THE PROBLEM OF CYCLE-SLIPPING FOR MULTIDIMENSIONAL PHASE CONTROL SYSTEMS

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Abstract

For distributed and discrete phase systems with vector nonlinearities the problem of cycle-slipping is developed. The method of a priori integral estimates and Lyapunov direct method are combined with a special technique. As a result certain frequency-domain estimates for the phase error are obtained.

Key words

Phase system, cycle-slipping, frequency-domain theorem.

1 Introduction

In this paper the problem of cycle-slipping for phase control systems is developed. For the first time this problem was set by J.J.Stoker [Stoker, 1950] for mathematical pendulum which underwent the resistance proportional to the square of its speed.

The problem proved to be rather important for various engineering systems, i.e. for phase locked loops, electric machines, synchronous tracking systems. The Stoker problem for differential equations of first and second order was investigated in a number of papers [Viterbi,1963],[Tausworthe, 1967],[Tausworthe, 1972], [Bozzoni, Marchetti, Mengali and Russo, 1970], where several formulae for mean cycle-slip time were obtained. In [Tausworthe, 1972] an asymptotic formula for differential equations of higher order was also offered.

For multidimensional phase control systems with a scalar periodic function the problem of cycle-slipping was studied in [Yershova, Leonov, 1983]. In this paper by means of Lyapunov direct method combined with special Bakaev-Guzh technique [Koryakin, Leonov, 1976] and Yakubovich-Kalman frequency theorem [Yakubovich, 1973] frequency-domain estimates for the number of slipped cycles were established. In monographs [Leonov, Reitmann and Smirnova, 1992], [Leonov, Smirnova, 2000] the frequency-domain theorems of [Yershova, Leonov, 1983] were extended to phase systems with distributed parameters described by Volterra integro-differential equations. In papers [Smirnova, Shepeljavyi and Utina, 2003],[Smirnova, Shepeljavyi and Utina, 2006] the ideas and methods of [Yershova, Leonov, 1983] were applied to discrete phase control systems described by difference equations. All frequency-domain theorems were formulated in terms of transfer function of the linear part of control system. They contain varying parameters. A frequency-domain theorem is the more effective, the more varying parameters it contains.

In this paper phase systems with vector control function are treated. We consider here systems described both by Volterra integro-differential equations and by difference equations. For phase Volterra equations we present an extension of the theorems proved in [Leonov, Reitmann and Smirnova, 1992],[Leonov, Smirnova, 2000] to the case of vector control function. The frequency-domain theorems are now formulated by means of the transfer matrix of the linear part of a control system. They contain matrix varying parameters. Just the same, for difference phase systems with vector control function matrix analogues of the theorems from [Smirnova, Shepeljavyi and Utina, 2006] are obtained.

2 Phase control systems with distributed parameters

Consider a system of integro-differential Volterra equations

$$\dot\sigma(t) = a(t) + R\varphi(\sigma(t-h)) - \\ -\int\limits_0^t \gamma(t-\tau)\varphi(\sigma(\tau))\,d\tau \quad (t>0). \tag{1}$$

Here $\sigma = ||\sigma_j||_{j=1,...,l}$, $a = ||a_j||_{j=1,...,l}$, $\varphi(\sigma) = ||\varphi_j(\sigma_j)||_{j=1,...,l}$ are vector-functions, R is an $l\times l$-matrix, $\gamma = ||\gamma_{ij}||_{i,j=1,...,l}$ is an $l\times l$-matrix-function and h is a nonnegative number. The initial condition for system(1) is as follows

$$\sigma(t)_{|t\in[-h,0]} = \sigma^0(t). \tag{2}$$

Every function $\varphi_j(\sigma_j)$ $(j=1,...,l)$ is $\mathbf{C}^1$, Δ_j-periodic and has finite number of simple zeros on $[0,\Delta_j)$. We suppose also that

$$\int\limits_0^{\Delta_j} \varphi_j(\sigma)\,d\sigma < 0 \ \ (j=1,...,l). \tag{3}$$

Let α_{1j}, α_{2j} be such numbers that

$$\alpha_{1j} \le \frac{d\varphi_j(\sigma)}{d\sigma} \le \alpha_{2j} \ \text{ for all } \ \sigma\in\mathbf{R}, \tag{4}$$

with $\alpha_{1j}<0<\alpha_{2j}$. Let $A_i = diag\{\alpha_{i1},...,\alpha_{il}\}$ $(i=1,2)$.

