terms to the right-hand part of that map. This is the case for the system (3), for which the map (4) delivers an approximate description (see [9] for some details).

IV. Complex analytic dynamics: Mandelbrot and Julia sets

The idea of physical realization of phenomena of complex analytic dynamics is based on the method of complex amplitudes. Let us turn to the following equations:

$$\begin{aligned}&\ddot{x} + F\cdot[g + \sin(2\pi t/T)]\dot{x} + \omega_0^2 x =\\ &\qquad = \varepsilon y \sin\omega_0 t + \lambda\sin(\omega_0 t + \varphi),\\ &\ddot{y} + F\cdot[g - \sin(2\pi t/T)]\dot{y} + 4\omega_0^2 y = \varepsilon x^2.\end{aligned} \tag{5}$$

A dissipation parameter in both oscillators varies slowly with period $T = 2\pi N/\omega_0$. Parameter g is positive, less then 1. Let us suppose that at the beginning of an epoch of activity for the second oscillator the first one oscillates with complex amplitude a: $x(t) \sim \mathrm{Re}[a(t)\mathrm{exp}(i\omega_0 t)]$. The coupling term in the second equation will be $x^2(t) = \frac{1}{2}|a(t)|^2 + \frac{1}{2}\mathrm{Re}[a^2\exp(2i\omega_0 t)]$. The germ for oscillations is delivered by the resonance component represented by the second harmonic, and the complex amplitude of the second oscillator during the next active stage will be proportional to a^2. A product of the signal from the second oscillator with the auxiliary signal of frequency ω_0 yields a component of difference frequency. A sum of that component with additional oscillatory term of frequency ω_0, amplitude λ and phase φ effects resonantly upon the first oscillator at the beginning of the next epoch of its activity. Hence, the stroboscopic map for the complex amplitude will correspond in certain normalization to the complex quadratic map $z_{k+1} = c + z_k^2$, where $c \sim \lambda e^{i\varphi}$, $z \sim a$.

In dependence on the complex parameter c it may occur that the solution for the coupled oscillators (5) with initial conditions in some domain remains bounded, or it may tend to infinity. In the left panel of Fig.2 the gray color designates zones of the bounded dynamics. Note a remarkable similarity of the plot to the classic picture of the Mandelbrot set. Labels 1, 2, 3 mark "leaves" of the "cactus" associated with dynamics of period T, $2T$, $3T$, respectively. For the points marked by the letters A, B, C basins of attraction are shown in the two-dimensional cross-section of the four-dimensional phase space of the Poincaré map with the plane

($y=0,\ \dot{y}=0$). They resemble Julia sets for the complex quadratic map.

More careful analysis shows that the correspondence does not extends over small-scale details of the Mandelbrot set. Indeed, the complex amplitude method is approximate. In higher orders of the perturbation theory corrections appear represented by non-analytic functions. Larger the period ratio N is, less these corrections are.

V. Arnold's cat map

Increasing a number of the alternately excited oscillators in the construction of the model systems we find an opportunity to get some other interesting types of complex dynamics.

Let us consider a system of four oscillators [7]

$$\begin{aligned}
&\ddot{x}-[A\cos(2\pi t/T)-x^2]\dot{x}+\omega_0^2 x=\varepsilon z\cos\omega_0 t,\\
&\ddot{y}-[A\cos(2\pi t/T)-y^2]\dot{y}+\omega_0^2 y=\varepsilon w,\\
&\ddot{z}-[-A\cos(2\pi t/T)-z^2]\dot{z}+4\omega_0^2 z=\varepsilon xy,\\
&\dot{w}-[-A\cos(2\pi t/T)-w^2]\dot{w}+\omega_0^2 w=\varepsilon x,
\end{aligned}\tag{6}$$

