COPRIME FACTORIZATION OF THE TRANSFER MATRIX 
OF A SINGULAR LINEAR SYSTEM

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Abstract
Given a linear dynamic time invariant represented by\newline
\[ x(t) = Ax(t)Bu(t), \ y(t) = Cx(t), \] \newline
we analyze conditions for obtention of a coprime factorization of\newline
transfer matrices of singular linear systems defined over\newline
commutative rings \( R \) with element unit. The problem\newline
presented is related to the existence of solutions of a\newline
matrix equation \( AX - NXA = Z \).

Key words
Singular systems, feedback, output injection, coprime\newline
factorizations.

1 Introduction
Let \( R \) be a commutative ring with unity and \( \mathbb{E}x(t) = Ax(t) + Bu(t), y(t) = Cx(t) \) be a singular\newline
system over \( R \), that we represent by \((E, A, B, C)\).\newline
Then, the transfer matrix of the system \((E, A, B, C)\) is\newline
given by \( H(s) = C(sE - A)^{-1}B \). \newline
This systems appear in literature when for example,\newline
one studies linear systems depending on a parameter or\newline
linear systems with delays.\newline
Let \((E, A, B, C)\) be a singular system with \( E = I_4 \),\newline\[ A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \]
then \((sI_4 - A)^{-1}\) is a rational matrix. Considering \[ F_E^B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_A^E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F_C^E = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \]
it is easy to compute \( \det(sI_4 + F_C^E + F_B^E) - (A + F_C^E + F_B^E)^{-1} \) is polynomial.\newline
We are interested in classify the singular systems \((E, A, B, C)\) for which there exist feedbacks \( F_E^B, F_A^E, F_C^E, \)\newline
and output injections \( F_C^E, F_B^E, F_A^C \), such that \((sI_4 + F_C^E + F_B^E) - (A + F_C^E + F_B^E)^{-1}\) is\newline
polynomial. We will call systems with polynomial transfer matrix by feedback \( (A, \quad F_C^E, \quad F_B^E, \quad F_A^C) \) and \( \frac{d}{dt} \).

Remark 1.1. Converse is not true as we can see in this example: \((E, A, B, C)\) with \( E = I_4, A = \)
\[ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \]
considering all possible feedbacks \( F_E^B, F_A^E, F_C^E, \) \( F_B^E, \) and output injections \( F_C^E, F_B^E, F_A^C \) matrix \( s(E + F_C^E + F_B^E) - (A + F_C^E + F_B^E)^{-1} \) is\newline
\[ \begin{pmatrix} s(1 + a) + a_1 & s(b + e) + (b_1 + c_1) & s(c + e_1) & s(d + d_1) \\ 0 & s(1 + f) + f_1 & 0 & 0 \\ 0 & s(1 + g) + g_1 & s & 1 \\ 0 & s(1 + h) + h_1 & 0 & s \end{pmatrix} \]
is easy to compute \( \det(s(E + F_C^E + F_B^E) - (A + F_C^E + F_B^E)^{-1}) \neq 0 \) for almost all \( s \in R \) and \( f = 0 \). Then \((E, A, B, C)\) is regularisable but not pbfoci.

In order to use a simple reduced system preserving these properties we consider the following equivalence relation deduced of to apply the standard transformations in state, input and output spaces \( x(t) = P x(t), \)
\( u(t) = Ru_1(t), \ y_1(t) = S y(t), \) premultiplication by an invertible matrix \( QE \) \( Ax(t) + Qu(t) \) making feedback \( u(t) = u_1(t) - Vx(t) \) and \( u(t) = u_1(t) - Ux(t) \) as well as output injection \( u(t) = u_1(t) - W y(t) \) and derivative output injection \( u(t) = u_1(t) - Z y_1(t) \).