We assume that functions a_j and γ_{ij} $(i,j=1,...,l)$ aquire the following properties:
1) a_j is $C[0,+\infty)$, $a_j(t)\to 0$ as $t\to+\infty$;
2) there exist such a positive constant r that functions $a_j(t)e^{rt}$, $\gamma_{ij}(t)e^{rt}$ are $L_2[0,+\infty)$.

Let us introduce the transfer matrix of (1) from the input φ to the output $(-\dot\sigma)$

$$\chi(p) = -Re^{-ph} - \int\limits_0^{+\infty} \gamma(t)e^{-pt}\,dt \ \ (p\in\mathbf{C}). \tag{5}$$

Let us introduce also several notations $(j=1,2,...,l)$:

$$\Omega_j^{(1)} = \{\sigma_j\in[0,\Delta_j) : \varphi_j(\sigma_j)>0\}, \tag{6}$$

$$\Omega_j^{(2)} = \{\sigma_j\in[0,\Delta_j) : \varphi_j(\sigma_j)<0\}, \tag{7}$$

$$\Gamma_j = \int\limits_{\Omega_j^{(2)}} |\varphi_j(\sigma)|\,d\sigma, \tag{8}$$

$$\gamma_j = \int\limits_{\Omega_j^{(1)}} \varphi_j(\sigma)\,d\sigma, \quad R_j = \frac{2\Gamma_j\gamma_j}{\Gamma_j+\gamma_j}, \tag{9}$$

$$\nu_j^{(i)}(æ,k,Q) = \frac{\gamma_j - \Gamma_j + \frac{(-1)^i}{kæ_j}(Q+\sum_{q=1,q\ne j}^{l} æ_q R_q)}{\gamma_j+\Gamma_j}, \\ (i=1,2) \tag{10}$$

where $æ = diag\{æ_1,...,æ_l\}$ with $æ_j>0$ $(j=1,...,l)$, $Q\in\mathbf{R}$, k is a natural number.

Consider an arbitrary solution of (1), (2) now. Let $\xi(t) = \varphi(\sigma(t))$ and $\mu(t)$, $\sigma_0(t)$ be defined by the formulae

$$\mu(t) = \begin{cases} 0, t<0, \\ t, 0\le t\le 1 \\ 1, t>1, \end{cases} \tag{11}$$

$$\sigma_0(t) = a(t) + (1-\mu(t-h))R\xi(t-h) - \\ -\int\limits_0^t (1-\mu(\tau))\gamma(t-\tau)\xi(\tau))\,d\tau \\ (\sigma_0 = ||\sigma_{0j}||_{j=1,...,l}). \tag{12}$$

Note that functions $\dot\sigma_j$, ξ_j, $\dot\xi_j$ are bounded on R_+ and $a_j(t)$, $\gamma_{ij}(t)$ are $L^1[0,+\infty)\cap L_2[0,+\infty)$. Let $|\dot\sigma_j|\le\bar\sigma_j$, $|\eta_j|<\bar\varphi_j$, $|\dot\eta_j|<\bar\varphi_{1j}$, $\int\limits_0^\infty a_j^2(t)\,dt = \bar a_j$, $\int\limits_0^\infty \gamma_{ij}^2(t)\,dt = \bar\gamma_{ij}$ $(i,j=1,...,l)$. Let us introduce $l\times l$-diagonal matrix parameters $æ,\delta,\tau$ and define the functions

$$f_1(t,æ,\eta,\tau) = \int\limits_0^t \{(1-\mu)\dot\sigma^* æ\xi + \\ +(1-\mu^2)\xi^*\eta\xi + (\dot{\widehat{\mu\xi}})^*\tau(\dot{\widehat{\mu\xi}}) - \dot\xi^*\tau\xi + \\ +(\dot{\widehat{\mu\xi}}-\xi)\tau A_2\dot\sigma + \dot\sigma^* A_1\tau(\dot{\widehat{\mu\xi}}-\xi)\}dt, \tag{13}$$

