Indirect control of the asymptotic states of a quantum dynamical semigroup

Raffaele Romano

***Abstract*— In the dynamics of open quantum systems, the interaction with the external environment usually leads to a contraction of the set of reachable states for the system as time increases, eventually shrinking to a single stationary point. In this contribution we describe to what extent it is possible to modify this asymptotic state by means of indirect control, that is by using an auxiliary system coupled to the target system in order to affect its dynamics, when there is a purely dissipative coupling between the two systems. We prove that, also in this restrictive case, it is possible to modify the asymptotic state of the relevant system, give necessary conditions for that and provide physical examples. Therefore, in indirect control schemes, the environmental action has not only a negative impact on the dynamics of a system, it is rather possible to make use of it for control purposes.**

INTRODUCTION

The study of quantum mechanical systems is relevant not only for a deep understanding of the fundamental physical laws, but also for its potential applications [1]. In particular, it has been proved that the use of quantum-based technologies would highly increase the performance of a computational device [2]. This is due to the mathematical structure of the theory, leading to peculiar features for the microscopic systems, absent in the macroscopic, classical world. Among them, the most relevant (and not completely understood) is represented by the quantum correlations known as *entanglement*, typical of quantum systems, whose complete characterization has been given only in the low dimensional cases. These correlations are the key of many recently proposed protocols, as teleportation [3] or quantum cryptography [4].

However, quantum systems are fragile: their relevant features are usually degraded by their interaction with the external environment, leading to irreversibility, dissipation and decoherence. Therefore closed systems, described by the Schrödinger equation, are only approximations of real systems, that are necessarily open since they exchange energy and information with the external world. To account for this, the standard dynamical model is given by a *quantum dynamical semigroup*, that is a one-parameter family of Markovian (i.e., satisfying the semigroup property) completely positive maps, transforming pure states into mixtures and destroying quantum coherence [5], [6]. Irreversibility is due to the fact that quantum dynamical semigroups are contractions in the state space of the system: this highly reduces the ability of manipulation on the system, since many transitions become forbidden. The extremal case is represented by the so-called *uniquely relaxing semigroups*, where there is a unique asymptotic state for every initial state of the system. Therefore, in order to fully make use of the potentialities of quantum systems, it is fundamental to understand the mechanisms leading to dissipation and to conceive methods to counteract them, and, more in general, to control the dynamics of the system.

With these motivations in mind, several solutions to the problem have been proposed. The study of the symmetries of the system-environment coupling has lead to the notion of *decoherence-free subspaces or subsystems*, that are quiet places, unaffected by decoherence, where to encode the relevant information (for a review on the topic, see [7]). From a more active perspective, a theory of quantum control has developed, dealing with the effect of external manipulations that can be performed on the system. In the standard setting (*coherent control*), the control parameters enter the Hamiltonian of the system, for example through external fields coupled to the system (for a geometric control perspective, in line with that developed in this work, refer to [8], [9], [10], [11], [12], [13], [14]). Although this is the most natural way to introduce external actions in the dynamics of a system, its ability to fight the unwished effects of the irreversible dynamics is limited [14], unless some information about the state of the system is collected, and then used to update in real time the controls. This information is usually obtained via an indirect continuous measurement and, because of this, the master equation describing the system becomes stochastic. This *quantum feedback* scheme represents a promising approach in many interesting cases [15], [16], [17].

A different approach to quantum control has been recently discussed, in which the control does not enter through the Hamiltonian of the system, but it is rather obtained by means of an auxiliary system (ancilla) [18], [19], [20], that can be manipulated and put in interaction with the relevant system, and finally discarded at the end of the procedure. The ability of driving the target system through the auxiliary one is determined by the correlations between them. The interest in this *indirect control* method is twofold. It is complementary to the coherent control approach, that is, it can be applied to experimental setups where the coherent control technique is not appropriate. Moreover, in the indirect control approach the environment does not only represent a source of noise, it can also be used for control purposes. In fact, it has been proved that the environment can correlate two systems

The author acknowledges support from the European grant ERG:044941-STOCH-EQ. Work in part supported by Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Italy.

R. Romano is with the Department of Theoretical Physics, University of Trieste, Strada Costiera 11, I-34014 Trieste, Italy; rromano@ts.infn.it

immersed in it via a noisy mechanism, not only destroy their existing correlations [21], [22]. This mechanism can be used to obtain total control of the target system even if there is not a direct interaction between the two parties [23].

