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COPRIME FACTORIZATION OF THE TRANSFER MATRIX OF A SINGULAR LINEAR SYSTEM

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Abstract
Given a linear dynamic time invariant represented by $x^+(t) = Ax(t)Bu(t)$, $y(t) = Cx(t)$, we analyze conditions for obtention of a coprime factorization of transfer matrices of singular linear systems defined over commutative rings R with element unit. The problem presented is related to the existence of solutions of a matrix equation $XE - NXA = Z$.

Key words
Singular systems, feedback, output injection, coprime factorizations.

1 Introduction

Let R be a commutative ring with unity and $Ex^+(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$) be a singular system over R, that we represent by (E, A, B, C). Then, the transfer matrix of the system (E, A, B, C) is given by $H(s) = C(sE - A)^{-1}B$.

This systems appear in literature when for example, one studies linear systems depending on a parameter or linear systems with delays.

Let (E, A, B, C) be a singular system with $E = I_4$, $A = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & 1 \\ & & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$, clearly $(sI_4 - A)^{-1}$ is a rational matrix. Considering $F_E^B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$, $F_A^B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, $F_E^C = 0$, $F_A^C = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, it is easy to compute $\det(s(E + F_E^C C + BF_E^B) - (A + F_A^C C + BF_A^B)) = 1 \neq 0$, $\forall s \in R$, consequently $(s(E + F_E^C C + BF_E^B) - (A + F_A^C C + BF_A^B))^{-1}$ is polynomial.

We are interested in classify the singular systems (E, A, B, C) for which there exist feedbacks F_E^B, F_A^B, and output injections F_E^C,F_A^C, such that $(s(E+F_E^C C+BF_E^B)-(A+F_A^C C+BF_A^B))^{-1}$ is polynomial. We will call systems with polynomial transfer matrix by feedback (proportional and derivative) and output injection (proportional and derivative) and we will write simply as pbfoi-systems, the systems verifying this property .

Notice that if this property holds the the system is regularisable, remember that a system (E, A, B, C) is regularisable if and only if there exist feedbacks F_E^B, F_A^B, and output injections F_E^C,F_A^C, such that $\det(s(E + F_E^C C + BF_E^B) - (A + F_A^C C + BF_A^B)) \neq 0$ for some $s \in R$.

Remark 1.1. *Converse is not true as we can see in this example: let (E, A, B, C) with $E = I_4$, $A = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & 1 \\ & & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$, considering all possible feedbacks F_E^B, F_A^B, and output injections F_E^C,F_A^C matrix $s(E + F_E^C C + BF_E^B) - (A + F_A^C C + BF_A^B)$ is*

$$\begin{pmatrix} s(1+a)+a_1 & s(b+e)+(b_1+e_1) & sc+c_1 & sd+d_1 \\ 0 & s(1+f)+f_1 & 0 & 0 \\ 0 & s(1+g)+g_1 & s & 1 \\ 0 & s(1+h)+h_1 & 0 & s \end{pmatrix}$$

is easy to compute $\det(s(E + F_E^C C + BF_E^B) - (A + F_A^C C + BF_A^B)) = (s(1+a)+a_1)(s(1+f)+f_1)s^2 \neq 0$ for almost all $s \in R$ and 0 for $s = 0$. Then (E, A, B, C) is regularisable but not pbfoi.