$$f_2(t,æ,\tau) = \int\limits_0^t \{\mu\sigma_0^* æ\xi + 2\dot\sigma^*\varepsilon\sigma_0 - \\ -2\sigma_0^* A_1\tau A_2\dot\sigma + \sigma_0^* A_1\tau\dot{\widehat{\mu\xi}} + (\dot{\widehat{\mu\xi}})^*\tau A_2\sigma_0\}dt, \tag{14}$$

where the symbol * is used for Hermitian conjugation. Both functions are bounded. So

$$|f_1(t,æ,\eta,\tau) + f_2(t,æ,\tau)| < Q(æ,\eta,\tau), \tag{15}$$

where Q may be calculated by means of $\bar\sigma_j$, $\bar\varphi_j$, $\bar\varphi_{1j}$, $\bar a_j$, $\bar\gamma_{ij}$ $(i,j=1,...,l)$.

Theorem 1. Suppose there exist such positive definite matrices $æ = diag\{æ_1,...,æ_l\}$ $\varepsilon = diag\{\varepsilon_1,...,\varepsilon_l\}$,

$\eta = diag\{\eta_1, ..., \eta_l\}$, $\tau = diag\{\tau_1, ..., \tau_l\}$ and such positive integers $m_1, m_2, ..., m_l$ that the following conditions are true:
1) for all $\omega \in \mathbf{R}$ the matrix

$$\begin{aligned}&\Pi(\omega) \equiv \Re e\{æ^*\chi(i\omega)\\&+(A_1\chi(i\omega) + i\omega E_l)^*\tau(i\omega E_l + A_2\chi(i\omega))\}\\&-\chi^*(i\omega)\varepsilon\chi(i\omega) - \eta \quad (i^2 = -1),\end{aligned} \tag{16}$$

is positive definite[1];
2)

$$4\varepsilon_j\delta_j > æ_j^2(\nu_j^{(i)})^2(æ, m_j, Q)\;(i = 1, 2; j = 1, 2, ..., l). \tag{17}$$

Then for any solution of (1), (2) the estimates

$$|\sigma_j(t) - \sigma_j(0)| < m_j\Delta_j \quad (t > 0, j = 1, ..., l) \tag{18}$$

are true.

Proof of theorem 1. Let $T > 0$ and $\sigma(t)$ be an arbitrary solution of (1), (2). Let us introduce the functions

$$\xi_T(t) = \begin{cases} \mu(t)\xi(t), t < T, \\ \mu(t)\xi(T)e^{c(T-t)}, t \geq T \;\; (c > 0), \end{cases} \tag{19}$$

$$\begin{aligned}&\sigma_T(t) = R\xi_T(t-h) - \int_0^t \gamma(t-\tau)\xi_T(\tau))\,d\tau\\&(\sigma_T = ||\sigma_{Tj}||_{j=1,\dots,l}).\end{aligned} \tag{20}$$

Note that

$$\dot\sigma(t) = \sigma_0(t) + \sigma_T(t) \quad \text{for } t \in [0, T]. \tag{21}$$

The properties of matrix-functions φ, a and γ imply that $\sigma_{Tj}, \xi_{Tj}, \dot\xi_{Tj} \in L_2[0, +\infty) \cap L_1[0, +\infty)$. Consider the functional

$$\begin{aligned}&\rho_T = \int_0^\infty \{\sigma_T^* æ\xi_T + \xi_T^*\eta\xi_T+\\&+(A_1\sigma_T - \dot\xi_T)^*\tau(\dot\xi_T - A_2\sigma_T) + \sigma_T^*\varepsilon\sigma_T\}\,dt.\end{aligned} \tag{22}$$