In this contribution we prove that, in the indirect control scheme, the environment induced correlations can be used to manipulate the asymptotic states of the target system. This result gives further evidence that, in this framework, it is possible to engineer the environmental action to get controllability. The work is organized as follows: in Section I we summarize some standard results about quantum dynamical semigroups and their stationary states, in Section II we provide necessary conditions for the indirect manipulation of the asymptotic states of the relevant system under a purely dissipative dynamics, in Section III we evaluate the stationary states and describe a concrete physical example of application, and finally we conclude in Section IV.

I. Quantum dynamical semigroups and their stationary states

In many situations (usually, when there is a weak interaction with the surrounding environment) it is possible to approximate the reduced dynamics of an open system S using a Markovian one-parameter family of completely positive maps $\{\gamma_t; t \geqslant 0\}$, satisfying the semigroup property $\gamma_{t+s} = \gamma_t \circ \gamma_s$, with $t, s \geqslant 0$, with

$$\rho_s(t) = \gamma_t[\rho_s(0)], \tag{1}$$

where the state of the system S is given by the Hermitian, positive, unit trace operator ρ_s (statistical operator), acting on the n-dimensional Hilbert space associated to S. Complete positivity is necessary in order to guarantee a physically consistent interpretation of the formalism when dealing with composite, entangled systems. The generator L of the dynamics is defined by $\dot{\rho}_s = L[\rho_s]$, and it is possible to prove that it has the general form (the so-called Lindblad-Kossakowski form)

$$L[\rho_s] = -i[H_s, \rho_s] + \sum_{i,j} c_{ij} \Big(F_i \rho_s F_j^\dagger - \frac{1}{2}\{F_j^\dagger F_i, \rho_s\} \Big), \tag{2}$$

where $H_s = H_s^\dagger$ is the Hamiltonian of S, and the set $\{F_i; i = 1, \ldots, n^2 - 1\}$, along with the n-dimensional identity operator, form a basis for the operators acting on the Hilbert space associated to S, satisfying $\mathrm{Tr} F_i = 0$, and $\mathrm{Tr}(F_i F_j^\dagger) = \delta_{ij}$. The $(n^2-1) \times (n^2-1)$ matrix $C = [c_{ij}]$ (Kossakowski matrix) fulfills $C^\dagger = C$ and $C \geqslant 0$, necessary and sufficient condition for the complete positivity of the dynamics [5], [6].

In the following, S is a bipartite system, $S = T + A$, where T is the target system (to be manipulated) and A the ancilla. We assume that T and A are two copies of the same two-level system, separately interacting with the same environment, assumed to be spatially homogeneous, according to the Markovian dynamics (2). We further assume that $H_s = 0$, since we want to study a purely dissipative dynamics. To model this system, it is sufficient to consider the basis $\{F_i; i = 1, \ldots, 6\}$, given by the local operators $F_i = \sigma_i \otimes \mathbb{I}$ for $i = 1, 2, 3$ and $F_i = \mathbb{I} \otimes \sigma_{i-3}$ for $i = 4, 5, 6$, where σ_i, $i = 1, 2, 3$ are the Pauli matrices. We consider the standard representation of these operators in which σ_3 is diagonal. The 6×6 matrix C has the form

$$C = \begin{bmatrix} A & B \\ B^\dagger & A \end{bmatrix}, \tag{3}$$

where $A = A^\dagger$ is the Kossakowski matrix for the system T (or A) alone, and B represents the dissipative coupling between the two parties. A and B are 3×3 blocks. The form (3) is not the most general joint Kossakowski matrix, as we have assumed that the two parties interact separately with the environment, and that the two local dissipative contributions are equal (homogenous environment). We will limit our attention to models well described by (3); moreover, for simplicity, we will further assume $B = B^\dagger$. This choice produces a significative simplification in the treatment and it is still of great phenomenological interest.