In order to use a simple reduced system preserving these properties we consider the following equivalence relation deduced of to apply the standard transformations in state, input and output spaces $x(t) = Px_1(t)$, $u(t) = Ru_1(t)$, $y_1(t) = Sy(t)$, premultiplication by an invertible matrix $QE\dot{x}(t) = QAx(t) + Qu(t)$ making feedback $u(t) = u_1(t) - Vx(t)$ and derivative feedback $u(t) = u_1(t) - U\dot{x}(t)$ as well as output injection $u(t) = u_1(t) - Wy(t)$ and derivative output injection $u(t) = u_1(t) - Z\dot{y}(t)$. Considering this equivalence relation and restricting out to the regularisable systems and for $R = \mathbb{C}$, it is possible to reduce the system to

(E_c, A_c, B_c, C_c) where

$$E_c = \begin{pmatrix} I_1 & & & & \\ & I_2 & & & \\ & & I_3 & & \\ & & & I_4 & \\ & & & & N_1 \end{pmatrix}$$

$$A_c = \begin{pmatrix} N_2 & & & & \\ & N_3 & & & \\ & & N_4 & & \\ & & & J & \\ & & & & I_5 \end{pmatrix}$$

$$B_c = \begin{pmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$C_c = \begin{pmatrix} C_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 & 0 \end{pmatrix}$$

and N_i denotes a nilpotent matrix in its reduced form $N_i = \text{diag}\,(N_{i_1}, \ldots, N_{i_t})$, $N_{i_j} = \begin{pmatrix} 0 & I_{n_{i_j}-1} \\ 0 & 0 \end{pmatrix} \in M_{n_{i_j}}(C)$,
J denotes the Jordan matrix $J = \text{diag}\,(J_1(\lambda_1), \ldots, J_m(\lambda_m))$, $J_i(\lambda_i) = \text{diag}(J_{i_1}(\lambda_i, \ldots, J_{i_t}(\lambda_i))$, $J_{i_j}(\lambda_i) = \lambda_i I + N$.

Notice that not all subsystems must appears in canonical reduced form.

Remark 1.2. *Canonical reduced form can be obtained directly using the complete set of invariants (see [6]).*

2 Coprime factorization

Two polynomial matrices $N(s) \in M_{p\times m}(R[s])$ and $D(s) \in M_m(R[s])$ are called (Bézout) right coprime if $\begin{pmatrix} N(s) \\ D(s) \end{pmatrix}$ is left-invertible, that is to say, if there exist $X(s) \in M_{m\times p}(R[s])$, $Y(s) \in M_m(R[s])$ satisfying the "Bézout identity"

$$X(s)N(s) + Y(s)D(s) = I_m$$

The polynomial matrices $X(s)$ and $Y(s)$ are called left Bézout factors for the pair $(N(s), D(s))$.

Let $H(s)$ be a rational matrix admitting a factorization $H(s) = N(s)D^{-1}(s)$, we will call this factorization a r.c.f. (right coprime factorization) of $H(s)$.

Theorem 2.1. *Let (E, A, B, C) a pbfoi system. Then there exist a coprime factorization of the transfer matrix associated to the system.*

Proof. Taking into account that (E, A, B, C) is a pbfoi system $(s(E + F_E^C C + BF_E^B) - (A + F_A^C C + BF_A^B))^{-1} = Q(s)$ is polynomial. The matrix pair $(N(s), D(s))$ with $N(s) = Q(s)$ and $D(s) = I - (s(BF_E^B + F_E^C C) + (BF_A^B + F_A^C C))Q(s)$ is coprime: $X(s)N(s) + Y(s)D(s) = I$ with $X(s) = s(BF_E^B + F_E^C C) + (BF_A^B + F_A^C C)$ and $Y(s) = I$.

$D(s) =$
$I - X(s)Q(s) + (sE + A)Q(s) - (sE + A)Q(s) =$
$I - (X(s) + (sE + A))Q(s) + (sE + A)Q(s) =$
$(sE + A)Q(s)$,
consequently $\det D(s) = \gamma \det(sE + A)$ for all $s \in R$ and $N(s)D^{-1}(s) = Q(s)((sE+A)Q(s))^{-1} = (sE+A)^{-1}$

$$H(s) = C(sE + A)^{-1}B = CN(s)D^{-1}(s)B.$$

□

Proposition 2.1. *Let (E, A, B, C) a pbfoi linear system, then there exist F_A^B,F_A^C, F_E^B,F_E^C, such that $A + BF_A^B + F_A^C C$ is invertible and $(E + BF_E^B + F_E^C C)(-A + BF_A^B + F_A^C C)^{-1}$ is nilpotent.*