Denote by $F[f](i\omega)$ the Fourier transform for a function $f \in L^1[0, +\infty) \cap L^2[0, +\infty)$. By means of Parseval equality we have

$$\begin{aligned}&\rho_T = \tfrac{1}{2\pi}\int_{-\infty}^{+\infty}\{F^*[\sigma_T]æF[\xi_T] + F[\xi_T]^*\eta F[\xi_T]+\\&+(A_1F[\sigma_T] - F[\dot\xi_T])^*\tau(F[\dot\xi_T] - A_2F[\sigma_T])+\\&+F[\sigma_T]^*\varepsilon F[\sigma_T]\}\,dt,\end{aligned} \tag{23}$$

[1] Hereafter $\Re eH = 1/2(H + H^*)$ for $l \times l$-matrix H

Since

$$F[\sigma_T](i\omega) = -\chi(i\omega)F[\xi_T](i\omega) \tag{24}$$

and

$$F[\dot\xi_T](i\omega) = i\omega F[\xi_T](i\omega) \tag{25}$$

we have

$$\rho_T = -\tfrac{1}{2\pi}\int_{-\infty}^{+\infty} F^*[\xi_T](i\omega)\Pi(\omega)F[\xi_T](i\omega)\,d\omega. \tag{26}$$

In virtue of condition 1) of the theorem

$$\rho_T < 0. \tag{27}$$

Let us represent the functional ρ_T as the following sum:

$$\begin{aligned}&\rho_T = I_T + J_T - f_1(T, æ, \eta, \tau) - f_2(T, æ, \tau)+\\&+f_3(T, \varepsilon, \tau) + f_4(T, æ, \varepsilon, \eta, \tau),\end{aligned} \tag{28}$$

where

$$I_T = \int_0^T\{\dot\sigma^*æ\xi + \xi^*\eta\xi + \dot\sigma^*\varepsilon\dot\sigma\}\,dt, \tag{29}$$

$$J_T = \int_0^T\{(A_1\dot\sigma - \dot\xi)^*\tau(\dot\xi - A_2\dot\sigma)\}\,dt, \tag{30}$$

$$\begin{aligned}&f_3(T, \varepsilon, \tau) = \int_T^\infty\{\sigma_T^*(\varepsilon - A_1^*\tau A_2)\sigma_T\}\,dt+\\&+\int_0^T\{\sigma_0^*\varepsilon\sigma_0\}\,dt,\end{aligned} \tag{31}$$

$$\begin{aligned}&f_4(T, æ, \varepsilon, \eta, \tau) = \int_T^\infty\{\sigma_T^*æ\xi_T + \xi_T^*\eta\xi_T+\\&+\sigma_T^*A_1^*\tau\dot\xi_T - \dot\xi_T^*\tau\dot\xi_T + \xi_T^*\tau A_2\sigma_T\}\,dt.\end{aligned} \tag{32}$$

Note that $f_3(T, \varepsilon, \tau) > 0$. Note also that

$$J_T = \sum_{j=1}^l \int_{\sigma_j(0)}^{\sigma_j(T)} \tau_j(\alpha_{1j} - \varphi_j'(\sigma))(\varphi_j'(\sigma) - \alpha_{2j})\,dt, \tag{33}$$

and so in virtue of (4)

$$J_T > 0. \tag{34}$$

Then it follows from (27) that for all $T > 0$

$$I_T < f_1 + f_2 - f_4. \tag{35}$$

Hence

$$I_T < Q + \varepsilon_0 \quad \text{for all } T > 0, \tag{36}$$

where ε_0 can be made as small as we wish by the choice of number c.

Let us introduce the functions

$$\Phi_j^{(i)}(\sigma) = \varphi_j(\sigma) - \nu_j^{(i)}(æ, m_j, Q + \varepsilon_0)|\varphi_j(\sigma)|, \tag{37}$$

$$(i = 1, 2; j = 1, ..., l). \tag{38}$$

The functional I_T can be represented as follows:

$$I_T = \sum_{j=1}^{l} æ_j \int\limits_{\sigma_j(0)}^{\sigma_j(T)} \Phi_j^{(i)}(\sigma)\, d\sigma + \sum_{j=1}^{l} \int\limits_0^T Z_j^{(i)}(t)\, dt, \tag{39}$$

where

$$\begin{aligned} Z_j^{(i)}(t) &= æ_j \dot{\sigma}_j \varphi_j(\sigma_j) + \varepsilon_j \sigma_j^2 + \\ &+ \eta_j \varphi_j^2(\sigma_j) - æ_j \Phi_j^{(i)}(\sigma_j)\dot{\sigma}_j \ \ (i = 1, 2). \end{aligned} \tag{40}$$