Considering this equivalence relation and restricting out the regularisable systems and for \( R = \mathbb{C} \), it is possible to reduce the system to
(E_c, A_c, B_c, C_c) where

\[
E_c = \begin{pmatrix} I_1 & \cdots & I_4 \\ \cdots & \cdots & \cdots \\ I_1 & \cdots & I_4 \\ \end{pmatrix}
\]

\[
A_c = \begin{pmatrix} N_2 \\ N_3 \\ N_4 \\ J \\ I_s \\ \end{pmatrix}
\]

\[
B_c = \begin{pmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \end{pmatrix}
\]

\[
C_c = \begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 \\ \end{pmatrix}
\]

and \(N_i\) denotes a nilpotent matrix in its reduced form

\(N_i = \text{diag}(N_{i_1}, \ldots, N_{i_s})\), \(N_{ij} = \begin{pmatrix} 0 & I_{n_{ij}} \\ 0 & 0 \\ \end{pmatrix} \in M_{n_{ij}}(C)\),

\(J\) denotes the Jordan matrix \(J = \text{diag}(J_1(\lambda), \ldots, J_m(\lambda_m))\), \(J_i(\lambda) = \text{diag}(J_{i_1}(\lambda_1), \ldots, J_{i_s}(\lambda_s))\), \(J_s(\lambda_s) = \lambda_s I + N_s\).

Notice that not all subsystems must appear in canonical reduced form.

**Remark 1.2.** Canonical reduced form can be obtained directly using the complete set of invariants (see [6]).

### 2 Coprime factorization

Two polynomial matrices \(N(s) \in M_{p \times m}(R[s])\) and \(D(s) \in M_{m \times n}(R[s])\) are called (Bézout) right coprime if \(\begin{pmatrix} N(s) \\ D(s) \end{pmatrix}\) is left-invertible, that is to say, if there exist \(X(s) \in M_{m \times p}(R[s])\), \(Y(s) \in M_{m \times n}(R[s])\) satisfying the “Bézout identity”

\[
X(s)N(s) + Y(s)D(s) = I_m
\]

The polynomial matrices \(X(s)\) and \(Y(s)\) are called left Bézout factors for the pair \((N(s), D(s))\).

Let \(H(s)\) be a rational matrix admitting a factorization \(H(s) = N(s)D^{-1}(s)\), we will call this factorization a r.c.f. (right coprime factorization) of \(H(s)\).

**Theorem 2.1.** Let \((E, A, B, C)\) a pbfoi system. Then there exist a coprime factorization of the transfer matrix associated to the system.

**Proof.** Taking into account that \((E, A, B, C)\) is a pbfoi system \((s(E + F^B_E C + BF^C_E) - (A + F^C_E C + BF^C_E))^{-1} = Q(s)\) is polynomial. The matrix pair \((N(s), D(s))\) with \(N(s) = Q(s)\) and \(D(s) = I - (s(BF^B_E + F^C_E C) + (BF^C_E + F^C_E C)Q(s))\) is coprime: \(X(s)N(s) + Y(s)D(s) = I\) with \(X(s) = s(BF^B_E + F^C_E C)\) and \(Y(s) = I\).

\[
D(s) = I - X(s)Q(s) + (sE + A)Q(s) - (sE + A)Q(s) = I - (X(s) + (sE + A))Q(s) + (sE + A)Q(s) = (sE + A)Q(s),
\]

consequently \(\det D(s) = \gamma \det(sE + A)\) for all \(s \in R\) and \(N(s)D^{-1}(s) = Q(s)((sE + A)Q(s))^{-1} = (sE + A)^{-1}\)

\[
H(s) = C(sE + A)^{-1}B = CN(s)D^{-1}(s)B.
\]

\[\square\]

**Proposition 2.1.** Let \((E, A, B, C)\) a pbfoi linear system, then there exist \(F^B_A, F^C_A, F^B_E, F^C_E\), such that \(A + BF^B_A + F^C_A C\) is invertible and \((E + BF^B_E + F^C_A C)^{-1}\) is nilpotent.

**Proof.** If \((E, A, B, C)\) is a pbfoi linear system, then there exist \(F^B_A, F^C_A, F^B_E, F^C_E\), such that \(P(s) = s(E + F^C_E C + BF^B_E) - (A + F^C_A C + BF^B_E)\) is invertible, so there exist \(Q(s) = sQ_1 + \ldots + sQ_1 + Q_0\) such that \(P(s)Q(s) = I_n\).