The first term in the right hand side of (2) represents the coherent part of the evolution and it generates a group of reversible, unitary transformations. The second term generates the irreversible dynamics, according to the matrix C whose entries depend on the microscopical details of the interaction between system and environment. It also leads to the appearance of attractors in the state space of S, and consequently relaxation to equilibrium of the states of the open system. A stationary state for the dynamics, ρ_s^∞, is defined by the condition on the generator $L[\rho_s^\infty] = 0$. Since the dynamics is linear in the state ρ_s, it is possible to fully characterize the asymptotic fate of the system by studying the eigenvalues of the dynamical matrix appearing in the coherence vector representation of (2) [24]. Although this treatment is very general, it is not suitable for the purposes of this work. We will rather refer to some necessary conditions for the existence of stationary states, and for the convergence of $\rho_s(t)$ to them, given in terms of the operators $\{V_i; i\}$ appearing in the diagonal form of (2),

$$L[\rho_s] = \sum_i \Big(V_i \rho_s V_i^\dagger - \frac{1}{2}\{V_i^\dagger V_i, \rho_s\} \Big). \tag{4}$$

These conditions are expressed by the following theorem [25], that has been adapted to the present context.

Theorem 1: Given the quantum dynamical semigroup (4), assume that it admits a stationary state ρ_0 of maximal rank. Defining $\mathcal{M} = \{V_i, V_i^\dagger; i\}'$, the commutant of the Hamiltonian plus the dissipative generators, the following conditions hold true:

1. If $\mathcal{M} = \mathrm{span}(\mathbb{I})$, then ρ_0 is the unique stationary state. Moreover, if $\{V_i; i\}$ is a self-adjoint set with $\{V_i; i\}' = \mathrm{span}(\mathbb{I})$, then for every initial condition $\rho_s(0)$

$$\lim_{t \to +\infty} \rho_s(t) = \rho_0.$$

2. If $\mathcal{M} \neq \mathrm{span}(\mathbb{I})$, then there exist a complete family $\{P_n; n\}$ of pairwise orthogonal projectors such that $\mathcal{Z} = \mathcal{M} \cap \mathcal{M}' = \{P_n; n\}''$. If $\{V_i; i\}' = \mathcal{M}$, two extreme cases together with their linear superpositions may occur.

i. If $\mathcal{Z} = \mathcal{M}$, then for every initial condition $\rho_s(0)$

$$\lim_{t\to+\infty} \rho_s(t) = \sum_n \mathrm{Tr}(P_n\rho_s(0)P_n)\frac{P_n\rho_0 P_n}{\mathrm{Tr}(P_n\rho_0 P_n)}.$$

ii. If $\mathcal{Z} = \mathcal{M}'$, then for every $\rho_s(0)$

$$\lim_{t\to+\infty} \rho_s(t) = \sum_n P_n\rho_s(0)P_n.$$

Therefore, the stationary states of a quantum dynamical semigroup can be characterized by means of the algebras $\mathcal{M}$, $\mathcal{M}'$, and $\mathcal{Z}$, if a maximal rank stationary state ρ_0 is available. These quantities are evaluated in the next section, depending on the form of the matrix C.

II. Relevant algebraic quantities

Following Theorem 1, we need to write C in diagonal form in order to find the operators V_i appearing in (4). This is achieved by means of the unitary transformation U such that

$$UCU^\dagger = \mathrm{diag}(\lambda_i, i = 1, \ldots, 6), \tag{5}$$

where λ_i are the eigenvalues of C. U has the form

$$U = \frac{1}{\sqrt{2}}\left[\begin{array}{cc} \tilde{U} & \tilde{U} \\ -\hat{U} & \hat{U}\end{array}\right] \tag{6}$$

and $\tilde{U}$, $\hat{U}$ are unitary transformations such that

$$\begin{aligned} \tilde{U}(A+B)\tilde{U}^\dagger &= \mathrm{diag}(\lambda_i^+, i = 1,2,3), \\ \hat{U}(A-B)\hat{U}^\dagger &= \mathrm{diag}(\lambda_i^-, i = 1,2,3). \end{aligned} \tag{7}$$

The eigenvalues of C are ordered as $\lambda_i = \lambda_i^+$ for $i = 1,2,3$ and $\lambda_i = \lambda_{i-3}^-$ for $i = 4,5,6$. Comparing the generator forms (2) and (4), and using the notation $U = [u_{ij}]$, we have

$$V_i = \sum_{k=1}^{6} u_{ik}^* F_k. \tag{8}$$

Following (6), it is possible to write

$$V_i = \begin{cases} \mathbb{I}\otimes\tilde{\sigma}_i + \tilde{\sigma}_i\otimes\mathbb{I}, & i = 1,\ 2,\ 3 \\ \\ \mathbb{I}\otimes\hat{\sigma}_{i-3} - \hat{\sigma}_{i-3}\otimes\mathbb{I}, & i = 4,\ 5,\ 6 \end{cases} \tag{9}$$

