ON QUALITATIVE ANALYSIS OF MECHANICAL SYSTEMS

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Abstract
The report presents several algorithms of qualitative analysis of dynamic systems. Efficiency of these techniques is demonstrated by examples of analysis of two problems of rigid body dynamics in gravitational and magnetic fields.

Key words
resonances, invariant manifolds, stability.

1 Introduction.

In this work the new results of investigation of the Kovalevskaya problem [Kowalewski, 1948] and the problem of a rigid body motion under influence of gravitational and magnetic forces [Bogoyavlensky, 1984] have been obtained. For this purpose, we applied the methods of analysis of differential equations of motion. These methods are based on the using "resonances" between first integrals of the problem, the solving of stationary equations of family of integrals with respect to part of phase variables and part of parameters of these families, and the constructing of the envelopes for families of first integrals. Similar problems on motion of a rigid body under influence of several force fields arise, for example, in cosmodynamics [Frik, 1970; Sarychev, Paglione , Guerman, 2008].

2 The Kowalewski Problem.

Let us write down equations of a rigid body motion in the Kowalewski case [Kowalewski, 1948].

$$\left.\begin{array}{l} 2\dot{p} = qr,\;\; 2\dot{q} = -rp + x_0\gamma_3,\;\; \dot{r} = -x_0\gamma_2, \\ \dot{\gamma_1} = r\gamma_2 - q\gamma_3,\;\; \dot{\gamma_2} = p\gamma_3 - r\gamma_1, \\ \dot{\gamma_3} = q\gamma_1 - p\gamma_2. \end{array}\right\} \quad (1)$$

This system admits the following two first integrals:

$$\left.\begin{array}{l} 2H = 2p^2 + 2q^2 + r^2 + 2x_0\gamma_1 = 2h, \\ V = (p^2 - q^2 - x_0\gamma_1)^2 + (2pq - x_0\gamma_2)^2 = k^2. \end{array}\right\} \quad (2)$$

Here p, q, r – projections of the angular velocity of the body on its main axes, $\gamma_1,\ \gamma_2,\ \gamma_3$ – direction cosines of the vertical in these axes.

We shall find relations between the problem variables under which the following equality

$$H^2|_0 = V|_0.$$

holds. For example, the latter is true when $p = r = \gamma_2 = 0$. By direct substitution into the differential equations we can verify that equations $p = r = \gamma_2 = 0$ define the invariant manifold (IM) of the system (1). In similar cases, we will speak that the resonance between first integrals H^2 and V takes place on the given IM. The equations of the vector field on the chosen IM look like:

$$2\dot{q} = x_0\gamma_3,\;\; \dot{\gamma_1} = -q\gamma_3,\;\; \dot{\gamma_3} = q\gamma_1.$$

They define pendulum-like oscillations of the body around its immobile horizontal principal inertia axis y.

The found IM satisfies the conditions of stationarity of nonlinear combination of integrals $K = H^2 - V$:

$$\begin{array}{l} \frac{\partial K}{\partial p} = 2(2p^2 + 2q^2 + r^2 + 2x_0\gamma_1)p - 4(p^2 - q^2 - \\ \quad x_0\gamma_1)p - 4(2pq - x_0\gamma_2)q = 0, \\ \frac{\partial K}{\partial q} = 2(2p^2 + 2q^2 + r^2 + 2x_0\gamma_1)q - \\ \quad 4(p^2 - q^2 - x_0\gamma_1)q - 4(2pq - x_0\gamma_2)p = 0, \\ \frac{\partial K}{\partial r} = 2(2p^2 + 2q^2 + r^2 + 2x_0\gamma_1)r = 0, \\ \frac{\partial K}{\partial \gamma_2} = 2(2pq - x_0\gamma_2)x_0 = 0, \\ \frac{\partial K}{\partial \gamma_1} = (2p^2 + 2q^2 + r^2 + 2x_0\gamma_1)x_0 - \\ \quad 2(p^2 - q^2 - x_0\gamma_1)x_0 = 0. \end{array}$$

Hence, this IM $p = r = \gamma_2 = 0$ is stationary. Such IMs we call invariant manifolds of stationary motions (IMSM).