Consequently:

\[
(A + BF^B_A + F^C_A C)Q_0 = I_n
\]

\[
(E + BF^B_E + F^C_A C)Q_0 - (A + BF^B_A + F^C_A C)Q_1 = 0
\]

\[
(E + BF^B_E + F^C_A C)Q_1 - (A + BF^B_A + F^C_A C)Q_2 = 0
\]

\[
\vdots
\]

\[
(E + BF^B_E + F^C_A C)Q_{\ell - 1} - (A + BF^B_A + F^C_A C)Q_{\ell} = 0
\]

\[
(E + BF^B_E + F^C_A C)Q_{\ell} = 0
\]

First equality says that \(- (A + BF^B_A + F^C_A C)^{-1} = Q_0\).

Since \(- (A + BF^B_A + F^C_A C)^{-1}\) is invertible we can obtain \(Q_1, \ell \geq i \geq 1\).

\[
Q_1 = - (A^{-1}E)^\ell A^{-1}
\]

where

\[
A = (A + BF^B_A + F^C_A C)
\]

\[
E = (E + BF^B_E + F^C_A C)
\]
The last equation

\[ 0 = (E + BF_R^B + F_C^E C)Q_t = \]
\[ -((E + BF_R^B + F_C^E C)(A + BF_A^B + F_C^A C)^{-1})t + 1 \]

consequently

\[(E + BF_R^B + F_C^E C)(A + BF_A^B + F_C^A C)^{-1} \] (1)

is a nilpotent matrix and taking into account that \(Q_t \neq 0\), the nilpotency order is \(t + 1\).

**Corollary 2.1.** If a system \((E, A, B, C)\) is pbfoi then it is repairable.

Remember that a system \((E, A, B, C)\) is repairable if and only if there exist \(F_R^B\) and \(F_A^B\) such that \(A + BF_R^B + F_C^A C\) is invertible, (for more information about repairable systems see [7]).

Notice that the system in remark 1.1 is not repairable.

**Remark 2.1.** Converse is not true as we can see in the following example: let \((E, A, B, C)\) with \(E = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\), \(A = I_2\), \(B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\), \(C = (0 \ 1 \ 0)\), considering all possible feedbacks \(F_R^B, F_A^B\), and output injections \(F_C^E, F_A^C\) matrix \(s(E + F_C^E C + BF_R^B) - (A + F_C^A C + BF_A^B)\) is

\[
\begin{pmatrix}
1 + s & 1 + s & 1 + s \\
0 & 1 + s & 1 + s \\
0 & 0 & 1 + s
\end{pmatrix}
\]

which inverse is not polynomial because of \(\det(s(E + F_C^E C + BF_R^B) - (A + F_C^A C + BF_A^B)) \notin \mathbb{C}_0\).

**Proposition 2.2.** Let \((E, A, B, C)\) be a pbfoi system. Then the equation \(XE - NXA = Z\) with \(N\) a nilpotent matrix has a solution \((X, Z)\) with \(X\) invertible.

**Proof.** Matrix 1 in proposition 2.1 is equivalent to a nilpotent matrix \(N\) in its reduced Jordan form

\[(E + BF_R^B + F_C^E C)(A + BF_A^B + F_C^A C)^{-1} = X^{-1}NX, \]
equivalently

\[X(E + BF_R^B + F_C^E C) = NX(A + BF_A^B + F_C^A C), \]

\[XE - NXA = -X(F_C^E C + BF_R^B) + NX(F_C^A C + BF_A^B) = Z. \]

The existence of \(F_R^B, F_C^E, F_A^B, F_C^A\), verifying proposition 2.1 implies that the equation \(XE - NXA = Z\) has a solution with \(X\) invertible and \(Z = -X(F_C^E C + BF_R^B) + NX(F_C^A C + BF_A^B)\).

