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LINEAR HAMILTONIAN CONTROL SYSTEMS. AN APPROACH UNDER LINEAR ALGEBRA POINT OF VIEW

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Abstract

Hamiltonian systems model a number of important problems in theoretical physics, mechanics, fluid dynamics and others disciplines.

Many important physical and engineering processes can be described by a suitable linear Hamiltonian formalism.

The properties of Hamiltonian systems like conservation of energy or volume in the phase space leads specific dynamical features.

This paper approaches the study, analysis and characterization of linear Hamiltonian systems through the linear algebra.

Key words

Hamiltonian systems, Controllability, Stability.

1 Introduction

Roughly speaking a Hamiltonian system is a mathematical formalism to describe the evolution equations of a physical system, and they are characterized by the existence of a symplectic structure on a smooth even-dimensional manifold ([D'Alessandro, 2001], [Massawe, 2016],[Seyranian, Mailybaev, 2003]). These kind of systems constitute a broad subject of study that can be treated from many different points of view.

We are interested in linear Hamiltonian systems that is say Hamilton system that are linear differential equations. For this kind of systems the linear algebra is the cornerstone of many of the results that can be obtained.

Controlling Hamiltonian systems has recently attracted the attention of researchers due to their applications in areas such as quantum control satellite control, mixing control or power system control, ([D'Alessandro, Dahleh, Mezic, 1999], [Bloch, Krishnaprasad, Marsden, Sánchez de Alvarez, 1992], [Brocken, Khaneja, 1999], [D'Alessandro, 2001], [Bloch, Krishnaprasad, Marsden, Sánchez de Alvarez, 1992], [Vaidya, Mezic, 2003]).

2 Preliminaries

2.1 Hamiltonian matrices

Let J be a real skew-symmetric matrix $2n \times 2n$ defined on the form

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$$

where O_n and I_n are zero and identity matrices.

Notice that $J^t = J^{-1} = -J$

Definition 2.1. *A real $2n \times 2n$ matrix A, is said a Hamiltonian matrix if the matrix JA is symmetric*

Attending at definition and J matrix properties, an well known equivalent definition of Hamiltonian character is given in the following proposition

Proposition 2.1. *The matrix A is Hamiltonian if and only if, it verifies the equation $A^tJ + JA = 0$*

Proof. JA is symmetric if and only if $JA = (JA)^t$

$$JA = (JA)^t \Longleftrightarrow JA - (JA)^t = 0$$

$$JA - A^tJ^t = 0 \Longleftrightarrow A^tJ + JA = 0$$

That is to say, A is a Hamiltonian matrix $\Longleftrightarrow A^tJ + JA = 0$

Definition 2.2. *The set of $2n \times 2n$ Hamiltonian matrices is expressed by*

$$\mathcal{H}^n = \{A \in \mathcal{M}(\mathbb{R})_{2n\times 2n} | \ A^tJ + JA = 0\}$$

Properties 2.1. *Let A be a Hamiltonian matrix.*

i) Suppose A written as a block matrix in the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where A_{11}, A_{12}, A_{21}, and A_{22} are n-order square matrices. Then, the matrices A_{12} and A_{21} are symmetric, and $A_{11} + A_{22}^t = 0$.

ii) $A = JS$ where S is a symmetric matrix. A^t is Hamiltonian.

iii) The sum (and any linear combination) of two Hamiltonian matrices is also Hamiltonian, as well as is their commutator.

iv) The space of all Hamiltonian matrices is a Lie algebra, denoted $sp(2n)$. The dimension of $sp(2n)$ is $2n^2 + n$. The corresponding Lie group is the symplectic group $Sp(2n)$. This group consists of the symplectic matrices, that is to say the set of matrices A which satisfy $A^t JA = J$.

3 Dynamical systems

Given a dynamical system with multiples inputs $u_1(t) \dots u_m(t)$, multiples outputs $y_1(t) \dots y_p(t)$ and $x_1(t) \dots x_n(t)$ state variables, it is possible model its behaviour with n first order differential equations