For the functions $Z_j^{(i)}(t)$ $(i = 1, 2, j = 1, ..., l)$ we have

$$Z_j^{(i)}(t) = \varepsilon_j \dot{\sigma}_j^2 + \eta_j \varphi_j^2(\sigma_j) + æ_j \nu_j^{(i)} |\varphi_j| \dot{\sigma}_j. \tag{41}$$

where $\nu_j^{(i)} = \nu_j^{(i)}(æ, m_j, Q + \varepsilon_0)$. According the condition 2) of the theorem the functions $Z_j^{(i)}(t)$ are positive definite if ε_0 is small enough. So

$$\sum_{j=1}^{l} \int\limits_0^T Z_j^{(i)}(t)\, dt > 0 \ \ (i = 1, 2). \tag{42}$$

Suppose now that there exists such a moment $t_1 > 0$ that for $j = k_1, k_2, ..., k_r (1 \le k_i \le l)$ $\sigma_j(t_1) = \sigma_j(0) + m_j \Delta_j$ though $\sigma_j(t) < \sigma_j(0) + m_j \Delta_j$ for $t \in [0, t_1)$. For all $j \ne k_i$ $(i = 1, 2, ..., r)$ $\sigma_j(t_1) < \sigma_j(0) + m_j \Delta_j$. Then for each $j = k_1, k_2, ..., k_r$ we have

$$\begin{aligned} &æ_j \int\limits_{\sigma_j(0)}^{\sigma_j(t_1)} \Phi_j^{(1)}(\sigma)\, d\sigma = \\ &= æ_j m_j \int\limits_0^{\Delta_j} \Phi_j^{(1)}(\sigma)\, d\sigma = æ_j m_j (\int\limits_0^{\Delta_j} \varphi_j(\sigma)\, d\sigma - \\ &- \nu_j^{(1)}(æ, m_j, Q + \varepsilon_0) \int\limits_0^{\Delta_j} |\varphi_j(\sigma)|\, d\sigma) = \\ &= Q + \varepsilon_0 + \textstyle\sum_{q=1, q\ne j}^{l} æ_q R_q. \end{aligned} \tag{43}$$

For each $j \ne k_1, k_2, ..., k_r$ we have

$$\begin{aligned} &æ_j \int\limits_{\sigma_j(0)}^{\sigma_j(t_1)} \Phi_j^{(1)}(\sigma)\, d\sigma = æ_j (\int\limits_{\sigma_j(0)}^{\sigma_j(t_1)} \varphi_j(\sigma)\, d\sigma - \\ &- \nu_j^{(1)}(æ, m_j, Q + \varepsilon_0) \int\limits_{\sigma_j(0)}^{\sigma_j(t_1)} |\varphi_j(\sigma)|\, d\sigma). \end{aligned} \tag{44}$$

Let $\sigma_j(t_1) = \sigma_j(0) + m_{0j}\Delta_j + \beta_j$, where $m_{0j} < m_j$ and $\beta_j \in [0, \Delta_j)$. Then

$$\begin{aligned} &\int\limits_{\sigma_j(0)}^{\sigma_j(t_1)} \varphi_j(\sigma)\, d\sigma = m_{0j} \int\limits_0^{\Delta_j} \varphi_j(\sigma)\, d\sigma + \int\limits_0^{\beta_j} \varphi_j(\sigma)\, d\sigma = \\ &= m_{0j}(\gamma_j - \Gamma_j) + \gamma_j' - \Gamma_j', \end{aligned} \tag{45}$$

where

$$\gamma_j' = \int\limits_{(0,\beta_j)\cap\Omega_j^{(1)}} \varphi_j(\sigma)\, d\sigma, \Gamma_j' = \int\limits_{(0,\beta_j)\cap\Omega_j^{(2)}} |\varphi_j(\sigma)|\, d\sigma, \tag{46}$$

and

$$\int\limits_{\sigma_j(0)}^{\sigma_j(t_1)} |\varphi_j(\sigma)|\, d\sigma = m_{0j}(\gamma_j + \Gamma_j) + (\gamma_j' + \Gamma_j'). \tag{47}$$