Suppose now, that the system \((E, A, B, C)\) is repairable and let \(F_R^B\) and \(F_A^B\) be such that \(A + BF_R^B + F_C^A C\) is invertible. If the equation \(XE - NXA = Z\) with \(N\) a nilpotent matrix, has a solution \(X, Z\) with \(X\) invertible, we can consider the matrix \(M = -X^{-1}Z + X^{-1}NX(F_C^A C + BF_A^B)\). If the equation \(F_C^E C + BF_R^B = M\) has a solution then the system is pbfoi, and

\[Q_t = -(A + BF_R^B + F_C^A C)^{-1}XNX^{-1}. \]

### 3 Characterization of systems pbfoi

In this section we will try to characterize pbfoi-systems.

**Proposition 3.1.** Let \((E, A, B, C)\) and \((E_1, A_1, B_1, C_1)\) be equivalent systems. There exist \(F_R^{E_1}, F_A^{E_1}, F_C^{E_1}, F_C^{E_1}\) such that

\[(s(E + F_C^E C + BF_R^B) - (A + F_C^A C + BF_A^B))^{-1}\]
is polynomial if and only if and There exist \(F_R^{E_1}, F_A^{E_1}, F_C^{E_1}, F_C^{E_1}\) such that

\[(s(E_1 + F_C^{E_1} C + B_1 F_R^{B_1}) - (A_1 + F_C^{A_1} C + B_1 F_A^{B_1}))^{-1}\]
is polynomial.

**Proof.**

\[E_1 = QEP + F_C^{E_1} CP + QBF_R^{B_1}, \]

\[A_1 = QAP + F_C^{A_1} CP + QBF_A^{B_1}, \]

\[B_1 = QBR, \]

\[C_1 = SCP, \]

\[(s(E_1 + F_C^{E_1} C_1 + B_1 F_R^{B_1}) - (A_1 + F_C^{A_1} C_1 + B_1 F_A^{B_1}))^{-1} =
\]
\[(s(QEP + F_C^{E_1} CP + QBF_R^{B_1} + F_C^{A_1} SCP + QBF_A^{B_1}) - (QAP + F_C^{A_1} CP + QBF_A^{B_1} + F_C^{E_1} SCP + QBF_R^{B_1}))^{-1} =
\]
\[Q(A + Q^{-1} F_C^{E_1} C + B_1 F_R^{B_1} P^{-1} + Q^{-1} F_C^{A_1} SCP + QBF_A^{B_1} P^{-1})^{-1} =
\]
\[P^{-1}(s(E + Q^{-1} F_C^{E_1} C + B_1 F_R^{B_1} P^{-1} + Q^{-1} F_C^{A_1} SCP + QBF_A^{B_1} P^{-1}))^{-1} =
\]
\[Q^{-1}(s(E + Q^{-1} F_C^{E_1} C + B_1 F_R^{B_1} P^{-1} + Q^{-1} F_C^{A_1} SCP + QBF_A^{B_1} P^{-1}))^{-1} \]

\[F_C^{E_1} = Q^{-1} F_C^{E_1} + Q^{-1} F_C^{A_1} SCP + QBF_A^{B_1} P^{-1} + Q^{-1} F_C^{A_1} SCP + QBF_A^{B_1} P^{-1} \]

\[C_1 = Q^{-1} F_C^{E_1} + Q^{-1} F_C^{A_1} SCP + B_1(F_R^{B_1} P^{-1} + PBF_A^{B_1} P^{-1})^{-1} \]

\[F_C^{E_1} = Q^{-1} F_C^{E_1} + Q^{-1} F_C^{A_1} SCP + B_1(F_R^{B_1} P^{-1} + PBF_A^{B_1} P^{-1})^{-1} \]

\[F_C^{E_1} = Q^{-1} F_C^{E_1} + Q^{-1} F_C^{A_1} SCP + B_1(F_R^{B_1} P^{-1} + PBF_A^{B_1} P^{-1})^{-1} \]

\[F_C^{E_1} = Q^{-1} F_C^{E_1} + Q^{-1} F_C^{A_1} SCP + B_1(F_R^{B_1} P^{-1} + PBF_A^{B_1} P^{-1})^{-1} \]

\[3.1 \text{ Cas } R = \mathbb{C} \]

Proposition 3.1 permit us to characterize the systems pbfoi.