where we have defined

$$\tilde{\sigma}_i = \sum_{k=1}^{3}\tilde{u}_{ik}^*\sigma_k, \qquad \hat{\sigma}_i = \sum_{k=1}^{3}\hat{u}_{ik}^*\sigma_k, \tag{10}$$

and we used the notation $\tilde{U} = [\tilde{u}_{ij}]$, $\hat{U} = [\hat{u}_{ij}]$. The operators in (10) satisfy $\mathrm{Tr}\,\tilde{\sigma}_i = \mathrm{Tr}\,\hat{\sigma}_i = 0$ and $\mathrm{Tr}(\tilde{\sigma}_i\tilde{\sigma}_j^\dagger) = \mathrm{Tr}(\hat{\sigma}_i\hat{\sigma}_j^\dagger) = \delta_{ij}$. They are self-adjoint if and only if the unitary operators $\tilde{U}$ and $\hat{U}$ are orthogonal.

The commutant of Theorem 1 can be expressed as

$$\mathcal{M} = \{V_i, V_i^\dagger; i|\lambda_i \neq 0\}' = \bigcap_{i|\lambda_i\neq 0}\{V_i, V_i^\dagger\}', \tag{11}$$

where only non-vanishing eigenvalues λ_i have to be considered, otherwise the corresponding V_i do not appear in the generator (4). Moreover, for a given i,

$$\{V_i, V_i^\dagger\}' = \{v | v\in\{V_i\}', v^\dagger\in\{V_i\}'\}, \tag{12}$$

therefore we can limit our attention to the sets $\{V_i\}'$. We find convenient to consider separately the two kinds of contributions defined in (9). To begin with, we consider a fixed index i such that $\lambda_i^+ \neq 0$, and assume that the corresponding $\tilde{\sigma}_i$ is non-singular. In this case it can be written as

$$\tilde{\sigma}_i = \tilde{\mu}_i R_i\sigma_3 R_i^{-1} \tag{13}$$

where it is possible to choose $R_i = R_i^{-1}$, and

$$\tilde{\mu}_i^2 = \sum_j(\tilde{u}_{ij}^*)^2. \tag{14}$$

Since $\mathbb{I}\otimes\tilde{\sigma}_i + \tilde{\sigma}_i\otimes\mathbb{I} = \tilde{\mu}_i\mathcal{R}_i(\mathbb{I}\otimes\sigma_3 + \sigma_3\otimes\mathbb{I})\mathcal{R}_i$, with $\mathcal{R}_i = R_i\otimes R_i$, it follows that

$$\{\mathbb{I}\otimes\tilde{\sigma}_i + \tilde{\sigma}_i\otimes\mathbb{I}\}' = \mathcal{R}_i\{\mathbb{I}\otimes\sigma_3 + \sigma_3\otimes\mathbb{I}\}'\mathcal{R}_i \tag{15}$$

and then, after the explicit computation,

$$\{V_i\}' = \mathrm{span}(\mathbb{I}\otimes\mathbb{I}, \mathbb{I}\otimes\tilde{\sigma}_i, \tilde{\sigma}_i\otimes\mathbb{I}, \tilde{\sigma}_i\otimes\tilde{\sigma}_i, \Omega^+, \Delta_i^-), \tag{16}$$

having defined the additional operators

$$\begin{aligned} \Omega^+ &= \sigma_1\otimes\sigma_1 + \sigma_2\otimes\sigma_2 + \sigma_3\otimes\sigma_3, \\ \Delta_i^- &= \mathcal{R}_i(\sigma_1\otimes\sigma_2 - \sigma_2\otimes\sigma_1)\mathcal{R}_i. \end{aligned} \tag{17}$$

Notice that, in general, the operators in the right hand side of (16) are not self-adjoint, nor orthogonal each other in the Hilbert-Schmidt metric, since the transformation $\mathcal{R}_i$ is not unitary (equivalently, self-adjoint). However, if the coefficients $\tilde{u}_{ij}^*$, $j = 1,2,3$, are real, $\tilde{\sigma}_i$ is self-adjoint and $\mathcal{R}_i$ unitary. Consequently, in this case the basis of $\{V_i\}$ is made of Hermitian, orthogonal operators.