2.1 Stability of IM

Let us investigate the stability of the found above IMSM using for this purpose the integral K. This integral has a stationary value on IM $p = r = \gamma_2 = 0$.

We write down integral K in the neighborhood of IM $p = r = \gamma_2 = 0$:

$$\begin{array}{l}\Delta K = 4x_0\gamma_1\xi_1^2 + (q^2 + x_0\gamma_1)\xi_2^2 - x_0^2\xi_3^2 \\ +4x_0 q\xi_1\xi_3 + \xi_1^2\xi_2^2 + \frac{1}{4}\xi_2^4,\end{array} \tag{3}$$

here $\xi_1,\ \xi_1,\ \xi_3$ are deviations from IM in the perturbed motion.

The conditions of sign-definiteness of quadratic form in ΔK (3) have the form:

$$-x_0^2 < 0,\ \ (q^2 + x_0\gamma_1) < 0,\ \ -x_0^2(q^2 + x_0\gamma_1) > 0.$$

When $x_0 \neq 0$ (it is supposed), the first condition of sign-definiteness of the quadratic form always holds. Because the energy integral has form $2H|_0 = q^2 + x_0\gamma_1 = 2h$ on IM $p = r = \gamma_2 = 0$, we can always satisfy the second condition $h < 0$ of sign-definiteness ΔK (3) on IM by the corresponding choice of starting conditions.

When the conditions of the sign-definiteness ΔK hold then IM $p = r = \gamma_2 = 0$ is stable. Hence, the deviations $\xi_1,\ \xi_1,\ \xi_3$ from this IM are small and do not break the conditions of the sign-definiteness ΔK obtained above.

Thus, we can consider that the requirement

$$h < 0$$

is sufficient for stability of the IM under investigation. Condition $h < 0$ has simple mechanical interpretation. During the pendulum oscillation, the center of gravity should be below, than the point of suspension of the body.

2.2 On Invariant Manifolds of Second Level

Following the algorithms proposed, we consider the problem of finding and analysis of IMs, on which integral V (2) assumes a stationary value. The necessary conditions of this first integral to have an extremum write:

$$\left.\begin{array}{l}\frac{\partial V}{\partial p} = 4(py_1 + qy_2) = 0,\ \frac{\partial V}{\partial \gamma_1} = -2x_0y_1 = 0, \\ \frac{\partial V}{\partial q} = -4(qy_1 - py_2) = 0,\ \frac{\partial V}{\partial \gamma_2} = -2x_0y_2 = 0,\end{array}\right\} \tag{4}$$

where $y_1 = p^2 - q^2 - x_0\gamma_1,\ \ y_2 = 2p\,q - x_0\gamma_2$.

From (4) it follows that one of IMSM, which corresponds to integral V (2), is defined by the equations:

$$\left.\begin{array}{l}y_1 = p^2 - q^2 - x_0\gamma_1 = 0, \\ y_2 = 2p\,q - x_0\gamma_2 = 0.\end{array}\right\} \tag{5}$$

This is well-known the Delone manifold . The equations of the vector field on this IMSM are derived from the initial system (1). They have the form:

$$\left.\begin{array}{l}2\dot{p} = qr,\ \ 2\dot{q} = -rp + x_0\gamma_3,\ \ \dot{r} = -2pq, \\ \dot{\gamma_3} = -q(p^2 + q^2)x_0^{-1}.\end{array}\right\} \tag{6}$$

The differential equations (6 have the following first integrals:

$$\left.\begin{array}{l}2\tilde{H} = 4p^2 + r^2 = 2h, \\ \tilde{V_1} = r\gamma_3 + 2p(p^2 + q^2)x_0^{-1} = m, \\ \tilde{V_2} = \gamma_3^2 + (p^2 + q^2)^2x_0^{-2} = 1.\end{array}\right\} \tag{7}$$

Let us formulate the problem of finding IMs on Delone's manifold (5) which provide a stationary value to the elements of algebra of the first integrals on this manifold. Such IMs we call second-level IMs.