**Lemma 3.1.** Let \((E, A, B, C)\) be a system equivalent to \((E_r, A_r, B_r, C_r)\) with \(E_r = \begin{pmatrix} I_2 & I_3 \\ I_N \end{pmatrix}, A_r = \begin{pmatrix} N_4 \\ N_4 \end{pmatrix}, B = \begin{pmatrix} B_2 \\ 0 \end{pmatrix}, C_r = \begin{pmatrix} 0 & C_2 \end{pmatrix} \). Then, the system is pbfoi.
Proof. It is easy to prove that the system is equivalent (see [7]) to \((E, \bar{A}, \bar{B}, \bar{C})\) with \(\bar{E} = \begin{pmatrix} N_3 \\ N_4 \\ N_1 \end{pmatrix}, \bar{A} = \begin{pmatrix} I_2 \\ I_3 \\ I_5 \end{pmatrix}, \bar{B} = B_r, \) and \(\bar{C} = C_r.\) Then, taking \(F^B_E = F^B_A = 0,\) and \(F^C_E = F^C_A = 0\) we have that \((s(\bar{E} + F^C_E \bar{C} + BF^B_E) - (\bar{A} + F^C_A \bar{C} + BF^B_A))\) is invertible. \(\square\)

Lemma 3.2. Let \((E, A, B, C)\) be a system equivalent to \((E_r, A_r, B_r, C_r)\) with \(E_r = \begin{pmatrix} I_2 \\ I_3 \\ I_4 \\ N_1 \end{pmatrix}, A_r = \begin{pmatrix} N_3 \\ N_4 \\ J \\ I_5 \end{pmatrix}, B = \begin{pmatrix} B_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, C_r = \begin{pmatrix} 0 \\ C_2 \\ 0 \\ 0 \end{pmatrix}.\) Then, the system can be not pbfoi.

Proof. It is easy to prove that the system is equivalent (see [7]) to \((E, \bar{A}, \bar{B}, \bar{C})\) with \(\bar{E} = \begin{pmatrix} N_3 \\ N_4 \\ I_4 \\ N_1 \end{pmatrix}, \bar{A} = \begin{pmatrix} I_2 \\ I_3 \\ J \\ I_5 \end{pmatrix}, \bar{B} = B_r, \) and \(\bar{C} = C_r.\) Then, for all \(F^B_E, F^B_A, F^C_E,\) and \(F^C_A\)

\[
det(s(\bar{E} + F^C_E \bar{C} + BF^B_E) - (\bar{A} + F^C_A \bar{C} + BF^B_A)) = s \begin{pmatrix} I_2 + E_{21}F_{11} & 0 & 0 & 0 \\ 0 & I_3 + G_{12}C_2 & 0 & 0 \\ 0 & G_{13}C_2 & I_4 & 0 \\ 0 & G_{14}C_2 & 0 & I_5 \end{pmatrix} + \begin{pmatrix} N_3 & B_2F_{21} & B_2F_{22} + G_{12}C_2 & B_2F_{23} & B_2F_{24} \\ 0 & N_4 + G_{23}C_2 & 0 & 0 \\ 0 & G_{24}C_2 & I_4 & 0 \\ 0 & G_{24}C_2 & 0 & N_1 \end{pmatrix} = p(s) \cdot \det(sI_4 + J) \notin C_0
\]

Theorem 3.1. Let \((E, A, B, C)\) be a repairable system verifying one of the following conditions

1. the system has not finite zeros
2. the number \(t\) of Jordan blocks is is less or equal than \(r = \text{rank } B_1 = \text{rank } C_1.\)

Then, the systems is pbfoi.