The commutants $\{V_i\}'$ are completely characterized for $i = 1,2,3$. Finally, $\{V_i, V_i^\dagger\}'$ can be found by considering (12):

$$\{V_i, V_i^\dagger\}' = \begin{cases} \{V_i\}', & \text{iff } \tilde{\sigma}_i = \tilde{\sigma}_i^\dagger; \\ \\ \mathrm{span}(\mathbb{I}\otimes\mathbb{I}, \Omega^+), & \text{otherwise.} \end{cases} \tag{18}$$

The corresponding sets for $i = 4,5,6$ can be found by applying the same procedure to $\hat{\sigma}_i$, assuming that $\lambda_i^- \neq 0$. The result is

$$\{V_i\}' = \mathrm{span}(\mathbb{I}\otimes\mathbb{I}, \mathbb{I}\otimes\hat{\sigma}_i, \hat{\sigma}_i\otimes\mathbb{I}, \hat{\sigma}_i\otimes\hat{\sigma}_i, \Omega_i^-, \Delta_i^+), \tag{19}$$

where

$$\begin{aligned} \Omega_i^- &= \mathcal{S}_i(\sigma_1\otimes\sigma_1 - \sigma_2\otimes\sigma_2)\mathcal{S}_i, \\ \Delta_i^+ &= \mathcal{S}_i(\sigma_1\otimes\sigma_2 + \sigma_2\otimes\sigma_1)\mathcal{S}_i, \end{aligned} \tag{20}$$

and $\mathcal{S}_i = S_i\otimes S_i$, with

$$\hat{\sigma}_i = \hat{\mu}_i S_i\sigma_3 S_i^{-1}, \tag{21}$$

where $S_i = S_i^{-1}$, and

$$\hat{\mu}_i^2 = \sum_j(\hat{u}_{ij}^*)^2. \tag{22}$$

Finally, in this case

$$\{V_i, V_i^\dagger\}' = \begin{cases} \{V_i\}', & \text{iff } \hat{\sigma}_i = \hat{\sigma}_i^\dagger; \\ \\ \mathrm{span}(\mathbb{I}\otimes\mathbb{I}), & \text{otherwise.} \end{cases} \tag{23}$$

If $\tilde{\sigma}_i$ (or $\hat{\sigma}_i$) is singular, the previous computations are not longer valid. In this case, the commutants must be evaluated by direct computation and it is not possible, in general, to express their structure in a compact form.

We have all the ingredients to evaluate the contribution related to the dissipative generators V_i in (11). We find convenient to denote by n_+ and n_- the number of non-vanishing eigenvalues of the type λ^+ and λ^- respectively. The non-trivial cases are summarized below, with the corresponding relevant algebras and set of projectors $\{P_i; i\}$, described in theorem 1, to be used to construct the set of stationary states. For further reference, the projectors Π_k, $k \in \{-,+,1,\dots,4\}$, are defined as

$$\begin{aligned} &\Pi_k = [\pi^k_{ij}], \quad \pi^k_{ij} = \delta_{ik}\delta_{jk}, \quad k = 1,\dots 4; \\ &\Pi_- = \frac{1}{4}(\mathbb{I}\otimes\mathbb{I} - \Omega^+), \quad \Pi_+ = \mathbb{I}\otimes\mathbb{I} - \Pi_-. \end{aligned} \tag{24}$$

A. Case 1

It is characterized by $n_+ = 1$, $A = A^T$, $B = A$. We notice that $A = B$ is equivalent to $n_- = 0$. The commutant is given by $\mathcal{M} = \mathrm{span}(\mathbb{I}\otimes\mathbb{I}, \mathbb{I}\otimes\tilde{\sigma}_i, \tilde{\sigma}_i\otimes\mathbb{I}, \tilde{\sigma}_i\otimes\tilde{\sigma}_i, \Omega^+, \Delta^-_i)$, and $\mathcal{Z} = \mathcal{M}' = \mathrm{span}(\mathbb{I}\otimes\mathbb{I}, \tilde{\sigma}_i\otimes\tilde{\sigma}_i, \mathbb{I}\otimes\tilde{\sigma}_i + \tilde{\sigma}_i\otimes\mathbb{I})$. The projectors are given by

$$\begin{aligned} P_1 = \mathcal{R}_i\Pi_1\mathcal{R}_i, \qquad P_2 = \mathcal{R}_i\Pi_4\mathcal{R}_i, \\ P_3 = \mathcal{R}_i(\Pi_2 + \Pi_3)\mathcal{R}_i. \end{aligned} \tag{25}$$