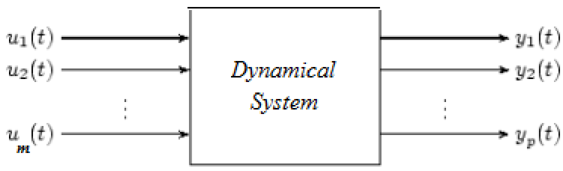

Figure 1. Dynamical system with multiples inputs and outputs

$$\begin{cases} \dot{x}_1 = f_1(x_1(t) \dots x_n(t), u_1(t) \dots u_m(t), t) \\ \vdots \\ \dot{x}_n = f_n(x_1(t) \dots x_n(t), u_1(t) \dots u_m(t), t) \end{cases}$$

and with the p output equations

$$\begin{Bmatrix} y_1 = g_1(x_1(t) \dots x_n(t), u_1(t) \dots u_m(t), t) \\ \vdots \\ y_p = g_p(x_1(t) \dots x_n(t), u_1(t) \dots u_m(t), t) \end{Bmatrix}$$

Taking

$$u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{pmatrix}; x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}; y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{pmatrix}$$

and rewriting in vectorial form

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), t) \\ y(t) &= g(x(t), u(t), t) \end{aligned} \quad (1)$$

where $\dot{x}(t) = \frac{d}{dt}x(t)$

it is obtained a general form to model dynamical system.

Characterization

At practice, the most of dynamical systems works like linear dynamical systems. And if not, in any case, it becomes necessary linearize them by zones to study. This is the reason for the interest in using a linear algebra techniques to study them.

So, the differential state equations most used to describe the behaviour of a system are

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \quad (2)$$

where $x(t) \in \mathbb{R}^n$ is called the state vector, $u(t) \in \mathbb{R}^m$ the input vector, $t \geq 0$, A, B, C and D are real matrices of appropriate sizes.

The most typical representation of a system modelled by (2), valid for all linear system, is using a blocks diagram as shown the figure (2)

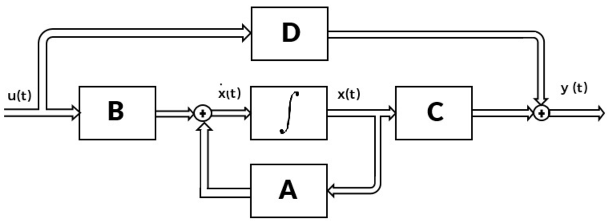

Figure 2. Blocks diagram of an open loop dynamical system.

It is important to note that (2) refers to linear time-invariant systems. By abuse of language, they only will be referred as linear systems while they make no mistake about it.

3.1 Linear Hamiltonian systems

Definition 3.1. *The system (2) is called linear Hamiltonian if and only if is A is a Hamiltonian matrix.*

In case of Linear Hamiltonian systems, the own Hamiltonian determines concretes sizes for matrices and vectors involved in the model. So,

Corollary 3.1. *Let us consider a linear Hamiltonian system given by (2) then,*

i) $A \in \mathcal{H}^n$
ii) $B \in \mathcal{M}(\mathbb{R})_{2n\times m}$
iii) $C \in \mathcal{M}(\mathbb{R})_{p\times 2n}$
iv) $D \in \mathcal{M}(\mathbb{R})_{p\times m}$

It is important to note and remember that m were the numbers of inputs, p the numbers of outputs and n the minimun number of states to describe the system.

3.2 Control of Hamiltonian Linear systems

Roughly speaking, controllability denotes the ability to move a system around in its entire configuration space using only certain admissible manipulations.

More specifically, the system (2) is controllable if and only if the controllability matrix has full row rank ([Chen, 1970]):

$$\text{rank}\,\left(B\ AB\ \ldots\ A^{n-1}B\right) = n$$

When the system is controllable, there exists a feedback K in such a way that the resultant closed loop system

$$\begin{array}{l}\dot{x}(t) = (A - BK)x(t)\\ y(t) = (C - DK)x(t)\end{array} \tag{3}$$

have a desired stable solution.

Given an open loop linear Hamiltonian, the interest of the study concerns in obtain the feedback making the closes loop system stable reaching the intended states and preserving the Hamiltonian structure of the system.

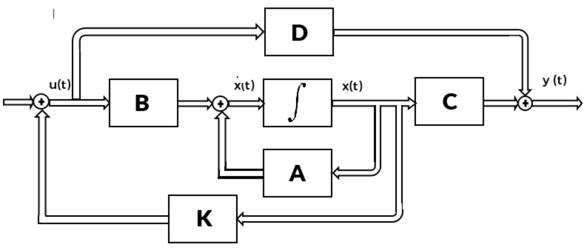

Figure 3. Closed loop Linear system

So, the system in closed loop (3) will be Linear Hamiltonian if and only if the system matrix is Hamiltonian, *i.e.* $A - BK \in \mathcal{H}^n$

Proposition 3.1. *Given a Linear Hamiltonian system in open loop, the closed loop system will continue being Hamiltonian if and only if BK is a Hamiltonian matrix.*

Theorem 3.1. *Let consider a Linear Hamiltonian System. The matrix $K = \left(K_1\ K_2\right)$ is a feedback matrix preserving the hamiltonian character if and only if K is a solution of the Sylvester generalized equation*

$$K_1^t B_1^t + B_2 K_2 = 0$$

in such away tat B_1K_2 and B_2K_1 are symmetrical matrices

Proof. It suffices to compute

$$M = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \left(K_1\ K_2\right)$$

and force it to be symmetrical: $M = M^t$.