We have

$$\begin{aligned} &æ_j \int\limits_{\sigma_j(0)}^{\sigma_j(t_1)} \Phi_j^{(1)}\, d\sigma > æ_j(\gamma_j' - \Gamma_j' - \tfrac{\gamma_j - \Gamma_j}{\gamma_j + \Gamma_j}(\gamma_j' + \Gamma_j')) = \\ &= \tfrac{2æ_j(\Gamma_j \gamma_j' - \Gamma_j' \gamma_j)}{\gamma_j + \Gamma_j} > -æ_j R_j. \end{aligned} \tag{48}$$

Consequently, in this case

$$\begin{aligned} &I_T > \textstyle\sum_{j=1}^{l} æ_j \int\limits_{\sigma_j(0)}^{\sigma_j(T)} \Phi_j^{(1)}(\sigma)\, d\sigma > r(Q + \varepsilon_0) + \\ &+ r \textstyle\sum_{q=1}^{l} æ_q R_q - \sum_{q=1}^{l} æ_q R_q > (Q + \varepsilon_0). \end{aligned} \tag{49}$$

But this inequality contradicts (36). So our assumption is false and

$$\sigma_j(t) < \sigma_j(0) + m_j \Delta_j \tag{50}$$

for all $t > 0$ and all $j = 1, 2, ..., l$.

Just in the similar way by means of functions $\Phi_j^{(2)}(\sigma)$ we can prove that for all $t > 0$ and all $j = 1, 2, ..., l$

$$\sigma_j(t) > \sigma_j(0) - m_j \Delta_j. \tag{51}$$

Theorem 1 is proved.

3 The discrete phase control system

Consider a multidimensional discrete phase system

$$\begin{aligned} z(n+1) &= Az(n) + B\xi(n), \\ \sigma(n+1) &= \sigma(n) + C^* z(n) + R\xi(n), \\ \xi(n) &= \varphi(\sigma(n)), \quad n = 0,1,2,\dots \end{aligned} \tag{52}$$

Here A, B, C, R are real matrices of order $(m \times m)$, $(m \times l)$, $(m \times l)$, $(l \times l)$ respectively. We suppose that the pair (A, B) is controllable, the pair (A, C) is observable and all eigenvalues of A lie inside the open unit circle. Vector function $\varphi(\sigma)$ satisfies all the assumptions of the previous section. We shall also use here all the designations concerning $\varphi(\sigma)$.

In this section we shall demonstrate for discrete system (52) several assertions similar to theorem 1. They are proved by Lyapunov direct method in the paper [Smirnova, Shepeljavyi and Utina, 2007].

The transfer matrix of the linear part of system (52) has the form

$$K(p) = C^*(A - pE_m)^{-1}B - R \ \ (p \in \mathbf{C}), \tag{53}$$

where E_m is an $(m \times m)$-unit matrix and a frequency-domain condition may be for instance as follows:

$$\Re e\{æK(p) - K^*(p)\varepsilon K(p) - \eta\} \geq 0 \ \ |p| = 1, \ p \in \mathbf{C}, \tag{54}$$

where $(l \times l)$-diagonal matrices æ, $\varepsilon > 0$, $\eta > 0$ are varying matrix parameters.

We shall need the notations

$$\mu_j^{(i)}(æ, k, Q) = \frac{\gamma_j - \Gamma_j + \frac{(-1)^i}{æ_j k}(Q + \sum_{j=1}^{l} |æ_j| R_j)}{\gamma_j + \Gamma_j}, \quad (i = 1,2) \tag{55}$$

We shall also need the following quadratic forms of $z \in \mathbf{R}^m$ and $\xi \in \mathbf{R}^l$:

$$\begin{aligned} &F(z,\xi) = \\ &= \xi^* æ(C^*z + R\xi) + \xi^*\eta\xi + (C^*z + R\xi)\varepsilon(C^*z + R\xi). \\ &\Phi(z,\xi) = (Az + B\xi)^* H(Az + B\xi) - z^*Hz + F(z,\xi), \end{aligned} \tag{56}$$

Here $H = H^*$ is a $(m \times m)$-matrix and $\varepsilon = diag\{\varepsilon_1, ..., \varepsilon_l\}$, $\eta = diag\{\eta_1, ..., \eta_l\}$, $æ = diag\{æ_1, ..., æ_l\}$ are real diagonal $(l \times l)$-matrices.