B. Case 2

It is characterized by $n_+ = 1$, $A \neq A^T$, $B = A$, or rather $n_+ > 1$, $B = A$. In this case $\mathcal{Z} = \mathcal{M} = \mathrm{span}(\mathbb{I}\otimes\mathbb{I}, \Omega^+)$, and there are only two projectors:

$$P_1 = \Pi_-, \qquad P_2 = \Pi_+. \tag{26}$$

C. Case 3

It is characterized by $n_+ = n_- = 1$, $A = A^T$, $B = \alpha A$, $\alpha \in \mathbb{R} \setminus \{-1,+1\}$. We observe that $[A,B] = 0$, thus it is possible to choose $\tilde{U} = \hat{U}$. Moreover, $B = \alpha A$ implies $\tilde{\sigma}_\xi = \hat{\sigma}_\xi$ for the index ξ such that $\lambda^+_\xi \neq 0$ and $\lambda^-_\xi \neq 0$. Finally, $\mathcal{Z} = \mathcal{M} = \mathrm{span}(\mathbb{I}\otimes\mathbb{I}, \tilde{\sigma}_i\otimes\tilde{\sigma}_i, \mathbb{I}\otimes\tilde{\sigma}_i, \tilde{\sigma}_i\otimes\mathbb{I})$, and the projectors are given by

$$P_j = \mathcal{R}_i\Pi_j\mathcal{R}_i, \qquad j = 1,\dots,4. \tag{27}$$

In all the remaining cases $\mathcal{M} = \mathrm{span}(\mathbb{I}\otimes\mathbb{I})$, part 1 of Theorem 1 applies and the maximal rank stationary state is unique (if there is one). Therefore, the aforementioned cases are necessary conditions for the indirect manipulation of the asymptotic state of the target system T via the auxiliary system A.

III. Stationary states

We separately explore the non-trivial cases described in Section II. Following Theorem 1, if a stationary state ρ_0 whose eigenvalues are all non-vanishing can be found, it is possible to build the whole family of stationary states ρ^∞_s, by using the projectors $\{P_i; i\}$. Finally, the corresponding stationary state of the target subsystem can be obtained from

$$\rho^\infty_T = \mathrm{Tr}_A\, \rho^\infty_s, \tag{28}$$

that is by a partial trace over the degrees of freedom of the auxiliary system. We consider two different choices for the initial state $\rho_s(0)$, depending on wether there are initial correlations or not. As the first choice we take into account the product state

$$\rho_s(0) = \rho_T(0)\otimes\rho_A(0), \tag{29}$$

where $\rho_T(0)$ and $\rho_A(0)$ are arbitrary states for the two subsystems, that will be written using a Bloch vector representation as

$$\rho_T(0) = \frac{1}{2}\Big(\mathbb{I} + \sum_{k=1}^{3}\rho^T_k\sigma_k\Big), \tag{30}$$

with real coefficients ρ^T_k, and analogously for $\rho_A(0)$, with real coefficients ρ^A_k. The choice (30) refers to initially uncorrelated systems, that will in general couple during their joint, dissipative evolution, because of the off-diagonal block B in the Kossakowski matrix C. Alternatively, we consider the pure initial state

$$\rho_s(0) = |\psi\rangle\langle\psi|, \quad |\psi\rangle = \sqrt{P}|\uparrow\rangle\otimes|\uparrow\rangle + \sqrt{1-P}|\downarrow\rangle\otimes|\downarrow\rangle, \tag{31}$$

where $P \in \mathbb{R}$, and $|\uparrow\rangle$, $|\downarrow\rangle$ are the $+1$, respectively -1 eigenvalues of the operator σ_3. This state is entangled if $P \neq 0,1$, and it is maximally entangled if $P = \frac{1}{2}$. It is not an arbitrary entangled state, nevertheless it can be used to test the impact of an initial quantum correlation on the manipulation of the stationary state of T.