To this end, we form the linear combination from integrals (7):

$$2\tilde{K} = 2\tilde{H} - 2\mu\tilde{V}_1 - \mu_1\tilde{V}_2. \tag{8}$$

where μ and μ_1 are some parameters.

Next, we write down the necessary conditions for integral $\tilde{K}$ (8) to have an extremum and compute the Jacobian of the resulted system of equations:

$$J = \frac{4}{x_0^2}(4\mu p x_0 + 2p^2(\mu_1 - 6\mu^2) + q^2(6\mu_1 + \mu^2) -$$
$$6\mu\mu_1 p(3q^2 + 2p^2) - 3\mu_1^2(3p^2q^2 + p^4 + q^4)(\mu_1 + \mu^2)$$

When $\mu_1 + \mu^2 = 0$, the Jacobian turns to zero and the system of stationary equations for the integral $\tilde{K}$ is degenerate. The following family of integrals with one parameter μ will now correspond to this system:

$$2\tilde{K}_1 = 2\tilde{H} - 2\mu\tilde{V}_1 + \mu^2\tilde{V}_2. \tag{9}$$

The conditions of stationarity for the function $\tilde{K}_1$ with respect to the phase variables when $\mu = \mu(p, q, r, \gamma_3)$ can be written as:

$$\begin{array}{l}\frac{\partial K}{\partial p} = 2(1 - \frac{\mu}{x_0}p)(2p - \frac{\mu}{x_0}(p^2 + q^2)) - \\ \quad \frac{\partial \mu}{\partial p}(V_1 - \mu V_2) = 0, \\ \frac{\partial K}{\partial q} = -\frac{2\mu q}{x_0}(2p - \frac{\mu}{x_0}(p^2 + q^2)) - \\ \quad \frac{\partial \mu}{\partial q}(V_1 - \mu V_2) = 0, \\ \frac{\partial K}{\partial r} = r - \mu\gamma_3 - \frac{\partial \mu}{\partial r}(V_1 - \mu V_2) = 0, \\ \frac{\partial K}{\partial \gamma_3} - \mu(r - \mu\gamma_3) - \frac{\partial \mu}{\partial \gamma_3}(V_1 - \mu V_2) = 0.\end{array}$$

Taking the parameter μ and the part of phase variables (p, r) as unknowns, we have the following solutions of the latter equations:

$$\left.\begin{array}{l}\mu = \tilde{V}_1/\tilde{V}_2,\ \ 2p - \mu x_0^{-1}(p^2 + q^2) = 0, \\ r - \mu\gamma_3 = 0.\end{array}\right\} \tag{10}$$

The last two equations of the system (10) define the family of IMs of differential equations (6). Having substituted the found value $\mu = \tilde{V}_1/\tilde{V}_2$ into these equations, we obtain the following expressions:

$$\gamma_3((p^2+q^2)r - 2px_0\gamma_3) = 0,$$
$$(p^2+q^2)x_0^{-2}((p^2+q^2)r - 2px_0\gamma_3) = 0.$$

They define two IMs of the initial system of differential equations (6):

$$(p^2+q^2)r - 2px_0\gamma_3 = 0, \tag{11}$$

and $\quad p = q = \gamma_3 = 0.$

Note that under the value μ (10) expression $\tilde{K}_1$ (9) becomes the first integral:

$$\tilde{\Omega} = 2\tilde{H} - \tilde{V}_1^2/\tilde{V}_2. \tag{12}$$

It is the envelope of the family of integrals $\tilde{K}_1$. After elimination μ from the last two equations of (10) we obtain again one equation

$$\tilde{V}_4 = (p^2+q^2)r - 2px_0\gamma_3 = 0, \tag{13}$$

which coincides with the solution (11). Moreover, the left side of this equation is the first integral of differential equations (6). In the given case, the constant of this integral is zero. It is obvious that IM (13) is stable as first integral.