Using Kronecker product ($\otimes$) and vectorializing operator (vec) the following corollary is obtained

Corollary 3.2. *Let consider a linear Hamiltonian system. The matrix $K = \left(K_1\ K_2\right)$ is a feedback matrix preserving the hamiltonian character if and only if K is a solution of the following linear system*

$$\left(B_1^t \otimes I_n\ I_n \otimes B_2\right) \begin{pmatrix} \text{vec}\, K_1^t \\ \text{vec}\, K_2 \end{pmatrix}$$

in such away tat B_1K_2 and B_2K_1 are symmetrical matrices

Proof. It suffices to remember (see [Lancaster, Tismenetsky, 1985], for more information), that

If $A = (a_j^i) \in M_{n\times m}(\mathbb{C})$ and $B \in M_{p\times q}(\mathbb{C})$ then,

$$A\otimes B = \begin{pmatrix} a_1^1B & a_2^1B & \ldots & a_m^1B \\ a_1^2B & a_2^2B & \ldots & a_m^2B \\ \vdots & \vdots & & \vdots \\ a_1^nB & a_2^nB & \ldots & a_m^nB \end{pmatrix} \in M_{np\times mq}(\mathbb{C}).$$

If $X = (x_j^i) \in M_{n\times m}(\mathbb{C})$, denoting $x_i = (x_1^i, \ldots, x_m^i)$ the i-th row of the matrix X.

$$\begin{array}{rl} vec: M_{n\times m}(\mathbb{C}) & \longrightarrow M_{nm\times 1}(\mathbb{C}) \\ X & \longrightarrow \left(x_1\ x_2\ \ldots\ x_n\right)^t. \end{array}$$

If $A \in M_{n\times m}(\mathbb{C})$, $X \in M_{m\times p}(\mathbb{C})$ and $B \in M_{p\times q}(\mathbb{C})$ then,

$$\text{vec}\,(AXB) = (B^t \otimes A)\text{vec}\,(X).$$

Particular cases

- Suppose now that $B_1 = 0$ then, $B_2 K_2 = 0$ and $B_2 K_1$ being a symmetric matrix
- Analogously, if $B_2 = 0$ then, $B_1 K_1 = 0$ and $B_1 K_2$ being a symmetric matrix
- If $\left(B_1 = B_2 \text{ then, } B_1^t \otimes I_n \; I_n \otimes B_1\right)\begin{pmatrix}\text{vec}\, K_1^t \\ K_2^t\end{pmatrix} = 0$ with $B_1 K_1$ and $B_1 K_1$ symmetric matrices.

Example 3.1. *Let B be a matrix with $B_1 = 0$ and $B_2 = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ Then, $K_2 = 0$ verifies $B_2 K_2 = 0$) and taking $K_1 = \begin{pmatrix}x & y\\ z & t\end{pmatrix}$ with $y + 2t = 3x + 4z$ $B_2 K_1$ is a symmetric matrix, and BK is a Hamiltonian matrix:*

$$\begin{pmatrix}0 & 0\\ 0 & 0\\ 1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}a & 3a+4c-2d & 0 & 0\\ c & d & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ a+2c & 3a+4c & 0 & 0\\ 3a+4c & 9a+12c-2d & 0 & 0\end{pmatrix}$$

and

$$\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ a+2c & 3a+4c & 0 & 0\\ 3a+4c & 9a+12c-2d & 0 & 0\end{pmatrix} = \begin{pmatrix}a+2c & 3a+4c & 0 & 0\\ 3a+4c & 9a+12c-2d & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

4 Conclusions

In this paper an analysis and characterization of linear Hamiltonian systems through the linear algebra has been done.

The study and existence of specific feedbacks that preserve the Hamiltonian properties is only function of the structure of control matrix, without influence of the own system matrix. The use of linear algebra techniques let us obtain the all possible feedbacks solving a linear system.

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