Although their algebraic structures are different, Cases 1 and 3 lead to the same results. Since $\mathcal{Z} = \mathcal{M}'$, in Case 1 there is not need of ρ_0, whereas the simplest maximal rank stationary state in Case 3 is given by the maximally mixed state $\rho_0 = \mathbb{I}\otimes\mathbb{I}$. If there is not correlation in the initial state, it is not possible to manipulate the stationary state of the system T by means of the ancilla A. In fact, the coefficients of Bloch vector representation of ρ^∞_T, denoted by ρ^∞_i, $i = 1,2,3$, depends only on $\rho_T(0)$:

$$\begin{aligned} \rho^\infty_1 &= u_{\xi 1}\Big(\rho^T_1 u_{\xi 1} - \rho^T_2 u_{\xi 2} + \rho^T_3 u_{\xi 3}\Big) \\ \rho^\infty_2 &= -u_{\xi 2}\Big(\rho^T_1 u_{\xi 1} - \rho^T_2 u_{\xi 2} + \rho^T_3 u_{\xi 3}\Big) \\ \rho^\infty_3 &= u_{\xi 3}\Big(\rho^T_1 u_{\xi 1} - \rho^T_2 u_{\xi 2} + \rho^T_3 u_{\xi 3}\Big), \end{aligned} \tag{32}$$

where $\xi \in \{1,2,3\}$ is such that $\lambda^+_\xi \neq 0$ in Case 1, $\lambda^+_\xi \neq 0$ and $\lambda^-_\xi \neq 0$ in Case 3. If we consider the (possibly entangled) initial state (31), the dependence on P is apparent:

$$\begin{aligned} \rho^\infty_1 &= (2P-1)u_{\xi 1}u_{\xi 3} \\ \rho^\infty_2 &= -(2P-1)u_{\xi 2}u_{\xi 3} \\ \rho^\infty_3 &= (2P-1)u^2_{\xi 3} \end{aligned} \tag{33}$$

Therefore, at different correlated initial states there correspond different stationary states ρ^∞_T. Manipulations of the asymptotic states of the target system are possible only when there is an initial correlation between T and A.

In Case 2, since the expression of the stationary state ρ^∞_T is more involved, we prefer to present a concrete example in which both uncorrelated and correlated initial states allow indirect manipulations of the asymptotic states. A simple case is given by the choice

$$A = B = \begin{bmatrix} a & ib & 0 \\ -ib & a & 0 \\ 0 & 0 & a \end{bmatrix}, \tag{34}$$

with the condition $a^2 - b^2 \geqslant 0$ expressing the complete positivity of the evolution. In this case, the maximal rank stationary state is found to be

$$\rho_0 = \frac{1}{4}\Big(\mathbb{I}\otimes\mathbb{I} + \frac{b}{a}(\mathbb{I}\otimes\sigma_3 + \sigma_3\otimes\mathbb{I}) + \Big(\frac{b}{a}\Big)^2\sigma_3\otimes\sigma_3\Big), \tag{35}$$

and the asymptotic state of T for the uncorrelated initial state has components

$$\begin{aligned} \rho_1^\infty &= 0 \\ \rho_2^\infty &= 0 \\ \rho_3^\infty &= \tau\Big(3 + \sum_{k=1}^{3}\rho_k^T\rho_k^A\Big), \end{aligned} \tag{36}$$

where

$$\tau = \frac{ab}{3a^2 + 2b^2}. \tag{37}$$

For the correlated initial states we get

$$\begin{aligned} \rho_1^\infty &= 0 \\ \rho_2^\infty &= 0 \\ \rho_3^\infty &= 4\tau\Big(1 + \sqrt{P(1-P)}\Big). \end{aligned} \tag{38}$$

Therefore, in both cases it is possible to manipulate ρ_T^∞. We stress that it is possible to drive the asymptotic state of T through the initial state of A even if the two system are initially uncorrelated, and there is not a Hamiltonian coupling between them.

IV. Discussion and Conclusions

We have explored the asymptotic performance of the indirect control method when both target and auxiliary systems are two-level systems, and they evolve under a purely dissipative dynamics. We have assumed that the two systems interact separately with an homogeneous environment, leading to a particular form of the Kossakowski matrix C for the composite system. We have found that the conditions expressed in Case 2 are necessary conditions for the indirect manipulation of the stationary state of T through the initial state of A when the initial state is a product state. We have given a numerical example in which this dependence is explicit. We have also proved that, in all the non-trivial cases considered, an initial entanglement between the two systems is effective for control purposes.