Proof. If (E, A, B, C) is a pbfoi linear system, then there exist F_A^B,F_A^C, F_E^B,F_E^C, such that $P(s) = s(E + F_E^C C + BF_E^B) - (A + F_A^C C + BF_A^B)$ is invertible, so there exist $Q(s) = s^\ell Q_\ell + \ldots + sQ_1 + Q_0$ such that $P(s)Q(s) = I_n$.

Consequently:

$$\begin{aligned}
&(A + BF_A^B + F_A^C C)Q_0 = I_n \\
&(E + BF_E^B + F_E^C C)Q_0 - \\
&\qquad (A + BF_A^B + F_A^C C)Q_1 = 0 \\
&(E + BF_E^B + F_E^C C)Q_1 - \\
&\qquad (A + BF_A^B + F_A^C C)Q_2 = 0 \\
&\vdots \\
&(E + BF_E^B + F_E^C C)Q_{\ell-1} - \\
&\qquad (A + BF_A^B + F_A^C C)Q_\ell = 0 \\
&(E + BF_E^B + F_E^C C)Q_\ell = 0
\end{aligned}$$

First equality says that $-(A+BF_A^B+F_A^C C)^{-1} = Q_0$.

Since $-(A+BF_A^B+F_A^C C)$ is invertible we can obtain Q_i, $\ell \geq i \geq 1$.

$$\begin{aligned}
&Q_i = -(\mathbb{A}^{-1}\mathbb{E})^i\mathbb{A}^{-1} \\
&\text{where} \\
&\mathbb{A} = (A + BF_A^B + F_A^C C) \\
&\mathbb{E} = (E + BF_E^B + F_E^C C)
\end{aligned}$$

The last equation

$$0 = (E + BF_E^B + F_E^C C)Q_\ell = \\ -((E + BF_E^B + F_E^C C)(A + BF_A^B + F_A^C C)^{-1})^{\ell+1}$$

consequently

$$(E + BF_E^B + F_E^C C)(A + BF_A^B + F_A^C C)^{-1} \quad (1)$$

is a nilpotent matrix and taking into account that $Q_\ell \neq 0$, the nilpotency order is $\ell + 1$. □

Corollary 2.1. *If a system (E, A, B, C) is pbfoi then it is repairable*

Remember that a system (E, A, B, C) is repairable if and only if there exist F_A^B and F_A^C such that $A + BF_A^B + F_A^C C$ is invertible, (for more information about repairable systems see [7]).

Notice that the system in remark 1.1 is not repairable.

Remark 2.1. *Converse is not true as we can see in the following example: let (E, A, B, C) with $E = \begin{pmatrix} 0&0&0\\0&0&0\\0&0&1 \end{pmatrix}$, $A = I_3$, $B = \begin{pmatrix} 1\\0\\0 \end{pmatrix}$, $C = \begin{pmatrix} 0&1&0 \end{pmatrix}$, considering all possible feedbacks F_E^B, F_A^B, and output injections F_E^C, F_A^C matrix $s(E + F_E^C C + BF_E^B) - (A + F_A^C C + BF_A^B)$ is*

$$\begin{pmatrix} 1 + c_1 + sa_1 & c_2 + d_1 + s(a_2 + b_1) & c_3 + sa_3 \\ 0 & 1 + d_2 + sb_2 & 0 \\ 0 & d_3 + sb_3 & 1 + s \end{pmatrix}$$

which inverse is not polynomial because of $\det(s(E + F_E^C C + BF_E^B) - (A + F_A^C C + BF_A^B)) \notin \mathbb{C}_0$.