If we shall write down necessary conditions for the enveloping integral (12) to have an extremum then, as it can be verified, this integral assumes a stationary value on IM (13). Hence, this IM is stationary.

We continue the analysis of equations (6) and investigate stationary sets of another combination of the integrals in which integral (13) is used:

$$\begin{aligned} 2\tilde{K}_2 = 2\tilde{H} - 2\mu\tilde{V}_4 + \mu^2\tilde{V}_2 = \\ 4p^2 + r^2 - 2\mu((p^2+q^2)rx_0^{-1} - 2p\gamma_3) \\ +\mu^2(\gamma_3^2 + (p^2+q^2)^2x_0^{-2}). \end{aligned} \tag{14}$$

The system of the stationarity conditions of $\tilde{K}_2$

$$\frac{\partial\tilde{K}_2}{\partial p} = 2(2p - \frac{\mu rp}{x_0} + \frac{\mu^2 p(p^2+q^2)}{x_0^2} + \mu\gamma_3) = 0,$$
$$\frac{\partial\tilde{K}_2}{\partial q} = -2q(\frac{\mu r}{x_0} - \frac{\mu^2(p^2+q^2)}{x_0^2}) = 0,$$
$$\frac{\partial\tilde{K}_2}{\partial r} = r - \frac{\mu(p^2+q^2)}{x_0} = 0,$$
$$\frac{\partial\tilde{K}_2}{\partial \gamma_3} = 2\mu p + \mu^2\gamma_3 = 0$$

has the following solution:

$$2p + \mu\gamma_3 = 0, \;\; rx_0 - \mu(p^2+q^2) = 0. \tag{15}$$

These equations, as it can be verified by IM definition, correspond to the stationary IM of system (6).

After elimination μ from equations (15) we have the following equation:

$$x_0\tilde{V}_1 = x_0r\gamma_3 + 2p(p^2+q^2) = 0,$$

which define the IM of system (6). The left side of the equation coincides (up to a constant factor) with the first integral $\tilde{V}_1$ of the initial differential equations. This IM is stationary. It provides a stationary value to the enveloping integral of the family's integrals $\tilde{K}_2$:

$$\tilde{Q} = 2\tilde{H} - \tilde{V}_4^2/\tilde{V}_2$$

The proposed procedure for constructing IMs on the base of a family of IM can be considered as an algorithm for constructing the IM which contains the given IM family.

The elements of IM family (15) can be investigated for stability. To this end, we introduce the deviations

$$y_1 = 2p + \mu\gamma_3, \;\; y_2 = x_0r - \mu(p^2+q^2).$$

from the IM in perturbed motion.

After elimination γ_3, r from integral $\tilde{K}_2$ (14) with the help of the latter expressions, we have:

$$\Delta\tilde{K}_2 = y_1^2 + \frac{y_2^2}{x_0^2}.$$

Because the variation of the integral is positive definite, IM (15) is stable.

The families of second level IMSM found in such a way can be "lifted up" into the initial phase space as invariant. To this end, it is necessary to add the IMSM equations to the equations of Delone's IM (5).

3 A Rigid Body in Gravitational and Magnetic Force Fields

Let us consider the problem of motion of a rigid body with constant magnetic momentum in uniform gravitational and magnetic fields [Bogoyavlensky, 1984]. Here the motion equations have the form:

$$\left.\begin{aligned} &2\dot{p} = qr + b\delta_3, \;\; 2\dot{q} = -rp + x_0\gamma_3, \\ &\dot{r} = -x_0\gamma_2 - b\delta_1, \; \dot{\gamma_1} = r\gamma_2 - q\gamma_3, \\ &\dot{\gamma_2} = p\gamma_3 - r\gamma_1, \; \dot{\gamma_3} = q\gamma_1 - p\gamma_2, \\ &\dot{\delta_1} = r\delta_2 - q\delta_3, \; \dot{\delta_2} = p\delta_3 - r\delta_1, \\ &\dot{\delta_3} = q\delta_1 - p\delta_2. \end{aligned}\right\} \tag{16}$$