Initial states with a different correlation between the two parties produce different stationary states for a reduced subsystem whenever some correlation survives to the decohering action of the environment. This is also true for more general models than the one described in this contribution.

For initially uncorrelated states, the dissipative evolution has to provide the necessary entanglement, that has to be preserved in the large time limit (for the asymptotic entanglement in a quantum dynamical semigroup with purely dissipative evolution, see the results presented in [26]). Therefore, the ability of varying the stationary state of T is a controlled dissipative mechanism. In this sense, in the indirect control approach the environmental action can be considered as a resource. This kind of behavior has already been observed when dealing with accessibility and controllability of a pair of qubits immersed in a bath of decoupled harmonic oscillators, in an exactly solvable model [23]. Therefore, it is not an artifact of the Markovian nature of the evolution.

Eq. (37) gives a limited ability of manipulation of ρ_T^∞. However, the case here discussed is intended to represent an example of explicit dependence, not a complete treatment of the asymptotic reachable set. Moreover, the relevant case $A = B$ is important in concrete experimental situations, for example in the study of the resonance fluorescence [27], [28], or in the analysis of the weak coupling of two atoms to an external quantum field [29].

References

[1] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information*, Cambridge, 2000
[2] P. Shor, SIAM J. Sci. Statist. Comput. 26, 1484 (1997)
[3] C.H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. Wootters, Phys. Rev. Lett. 70, 1895 (1993)
[4] C.H. Bennett, F. Bessette, G. Brassard, L. Salvail, and J. Smolin, Jour. Crypt. 5, 13 (1992)
[5] R. Alicki and K. Lendi, *Quantum Dynamical Semigroups and Applications*, Lecture Notes in Physics, Vol. 286, Springer, Berlin, 1987
[6] H. P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems*, Oxford University Press, Oxford, 2002
[7] D.A. Lidar, K.B. Whaley, *Decoherence-Free Subspaces and Subsystems*, Lecture Notes in Physics, Vol. 622, Springer, Berlin, 2003
[8] S. G. Schirmer, H. Fu and A. I. Solomon, Phys. Rev. A 63, 063410 (2001)
[9] F. Albertini and D. DAlessandro, IEEE Transactions on Automatic Control 48, 1399 (2003)
[10] L. Viola, S. Lloyd and E. Knill, Phys. Rev. Lett. 83, 4888 (1999)
[11] S. Lloyd and L. Viola, Phys. Rev. A 65, 010101(R) (2001)
[12] G. M. Huang, T. J. Tarn and J. W. Clark, J. Math. Phys. 24, 2608 (1983)
[13] V. Ramakrishna, M. V. Salapaka, M. Dahleh, H. Rabitz and A. Peirce, Phys. Rev. A 51, 960 (1995)
[14] C. Altafini, J. Math. Phys. 44, 2357 (2003)
[15] V.P. Belavkin, J. Multivariate. Anal. 42, 171 (1992)
[16] H.M. Wiseman and G. J. Milburn, Phys. Rev. Lett. 70, 548 (1993)
[17] S. Mancini, D. Vitali, and P. Tombesi , Phys. Rev. Lett. 80, 688 (1998)
[18] A. Mandilara and J. W. Clark, Phys. Rev. A 71, 013406 (2005)
[19] R. Romano and D. D'Alessandro, Phys. Rev. A 73, 022323 (2006)
[20] H.C. Fu, H. Dong, X.F. Liu and C.P. Sun, Phys. Rev. A 75, 052317 (2007)
[21] D. Braun, Phys. Rev. Lett. 89, 277901 (2002)
[22] F. Benatti, R. Floreanini and M. Piani, Phys. Rev. Lett. 91, 070402 (2003)
[23] R. Romano and D. D'Alessandro, Phys. Rev. Lett. 97, 080402 (2006)
[24] K. Lendi, J. Phys. A 20, 15 (1987)
[25] A. Frigerio, Comm. Math. Phys. 63, 269 (1977)
[26] F. Benatti and R. Floreanini, Int. J. Quant. Inf. 4, 395 (2006)
[27] G.S. Argawal, A.C. Brown, and G. Vetri, Phys. Rev. A 15, 1613 (1977)
[28] R.R. Puri, *Mathematical Methods of Quantum Optics*, Springer, Berlin, 2001
[29] F. Benatti and R. Floreanini, J. Opt. B 7, 429 (2005)