Proposition 2.2. *Let (E, A, B, C) be a pbfoi system. Then the equation $XE - NXA = Z$ with N a nilpotent has a solution (X, Z) with X invertible.*

Proof. Matrix 1 in proposition 2.1 is equivalent to a nilpotent matrix N in its reduced Jordan form

$$(E + BF_E^B + F_E^C C)(A + BF_A^B + F_A^C C)^{-1} = X^{-1}NX,$$

equivalently

$$X(E + BF_E^B + F_E^C C) = NX(A + BF_A^B + F_A^C C),$$

$$XE - NXA = \\ -X(F_E^C C + BF_E^B) + NX(F_A^C C + BF_A^B) = Z.$$

The existence of F_E^B, F_E^C, F_A^B, F_A^C, verifying proposition 2.1 implies that the equation $XE - NXA = Z$ has a solution with X invertible and $Z = -X(F_E^C C + BF_E^B) + NX(F_A^C C + BF_A^B)$. □

Suppose now, that the system (E, A, B, C) is repairable and let F_A^B and F_A^C be such that $A + BF_A^B + F_A^C C$ is invertible. If the equation $XE - NXA = Z$ with N a nilpotent matrix, has a solution X, Z with X invertible, we can consider the matrix $M = -X^{-1}Z + X^{-1}NX(F_A^C C + BF_A^B)$.

If the equation $F_E^C C + BF_E^B = M$ has a solution then the system is pbfoi, and

$$Q_i = -(A + BF_A^B + F_A^C C)^{-1} XNX^{-1}.$$

3 Characterization of systems pbfoi

In this section we will try to characterize pbfoi-systems.

Proposition 3.1. *Let (E, A, B, C) and (E_1, A_1, B_1, C_1) be equivalent systems. There exist F_E^B, F_A^B, F_E^C,F_A^C, such that $(s(E + F_E^C C + BF_E^B) - (A + F_A^C C + BF_A^B))^{-1}$ is polynomial if and only if and There exist $F_{E_1}^{B_1}$, $F_{A_1}^{B_1}$, $F_{E_1}^{C_1}$, $F_{A_1}^{C_1}$, such that $(s(E_1 + F_{E_1}^{C_1} C_1 + B_1 F_{E_1}^{B_1}) - (A_1 + F_{A_1}^{C_1} C_1 + B_1 F_{A_1}^{B_1}))^{-1}$ is polynomial.*

Proof.

$$\begin{aligned} E_1 &= QEP + \bar{F}_E^C CP + QB\bar{F}_E^B, \\ A_1 &= QAP + \bar{F}_A^C CP + QB\bar{F}_A^B, \\ B_1 &= QBR, \\ C_1 &= SCP, \end{aligned}$$

$$\begin{aligned} &(s(E_1 + F_{E_1}^{C_1} C_1 + B_1 F_{E_1}^{B_1}) - (A_1 + F_{A_1}^{C_1} C_1 + B_1 F_{A_1}^{B_1}))^{-1} = \\ &(s(QEP + \bar{F}_E^C CP + QB\bar{F}_E^B + F_{E_1}^{C_1} SCP + QBRF_{E_1}^{B_1}) - \\ &(QAP + \bar{F}_A^C CP + QB\bar{F}_A^B + F_{A_1}^{C_1} SCP + QBRF_{A_1}^{B_1})^{-1} = \\ &(sQ(E + Q^{-1}\bar{F}_E^C C + B\bar{F}_E^B P^{-1} + Q^{-1}F_{E_1}^{C_1} SC + BRF_{E_1}^{B_1} P^{-1})P - \\ &Q(A + Q^{-1}\bar{F}_A^C C + B\bar{F}_A^B P^{-1} + Q^{-1}F_{A_1}^{C_1} SC + BRF_{A_1}^{B_1} P^{-1})P)^{-1} = \\ &P^{-1}(s(E + Q^{-1}\bar{F}_E^C C + B\bar{F}_E^B P^{-1} + Q^{-1}F_{E_1}^{C_1} SC + BRF_{E_1}^{B_1} P^{-1}) - \\ &(A + Q^{-1}\bar{F}_A^C C + B\bar{F}_A^B P^{-1} + Q^{-1}F_{A_1}^{C_1} SC + BRF_{A_1}^{B_1} P^{-1})^{-1} Q^{-1} = \\ &P^{-1}(s(E + (Q^{-1}\bar{F}_E^C + Q^{-1}F_{E_1}^{C_1} S)C + B(\bar{F}_E^B P^{-1} + RF_{E_1}^{B_1} P^{-1})) - \\ &(A + (Q^{-1}\bar{F}_A^C + Q^{-1}F_{A_1}^{C_1} S)C + B(\bar{F}_A^B P^{-1} + RF_{A_1}^{B_1} P^{-1})))^{-1} Q^{-1} \end{aligned}$$