This system of differential equations admits the following two first integrals:

$$\left.\begin{aligned} &2H = 2(p^2+q^2 - x_0\gamma_1 - b\delta_2) + r^2 = 2h, \\ &V = (p^2 - q^2 + x_0\gamma_1 - b\delta_2)^2 + \\ &(2pq + x_0\gamma_2 + b\delta_1)^2 = k^2. \end{aligned}\right\} \tag{17}$$

We shall find stationary IMs which correspond to the integral V (17). The conditions of stationarity for the integral V with respect to the phase variables write:

$$
\begin{aligned}
&\frac{\partial V}{\partial p} = 4(p^2 - q^2 + x_0\gamma_1 - b\delta_2)p + \\
&\quad 4(2pq + x_0\gamma_2 + b\delta_1)q = 0, \\
&\frac{\partial V}{\partial q} = -4(p^2 - q^2 + x_0\gamma_1 - b\delta_2)q + \\
&\quad 4(2pq + x_0\gamma_2 + b\delta_1)p = 0, \\
&\frac{\partial V}{\partial \gamma_1} = 2(p^2 - q^2 + x_0\gamma_1 - b\delta_2)x_0 = 0, \\
&\frac{\partial V}{\partial \gamma_2} = 2(2pq + x_0\gamma_2 + b\delta_1)x_0 = 0, \\
&\frac{\partial V}{\partial \delta_1} = 2(2pq + x_0\gamma_2 + b\delta_1)b = 0, \\
&\frac{\partial V}{\partial \delta_2} = -2(p^2 - q^2 + x_0\gamma_1 - b\delta_2)b = 0.
\end{aligned}
$$

One of the solution of the latter equations is

$$p^2-q^2+x_0\gamma_1-b\delta_2 = 0,\ 2pq+x_0\gamma_2+b\delta_1 = 0. \quad (18)$$

These equations, as it can be verified, define the IMSM of differential equations system (16) (this IMSM is similar to the Delone manifold for the Kowalewski problem). The IM is stable. Indeed, introduce the deviations

$$z_1 = p^2 - q^2 + x_0\gamma_1 - b\delta_2,\ \ z_2 = 2pq + x_0\gamma_2 + b\delta_1$$

from IM (18). As a result, we have the sign-definite variation

$$\Delta V = z_1^2 + z_2^2.$$

of integral V (17) in the neighborhood of the IM. From the latter, we conclude that IM (18) is stable.

Consider the vector field on the IM (18):

$$
\left.
\begin{aligned}
&2\dot{p} = qr + b\delta_3,\ \ 2\dot{q} = -rp + x_0\gamma_3, \\
&\dot{r} = -2(pq + b\delta_1),\ \dot{\delta_1} = r\delta_2 - q\delta_3, \\
&\dot{\delta_2} = p\delta_3 - r\delta_1,\ \dot{\delta_3} = q\delta_1 - p\delta_2, \\
&\dot{\gamma_3} = (q(p^2 + q^2) + b(p\delta_1 + q\delta_2))/x_0.
\end{aligned}
\right\} \quad (19)
$$

The first integrals of differential equations (19) are:

$$
\begin{aligned}
&2H = 4p^2 + r^2 - 4b\delta_1,\ F_{02} = \gamma_3^2 + \\
&((2pq + b\delta_1)^2 + (-p^2 + q^2 + b\delta_2)^2)/x_0^2 = 1, \\
&F_{03} = \delta_1^2 + \delta_2^2 + \delta_3^2 = 1, \\
&F_{04} = ((p^2 - q^2)\delta_1 + 2pq\delta_2 + x_0\delta_3\gamma_3)/x_0. \\
&2F_{05} = (p^2 + q^2)r - 2p\gamma_3 x_0 + 2bq\delta_3,
\end{aligned}
$$