Proof. If the system \((E, A, B, C)\) is pbfoi it is repairable. So the system is equivalent (see [7]) to \((E_1, A_1, B_1, C_1)\) with

\[
E_1 = \begin{pmatrix} N_1 \\ N_2 \\ J \end{pmatrix}, A_1 = \begin{pmatrix} I_1 \\ I_2 \end{pmatrix},
B_1 = \begin{pmatrix} B_1 \\ B_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, C_1 = \begin{pmatrix} 0 \\ C_2 \\ 0 \end{pmatrix}
\]

with \(E = \begin{pmatrix} 0 \\ I \end{pmatrix}, J = \begin{pmatrix} J \\ N_3 \end{pmatrix}, A = \begin{pmatrix} 0 \\ N \end{pmatrix}\)

\(B_1 = \begin{pmatrix} I \\ 0 \end{pmatrix}, C_1 = \begin{pmatrix} I \end{pmatrix}, J = \text{diag}(J_1, \ldots, J_t)\)

\(J_t\) non derogatory with simple non-zero eigenvalue (different \(J_t\) may be the same eigenvalue). After lemmas it suffices to consider systems in the form \(\begin{pmatrix} 0 \\ I \\ J \\ J \end{pmatrix}, \begin{pmatrix} 0 \\ I \\ J^{-1} \\ J \end{pmatrix}\) which are equivalent to \(\begin{pmatrix} 0 \\ I \\ J^{-1} \\ J \end{pmatrix}, \begin{pmatrix} 0 \\ I \\ J \end{pmatrix}\)

Suppose now \(t = 1,\) that is to say

\[
J^{-1} = \begin{pmatrix} a & 1 \\ \vdots & \ddots \end{pmatrix},
\]

and taking \(F^B_A = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix},
F^C_A = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix},
\]

and \(F^C_E = 0, F^B_E = 0.\) So

\[
det(s(E + BF_E + FC_E) + A + BF_A + FC_A) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]

For \(1 < t \leq r = \text{rank } B_1 = \text{rank } C_1,\) the system \((E, A, B, C)\) with \(E = \begin{pmatrix} J_1 \\ \vdots \\ J_t \end{pmatrix},\)
the transfer matrix of the system. On the other hand, if we can construct a coprime factorization of a system, then we have an equivalent system to a system in the previous form, then we can construct a coprime factorization of a system. Proposition 3.2. Let \( A, B \) be a system over a principal ideal domain. Then there are equivalent conditions:

1. There exist \( F_E \) and \( F_A \) such that \( P(s) = (sI_n - A + sBF_E + BF_A) \) is an unimodular matrix.

2. The system is repairable, it is, there exist \( F_A \) such that \( A - BF_A \) is invertible. The equation \( XE + NXA = BY \), with \( N \) nilpotent, has a solution \((X, Y)\) with \( X \) invertible.

Proof. First implication is direct by corollary 3.1 and proposition 3.2. Reciprocally, we consider \( F_E = (F_A X N - Y)X^{-1} \in M_{m \times n}(\mathbb{R}) \), then \( (I_n + BF_E)(-A + BF_A)^{-1} \) is nilpotent of order \( r \): \( ((I_n + BF_E)(-A + BF_A)^{-1})^r = T N^T T^{-1} = 0 \), where \( T = ((-A + BF_A))X \). Furthermore, since \( ((I_n + BF_E)(-A + BF_A)^{-1})^r \neq 0 \), we define

\[
Q_i = (-1)^i((-A + BF_A)^{-1}((I_n + BF_E)))(-A + BF_A)^{-1},
\]

for all \( i = 0, 1, \ldots, r - 1 \). So, we have \( (I_n + BF_E)Q_r = 0 \) and \( Q_{r-1} \neq 0 \). Finally, we consider polynomial matrix \( Q(s) = \sum_{i=0}^{r-1} Q_i s^i \) verifying \( P(s)Q(s) = I_n \). Note that \( r = \ell + 1 \).

Corollary 3.1. Let \( (A, B) \) be a repairable system. If equation \( XE + NXA = BY \), with \( N \) nilpotent, has a solution \((X, Y)\) with \( X \) invertible, then there exist a coprime factorization of the transfer matrix associated to the system.

Proof. By theorem 3.2 and proposition 3.2, \( (N(s) = \sum_{i=0}^{l} N_i s^i, D(s) = \sum_{i=0}^{l} D_i s^i) \) with \( N_0 = XC, N_i = (-1)^i X N^i C \) for all \( i = 1, \ldots, l \). \( D_0 = I_n - F_A(-A + BF_A)^{-1}B, D_1 = -Y C \) and \( D_{i+1} = (-1)^{i+1} Y N^i C \) for all \( i = 1, \ldots, l \), where \( C = X^{-1}(-A + BF_A)^{-1}B \), is a coprime factorization of the transfer matrix associated to the system \((A, B)\).