$F_E^C = Q^{-1}\bar{F}_E^C + Q^{-1}F_{E_1}^{C_1} S$, $F_E^C = \bar{F}_E^B P^{-1} + RF_{E_1}^{B_1} P^{-1}$, $F_A^C = Q^{-1}\bar{F}_A^C + Q^{-1}F_{A_1}^{C_1} S$, $F_A^B = \bar{F}_A^B P^{-1} + RF_{A_1}^{B_1} P^{-1}$ □

3.1 Cas $R = \mathbb{C}$

Proposition 3.1 permit us to characterize the systems pbfoi.

Lemma 3.1. *Let (E, A, B, C) be a system equivalent to (E_r, A_r, B_r, C_r) with $E_r = \begin{pmatrix} I_2 && \\ & I_3 & \\ && N_1 \end{pmatrix}$, $A_r = \begin{pmatrix} N_3 && \\ & N_4 & \\ && I_5 \end{pmatrix}$, $B = \begin{pmatrix} B_2 \\ 0 \\ 0 \end{pmatrix}$, $C_r = \begin{pmatrix} 0 & C_2 & 0 \end{pmatrix}$. Then, the system is pbfoi.*

Proof. It is easy to prove that the system is equivalent (see [7]) to $(\bar{E}, \bar{A}, \bar{B}, \bar{C})$ with $\bar{E} = \begin{pmatrix} N_3 & & \\ & N_4 & \\ & & N_1 \end{pmatrix}$, $\bar{A} = \begin{pmatrix} I_2 & & \\ & I_3 & \\ & & I_5 \end{pmatrix}$ $\bar{B} = B_r$, and $\bar{C} = C_r$. Then, taking $F^{\bar{B}}_{\bar{E}} = F^{\bar{B}}_{\bar{A}} = 0$ and $F^{\bar{C}}_{\bar{E}} = F^{\bar{C}}_{\bar{A}} = 0$ we have that $(s(\bar{E} + F^{\bar{C}}_{\bar{E}}\bar{C} + \bar{B}F^{\bar{B}}_{\bar{E}}) - (\bar{A} + F^{\bar{C}}_{\bar{A}}\bar{C} + \bar{B}F^{\bar{B}}_{\bar{A}}))$ is invertible. □

Lemma 3.2. *Let (E, A, B, C) be a system equivalent to (E_r, A_r, B_r, C_r) with $E_r = \begin{pmatrix} I_2 & & & \\ & I_3 & & \\ & & I_4 & \\ & & & N_1 \end{pmatrix}$, $A_r = \begin{pmatrix} N_3 & & & \\ & N_4 & & \\ & & J & \\ & & & I_5 \end{pmatrix}$, $B = \begin{pmatrix} B_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $C_r = \begin{pmatrix} 0 & C_2 & 0 & 0 \end{pmatrix}$. Then, the system can be not pbfoi.*