Let us construct a bundle of these first integrals:

$$
\left.
\begin{aligned}
&K = 4p^2 + r^2 - 4b\delta_2 - 2\lambda_1(-2p\gamma_3 + \\
&((p^2 + q^2)r + 2bq\delta_3)/x_0) - \lambda_2(\gamma_3^2 + \\
&(2pq + b\delta_1)^2/x_0^2 + (-p^2 + q^2 + b\delta_2)^2/x_0^2) - \\
&\left(\left(p^2 - q^2\right)\delta_1 + 2pq\delta_2 + x_0\gamma_3\delta_3\right)\lambda_3/x_0 + \\
&\lambda_4\left(\delta_1^2 + \delta_2^2 + \delta_3^2\right).
\end{aligned}
\right\} \quad (20)
$$

The stationary conditions for the bundle of first integrals K (20) with respect to the phase variables

$$
\begin{aligned}
&\frac{\partial K}{\partial p} = -2x_0^2\left(2p + \gamma_3\lambda_1\right) + 2(bq\delta_1 + p(p^2 + q^2 - \\
&\quad b\delta_2))\lambda_2 + x_0\left(2pr\lambda_1 + \left(p\delta_1 + q\delta_2\right)\lambda_3\right) = 0, \\
&\frac{\partial K}{\partial q} = 2\left(bp\delta_1 + q\left(p^2 + q^2 + b\delta_2\right)\right)\lambda_2 + \\
&\quad x_0\left(2\left(qr + b\delta_3\right)\lambda_1 + \left(-q\delta_1 + p\delta_2\right)\lambda_3\right) = 0, \\
&\frac{\partial K}{\partial r} = rx_0 - \left(p^2 + q^2\right)\lambda_1 = 0, \\
&\frac{\partial K}{\partial \gamma_3} = 4p\lambda_1 - 2\gamma_3\lambda_2 - \delta_3\lambda_3 = 0, \\
&\frac{\partial K}{\partial \delta_1} = 2b\left(2pq + b\delta_1\right)\lambda_2 + \\
&\quad x_0((p^2 - q^2)\lambda_3 - 2x_0\delta_1\lambda_4) = 0, \\
&\frac{\partial K}{\partial \delta_2} = b\left(-p^2 + q^2 + b\delta_2\right)\lambda_2 + pqx_0\lambda_3 + \\
&\quad x_0^2\left(2b - \delta_2\lambda_4\right) = 0, \\
&\frac{\partial K}{\partial \delta_3} = 4bq\lambda_1 + x_0\left(\gamma_3\lambda_3 - 2\delta_3\lambda_4\right) = 0,
\end{aligned}
$$

when the phase variables $r, \gamma_3,\ \delta_1,\ \delta_2, \delta_3$ and parameter λ_4 are considered unknown, have the following family of solutions:

$$
\left.
\begin{aligned}
&r=\frac{\left(p^2+q^2\right)\lambda_1}{x_0},\ \lambda_4= -\ \frac{-4b^2\lambda_1^2\lambda_2+x_0^2\lambda_3^2}{4x_0^2\left(\lambda_1^2+\lambda_2\right)}, \\
&\gamma_3=\frac{2\left(-2bqx_0\left(\lambda_1^2+\lambda_2\right)\lambda_3+p\left(4b^2\lambda_1^2\lambda_2-x_0^2\lambda_3^2\right)\right)}{\lambda_1\left(4b^2\lambda_2^2+x_0^2\lambda_3^2\right)}, \\
&\delta_3=\frac{4x_0\left(\lambda_1^2+\lambda_2\right)(2bq\lambda_2+px_0\lambda_3)}{\lambda_1\left(4b^2\lambda_2^2+x_0^2\lambda_3^2\right)}, \\
&\delta_2= -\ \frac{4\left(\lambda_1^2+\lambda_2\right)\left(2bx_0^2-bp^2\lambda_2+bq^2\lambda_2+pqx_0\lambda_3\right)}{4b^2\lambda_2^2+x_0^2\lambda_3^2}, \\
&\delta_1= -\ \frac{2\left(\lambda_1^2+\lambda_2\right)\left(4bpq\lambda_2+\left(p^2-q^2\right)x_0\lambda_3\right)}{4b^2\lambda_2^2+x_0^2\lambda_3^2}
\end{aligned}
\right\} \quad (21)
$$