If the condition (54) is fulfilled for some matrices æ, $\varepsilon > 0$, $\eta > 0$then according to Yakubovich-Kalman frequency-domain theorem [Yakubovich, 1973] there exists a matrix $H = H^*$, which guarantees that the inequality $\Phi(z,\xi) \leq 0$ is valid for all $z \in \mathbf{R}^m$, $\xi \in \mathbf{R}^l$.

Theorem 2. Let there exist such diagonal matrices $\varepsilon > 0$, $\eta > 0$, æ and such positive integers $m_1, m_2, ..., m_l$ that the following relations hold:
1) For all $p \in \mathbf{C}$, $|p| = 1$ the matrix

$$\Re e\{æK(p) - K^*(p)\varepsilon K(p) - \eta\} \tag{57}$$

is positive definite.
2) The inequalities

$$\begin{aligned} &4\eta_j \left[\varepsilon_j - \tfrac{æ_j \alpha_{0j}}{2}(1 + |\mu_j^{(i)}(æ, m_j, z^*(0)Hz(0))|) \right] > \\ &> \left[æ_j \mu_j^{(i)}(æ, m_j, z^*(0)Hz(0)) \right]^2 \end{aligned} \tag{58}$$

$(j = 1,2,...,l, i = 1,2)$ with $\alpha_{0j} = \alpha_{2j}$ if $æ_j > 0$, and $\alpha_{0j} = \alpha_{1j}$ if $æ_j < 0$ are true. Here $H = H^*$ is just such a $(m \times m)$-matrix that $\Phi(z,\xi) \leq 0$, $\forall z \in \mathbf{R}^m$, $\xi \in \mathbf{R}^l$.

Then for any solution $(z(n), \sigma(n))$ of (52) with initial data $(z(0), \sigma(0))$ the estimates

$$|\sigma_j(n) - \sigma_j(0)| < m_j \Delta_j \ \ (j = 1,2,...,l) \tag{59}$$

are true for all natural n.

The proof of theorem 2 is based on a special Lyapunov-type lemma with Lyapunov functions of the form "a quadratic form plus integral of a nonlinearity". The nonlinearity in Lyapunov function is constructed by Bakaev-Guzh technique [Leonov, Smirnova, 2000] intended specially for phase control systems.

Let us extend the state space of system (52) [Leonov, Smirnova, 2000], [Koryakin, Leonov, 1976]. For the purpose we introduce the notations

$$y = \left\| \begin{matrix} z \\ \varphi(\sigma) \end{matrix} \right\|, \ P = \left\| \begin{matrix} A & B \\ 0 & E_l \end{matrix} \right\|, \ L = \left\| \begin{matrix} 0 \\ E_l \end{matrix} \right\|, \tag{60}$$

$C_1^* = ||C^*, R||$, $\quad \xi_1(n) = \varphi(\sigma(n+1)) - \varphi(\sigma(n))$.
Here P is a $((m+l) \times (m+l))$ - matrix, L is a $((m+l) \times l)$ - matrix, C_1^* is a $(l \times (m+l))$ - matrix, y is a $(m+l)$-vector and ξ_1 is a l-vector. Then system (52) can be written as follows

$$\begin{aligned} y(n+1) &= Py(n) + L\xi_1(n), \\ \sigma(n+1) &= \sigma(n) + C_1^* y(n), \ n = 0,1,2,\dots \end{aligned} \tag{61}$$