and suppose that the first and the second oscillators during the stage of activity have some phases φ_x and φ_y: $x\sim\cos(\omega_0 t+\varphi_x)$, $y\sim\cos(\omega_0 t+\varphi_y)$. The coupling term in the third equation is proportional to $\frac{1}{2}\cos(\varphi_x-\varphi_y)+\frac{1}{2}\cos(2\omega_0 t+\varphi_x+\varphi_y)$. The component of the doubled frequency effects resonantly the oscillator z, and on the activity stage it accepts the phase $\varphi_z\approx\varphi_x+\varphi_y$. In the same time, the fourth oscillator simply inherits the phase of the first one: $\varphi_w\approx\varphi_x$. At the beginning of the next activity epoch for x and y the term $z\cos\omega_0 t\sim\frac{1}{2}\cos(3\omega_0 t+\varphi_x+\varphi_y)+\frac{1}{2}\cos(\omega_0 t+\varphi_x+\varphi_y)$ ensures the phase $\varphi_x'\approx\varphi_z\approx\varphi_x+\varphi_y$ for the oscillator x, and the oscillator y inherits the phase of the oscillator w: $\varphi_y'\approx\varphi_w\approx\varphi_x$. Thus, we arrive at the map

$$\varphi_x'=\varphi_x+\varphi_y,\ \ \varphi_y'=\varphi_x\ \ (\mathrm{mod}\,2\pi).\tag{7}$$

Two iterations of this map correspond to the Arnold cat map, a well known example of hyperbolic chaotic map on torus. Fig.3 illustrates transformation of the traditional picture of the cat face on two steps of iterations obtained from computer solution of Eqs. (6).

VI. Hyperchaos

Our last example is a system

$$\begin{aligned}
&\ddot{x}-[A\cos(2\pi t/T)-x^2]\dot{x}+\omega_0^2 x=\varepsilon z\cos\omega_0 t,\\
&\ddot{y}-[A\cos(2\pi t/T)-y^2]\dot{y}+\omega_0^2 y=\varepsilon w\cos 2\omega_0 t,\\
&\ddot{z}-[-A\cos(2\pi t/T-z^2]\dot{z}+4\omega_0^2 z=\varepsilon xy,\\
&\ddot{w}-[-A\cos(2\pi t/T)-w^2]\dot{w}+9\omega_0^2 w=\varepsilon x^3,
\end{aligned}\tag{8}$$

demonstrating hyperchaos, a dynamical regime with two positive Lyapunov exponents. Dynamics of phases of the active oscillators is governed by stroboscopic map

$$\varphi_x'=\varphi_x+\varphi_y,\ \ \ \varphi_y'=3\varphi_x\ \ (\mathrm{mod}2\pi).\tag{9}$$

Two Lyapunov exponents expressed via the eigenvalues of the associated 2×2 matrix are both positive: $\Lambda_1=\ln\left((7+\sqrt{13})/2\right)=1.67$ and $\Lambda_2=\ln\left((7-\sqrt{13})/2\right)=0.51$. Fig.6 shows a plot for all Lyapunov exponent of the original coupled oscillator model versus parameter A. Observe that the largest two exponents are close to the approximate values for the phase map.

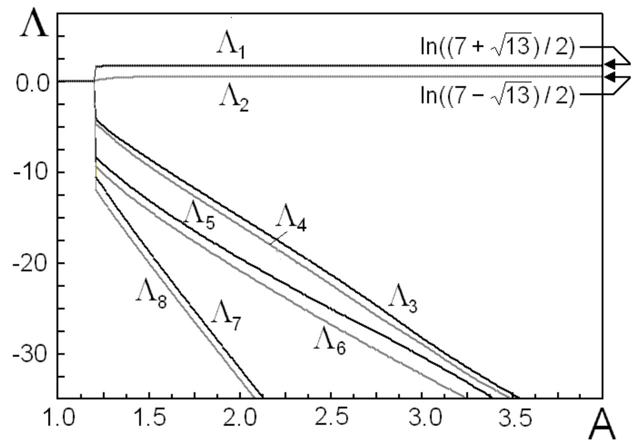

Fig. 4. Spectrum of Lyapunov exponents (normalized to time T) for the model (8) versus parameter A at ω_0=2π, T=20, ε=0.5. Note presence of two positive exponents in a wide range

VII. Conclusion

As follows from the studies we undertake, the systems of coupled non-autonomous oscillators alternately passing excitation each other, are of great interest from the point of view of realization of many phenomena of nonlinear dynamics, till now represented by mathematical constructions. On a basis of this general idea we suggested examples of the systems governed in some approximation by maps of Bernoully and Arnold, a system with robust SNA of Hunt and Ott, a system manifesting phenomena of complex analytic dynamics (Mandelbrot and Julia sets). All the schemes we propose allow physical realization e.g. as electron devices, or as systems of other physical nature (in mechanics, laser physics etc.).

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