Here the expressions for $r, \gamma_3,\ \delta_1,\ \delta_2, \delta_3$ define the IM family of codimension five of system (19). On the elements of IM family (21) the equations of vector field write:

$$
\left.
\begin{aligned}
&\dot{p} = \frac{q\left(p^2+q^2\right)\lambda_1}{2x_0} + (2bx_0\left(\lambda_1^2 + \lambda_2\right)(2bq\lambda_2 + \\
&px_0\lambda_3))/(\lambda_1\left(4b^2\lambda_2^2 + x_0^2\lambda_3^2\right)), \\
&\dot{q} = -\frac{p\left(p^2+q^2\right)\lambda_1}{2x_0} + (x_0(4b^2p\lambda_1^2\lambda_2 - 2bqx_0(\lambda_1^2 + \\
&\lambda_2)\lambda_3 - px_0^2\lambda_3^2))/(\lambda_1\left(4b^2\lambda_2^2 + x_0^2\lambda_3^2\right)).
\end{aligned}
\right\} \quad (22)
$$

Equations (22) have the first integral:

$$
\begin{aligned}
&F_0 = \left(p^2 + q^2\right)^2\lambda_1^2\left(4b^2\lambda_2^2 + x_0^2\lambda_3^2\right) + 4x_0^2(4b^2(q^2 - \\
&p^2)\lambda_1^2\lambda_2 + 4bpqx_0\lambda_1^2\lambda_3 + \left(2bq\lambda_2 + px_0\lambda_3\right)^2).
\end{aligned}
$$

3.1 On the embedding of invariant manifolds

Let us find a stationary value of λ_4 from (21) with respect to λ_3. To this end let us write down the corresponding necessary conditions of extremum:

$$\frac{\partial\lambda_4}{\partial\lambda_3} = -\frac{-\lambda_3}{2(\lambda_1^2 + \lambda_2)} = 0$$

From this follows the desired value of $\lambda_3 = 0$. Under this value of the parameter characterizing the family of solutions (21) we have:

$$\left.\begin{array}{l} r=\frac{\left(p^2+q^2\right)\lambda_1}{x_0}, \gamma_3=\frac{2p\lambda_1}{\lambda_2}, \delta_3=\frac{2qx_0\left(\lambda_1^2+\lambda_2\right)}{b\lambda_1\lambda_2}, \\ \delta_2=-\frac{(\lambda_1^2+\lambda_2)(2x_0^2+(-p^2+q^2)\lambda_2)}{b\lambda_2^2}, \\ \delta_1=-\frac{2pq(\lambda_1^2+\lambda_2)}{b\lambda_2},\ \ \lambda_4=\frac{b^2\lambda_1^2\lambda_2}{x_0^2(\lambda_1^2+\lambda_2)}. \end{array}\right\} \quad (23)$$

If, for example, parameter λ_1 is removed from these equations, then we obtain the following system of equalities:

$$\begin{array}{l} 2prx_0-\left(p^2+q^2\right)\gamma_3\lambda_2=0, \\ bp\gamma_3\delta_3\lambda_2-qx_0\left(4p^2+\gamma_3^2\lambda_2\right)=0, \\ 4bp^2\delta_2\lambda_2+\left(2x_0^2-\left(p^2-q^2\right)\lambda_2\right)\left(4p^2+\gamma_3^2\lambda_2\right)=0, \\ 4p^2q+2bp\delta_1+q\gamma_3^2\lambda_2=0\ \ (\lambda_1=rx_0(p^2+q^2)^{-1}). \end{array}$$

They define one-parameter family of IMs of the system (19) of codimension four.