Consider the forms of $y \in \mathbf{R}^{m+l}$ and $\xi_1 \in \mathbf{R}^l$

$$\begin{aligned} &\Phi_1(y,\xi_1) = (Py + L\xi_1)^* H_1 (Py + L\xi_1) - \\ &- y^* H_1 y + F_1(y,\xi_1), \\ &F_1(y,\xi_1) = y^* L æ C_1^* y + y^* C_1 \varepsilon C_1^* y + \\ &+ y^* L\eta L^* y + (A_1 C_1^* y - \xi_1)^* \tau (\xi_1 - A_2 C_1^* y), \end{aligned} \tag{62}$$

where $A_i = diag\{\alpha_{i1}, \alpha_{i2}, ..., \alpha_{il}\}$ $(i = 1,2)$, $H_1 = H_1^*$ is a $((m+l) \times (m+l))$ - matrix, and $\varepsilon, \eta, æ, \tau$ are real diagonal matrices with varied elements.

Theorem 3. Suppose there exist such diagonal matrices $\varepsilon > 0$, $\tau > 0$, $\eta > 0$, æ and such positive integers $m_1, m_2, ..., m_l$ that the following relations hold:

1) For all $p \in \mathbf{C}, |p| = 1$ the matrix

$$\Re e\{æK(p) - K^*(p)\varepsilon K(p) - \eta \\ +(A_1K(p) + (p-1)E_l)^*\tau((p-1)E_l + A_2K(p))\} \tag{63}$$

is positive definite.

2) The inequalities

$$4\eta_j \left[\varepsilon_j - \frac{æ_j\alpha_{0j}}{2}(1 + |\mu_j^{(i)}(æ, m_j, y^*(0)H_1y(0) - \\ -r)|)\right] > \left[æ_j\mu_j^{(i)}(æ, m_j, y^*(0)H_1y(0) - r)\right]^2 \tag{64}$$

$(j = 1, 2, ..., l, i = 1, 2)$ are valid, where $H_1 = H_1^*$ is such a $((m+l) \times (m+l))$-matrix that $\Phi_1(y, \xi_1) \leq 0$ $(y \in \mathbf{R}^{m+l}, \xi_1 \in \mathbf{R}^l)$ and

$$r \leq \inf_{n=0,1,2,...} y^*(n)H_1y(n). \tag{65}$$

Then for solution $(z(n), \sigma(n))$ of (52) with initial data $(z(0), \sigma(0))$ the estimates (59) are true for all natural n.
Remark 1. Notice that if the relation 1) of the theorem is fulfilled then according to Yakubovich–Kalman frequency–domain theorem [Yakubovich, 1973] there exists a matrix $H_1 = H_1^*$, which guarantees that the inequality $\Phi_1(y, \xi_1) \leq 0$ is valid for all $y \in \mathbf{R}^{m+l}$, $\xi_1 \in \mathbf{R}^l$.
Theorem 4. Let all the relations of theorem 3 be fulfilled, except relation 2) which is substituted by the requirement
2') inequalities

$$4\eta_j \left[\varepsilon_j - \frac{æ_j\alpha_{0j}}{2}(1 + |\mu_j^{(i)}(æ, m_j, |y^*(0)H_1y(0)|)|)\right] > \\ > \left[æ_j\mu_j^{(i)}(æ, m_j, |y^*(0)H_1y(0)|)\right]^2 \tag{66}$$

$(j = 1, 2, ..., l, i = 1, 2)$ are valid with $H_1 = H_1^*$ satisfying $\Phi_1 \leq 0$ $(y \in \mathbf{R}^{m+l}, \xi_1 \in \mathbf{R}^l)$.
Then for any solution $(z(n), \sigma(n))$ of (52) with initial data $(z(0), \sigma(0))$ the following limit relations are true:

$$z(n) \to 0, \ \ \sigma_j(n) \to \hat{\sigma}_j \text{ as } n \to +\infty, \tag{67}$$

where $\varphi_j(\hat{\sigma}_j) = 0$, and

$$|\sigma_j(0) - \hat{\sigma}_j| < m_j\Delta_j \ \ (j = 1, 2, ..., l). \tag{68}$$

4 Conclusion

Two types of multidimensional phase systems, namely distributed systems and discrete ones are considered. By means of a priori integral estimates method and Lyapunov direct method combined with Bakaev-Guzh special technique frequency-domain estimates for the phase coordinates are established.

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