Remark 3.1. We can write a procedure with Input \((A, B)\) \( n \)-dimensional \( m \)-input reachable system, and Output \((N(s), D(s))\) coprime matrix fraction description of the transfer matrix of the system. In particular, \( H(s) = (sI_n - A + sBF_E + BF_A)^{-1}B \) is a polynomial transfer matrix.

Step 1. Give canonical form \((A_1, B_1) = (A P^{-1} A P^{-1} B F, P^{-1} B Q)\).

Step 2. Find \( F' \) such that \( A_1 + B_1 F' \) is invertible.

Step 3. Solve equation \( A_1 X_1 N + X_1 = B_1 Y_1 \).

Step 4. Calculate \( X = PX_1 \) and \( Y = Q Y_1 - F X_1 N \).

Step 5. Calculate \( F_A = (F + Q F')P^{-1} \) and \( F_E = (F_A X N - Y)X^{-1} \).

Step 6. Return polynomial coeff. of \( N(s) \) and \( D(s) \):

\[
\begin{align*}
N_0 = XC, & \quad N_i = (-1)^i X N^i C, \\
C = X^{-1}(-A + BF_A)^{-1}B, & \quad D_0 = I_n - F_A(-A + BF_A)^{-1}B, \quad D_1 = -Y C, \\
D_{i+1} = (-1)^{i+1} Y N^i C
\end{align*}
\]
3.2.1 Single input reachable system

Theorem 3.2. Let $(A, B)$ be a single input reachable system. If $N$ is nilpotent of order $n$, then there exist $Y$ such that $AXN + X = BY$ equation has a solution $(X, Y)$ with $X$ invertible.

Proof. First, by proposition 3.1, we can consider an equivalent canonical system.

$$(A_R, B_R) = \left( \begin{array}{cc} 0 & 0 \\ 1_{n-1} & 0 \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \end{array} \right)$$

Second, if $N$ has nilpotent order $r < n$ then $X$ is no invertible: $X = (B \ldots (-1)^{r-1} A^{r-1} B \ldots (-1)^n A^{n-1} B) (Y \ldots Y N^{r-1} \ldots 0) = (B \ldots (-1)^{r-1} A^{r-1} B) (Y \ldots Y N^{r-1})$, so

$$X = \begin{pmatrix} 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & (1)^{n-r} \end{pmatrix} \begin{pmatrix} Y \\ \vdots \\ Y N^{r-1} \end{pmatrix}$$

is no invertible. Hence, we suppose $N$ of order $n$ and reduced triangular form (see [?]), $N = (a_{ij})$ with $a_{ij} = 0 \forall j \leq i$. In this case

$$X = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & -1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & (1)^{n-1} \end{pmatrix} \begin{pmatrix} Y \\ \vdots \\ Y N^{r-1} \end{pmatrix}$$

Since $N$ is of order $n$, $a_{ii+1} \neq 0$ for all $i = 1, \ldots, n-1$, so, we can consider $Y$ such that $y_i \neq 0$. $\square$

Corollary 3.2. Let $(A, B)$ be a single input reachable system. Then $(A, B)$ is a pfboi-system.

Proof. We suppose $(A, B)$ reduced canonical system. If we consider $F_A = (0 \ldots 0 1)$ and $F_B = (FAXN \ldots Y)X^{-1}$, then $A + BF_A$ and $F(s) = (sI_n - A + sBF_E + BF_A)$ are invertible matrices. $\square$

4 Conclusions

The goal of this paper is the study of the coprime factorization of the transfer matrix of a singular linear system $(E, A, B)$, throughout repairable property and solutions of a particular equation $XE - NXA = Z$. In particular, repairable property has been study over principal ideal domains (see [M. Carriegos, 1999]) and stable rings (see [J.A. Hermita-Alonso, M.M. López-Cabeceira and M.T. Trobajo, 2005]). Currently, we are developing our study over no single input systems over principal ideal domains.

References