Continuing this process, let as exclude from the latest equations parameter λ_2. As a result we get the following three equations:

$$\begin{array}{l} bpr\delta_3-q\left(2p\left(p^2+q^2\right)+rx_0\gamma_3\right)=0, \\ 2p^2\left(q^4-p^4\right)r+p(2\left(p^2+q^2\right)^2-\left(p^2-q^2\right)r^2)x_0\gamma_3 \\ +\left(p^2+q^2\right)rx_0^2\gamma_3^2+2bp^2\left(p^2+q^2\right)r\delta_2=0, \\ q\left(2p\left(p^2+q^2\right)+rx_0\gamma_3\right)+b\left(p^2+q^2\right)\delta_1=0, \\ (\lambda_2=2prx_0/(\gamma_3(p^2+q^2))). \end{array}$$

These equations also will define IM of codimension three of the system (19).

Equations of the vector fields on the elements of the investigated IMs in general cases cannot be integrated in elementary functions.

3.2 IM stability

Let as investigate the stability of elements of IM family (21). Introduce deviations in the perturbed motion from elements of this IM family :

$$\begin{array}{l} x_r=r-\frac{\left(p^2+q^2\right)\lambda_1}{x_0}, \\ x_\gamma=\gamma_3-\frac{\left(2\left(4b^2p\lambda_1^2\lambda_2-2bqx_0\left(\lambda_1^2+\lambda_2\right)\lambda_3-px_0^2\lambda_3^2\right)\right)}{\lambda_1\left(4b^2\lambda_2^2+x_0^2\lambda_3^2\right)}, \\ x_3=\delta_3-\frac{4x_0\left(\lambda_1^2+\lambda_2\right)(2bq\lambda_2+px_0\lambda_3)}{\lambda_1\left(4b^2\lambda_2^2+x_0^2\lambda_3^2\right)}, \\ x_2=\delta_2+\frac{\left(4\left(\lambda_1^2+\lambda_2\right)\left(2bx_0^2+b\left(-p^2+q^2\right)\lambda_2+pqx_0\lambda_3\right)\right)}{\left(4b^2\lambda_2^2+x_0^2\lambda_3^2\right)}, \\ x_1=\delta_1+\frac{2\left(\lambda_1^2+\lambda_2\right)\left(4bpq\lambda_2+\left(p^2-q^2\right)x_0\lambda_3\right)}{4b^2\lambda_2^2+x_0^2\lambda_3^2} \end{array}$$

The expression for the integral K (20) in perturbed motion has the form

$$\begin{array}{l} \Delta K=-\frac{(4b^4\lambda_2^2+x_0^2\lambda_3^2)(x_1^2+x_2^2)}{4x_0^2(\lambda_1^2+\lambda_2)}+ \\ x_r^2+\frac{(4b^4\lambda_1^2\lambda_2-x_0^2\lambda_3^2)x_3^2}{4x_0^2(\lambda_1^2+\lambda_2)}-\lambda_3x_3x_\gamma-\lambda_2x_\gamma^2. \end{array}$$

The conditions of sign-definiteness of the last quadratic form look like

$$\lambda_2<0,\ \ -\frac{(4b^4\lambda_2^2+x_0^2\lambda_3^2)}{4x_0^2(\lambda_1^2+\lambda_2)}>0. \quad (24)$$

This is equivalent to the only inequality:

$$\lambda_1^2+\lambda_2<0. \quad (25)$$

Note that the obtained inequality does not contain constants b, x_0, but sets constraints only to parameters $\lambda_i,\ i=1,2,3$ of IM family, thus separating subfamily of the investigated IM family. According to Lyapunov method [Lyapunov, 1956] the conditions (25) are sufficient conditions of stability of elements of the selected IM subfamily. It is easy to check that for obtaining sufficient conditions of elements of IM family (23) it is enough to set $\lambda_3=0$ in conditions (24). As a result, sufficient conditions of stability (25) of IM family (23) will match to conditions of stability of the elements of IM family (21).

4 Conclusion

The authors have proposed several procedures for selection and qualitative investigation of invariant manifolds of motion equations of dynamic systems. All they to the certain degree reduce to selection of invariant manifolds providing the stationary value for some function. This allows later use this function to further investigate obtained IMs, in particular to analyse their stability.

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