Proof. It is easy to prove that the system is equivalent (see [7]) to $(\bar{E}, \bar{A}, \bar{B}, \bar{C})$ with $\bar{E} = \begin{pmatrix} N_3 & & & \\ & N_4 & & \\ & & I_4 & \\ & & & N_1 \end{pmatrix}$, $\bar{A} = \begin{pmatrix} I_2 & & & \\ & I_3 & & \\ & & J & \\ & & & I_5 \end{pmatrix}$ $\bar{B} = B_r$, and $\bar{C} = C_r$. Then, for all $F^{\bar{B}}_{\bar{E}}$, $F^{\bar{B}}_{\bar{A}}$, $F^{\bar{C}}_{\bar{E}}$ and $F^{\bar{C}}_{\bar{A}}$

$$\det(s(\bar{E} + F^{\bar{C}}_{\bar{E}}\bar{C} + \bar{B}F^{\bar{B}}_{\bar{E}}) - (\bar{A} + F^{\bar{C}}_{\bar{A}}\bar{C} + \bar{B}F^{\bar{B}}_{\bar{A}})) =$$
$$\det\left(\begin{pmatrix} I_2 + B_2F_{1_A} & B_2F_{2_A} + G_{1_A}C_2 & B_2F_{3_A} & B_2F_{4_A} \\ 0 & I_3 + G_{2_A}C_2 & 0 & 0 \\ 0 & G_{3_A}C_2 & J & 0 \\ 0 & G_{4_A}C_2 & 0 & I_5 \end{pmatrix} + s\begin{pmatrix} N_3 + B_2F_{1_E} & B_2F_{2_E} + G_{1_E}C_2 & B_2F_{3_E} & B_2F_{4_E} \\ 0 & N_4 + G_{2_E}C_2 & 0 & 0 \\ 0 & G_{3_E}C_2 & I_4 & 0 \\ 0 & G_{4_E}C_2 & 0 & N_1 \end{pmatrix}\right) =$$
$$= p(s) \cdot \det(sI_4 + J) \notin \mathbb{C}_0$$

□

Theorem 3.1. *Let (E, A, B, C) be a repairable system verifying one of the following conditions*

1. *the system has not finite zeros*
2. *the number t of Jordan blocks is is less or equal than $r = \operatorname{rank} B_1 = \operatorname{rank} C_1$.*

Then, the systems is pbfoi.

Proof. If the system (E, A, B, C) is pbfoi it is repairable. So the system is equivalent (see [7]) to (E_1, A_1, B_1, C_1) with

$$E_1 = \begin{pmatrix} \bar{E} & & & \\ & N_1 & & \\ & & N_2 & \\ & & & \bar{J} \end{pmatrix}, A_1 = \begin{pmatrix} \bar{A} & & & \\ & I_1 & & \\ & & I_2 & \\ & & & I \end{pmatrix},$$

$$B_r = \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, C_1 = \begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with $\bar{E} = \begin{pmatrix} 0 & \\ & I \end{pmatrix}$, $\bar{J} = \begin{pmatrix} J & \\ & N_3 \end{pmatrix}$, $\bar{A} = \begin{pmatrix} 0 & \\ & N \end{pmatrix}$ $B_1 = \begin{pmatrix} I \\ 0 \end{pmatrix}$, $C_1 = \begin{pmatrix} I & 0 \end{pmatrix}$ and $J = \operatorname{diag}(J_1, \ldots, J_\ell)$ J_i non derogatory with simple non-zero eigenvalue (different J_i may be the same eigenvalue). After lemmas it suffices to consider systems in the form $\left(\begin{pmatrix} 0 & \\ & J \end{pmatrix}, \begin{pmatrix} I & \\ & I \end{pmatrix}, \begin{pmatrix} I \\ 0 \end{pmatrix}, \begin{pmatrix} I & 0 \end{pmatrix}\right)$ which are equivalent to $\left(\begin{pmatrix} 0 & \\ & I \end{pmatrix}, \begin{pmatrix} I & \\ & J^{-1} \end{pmatrix}, \begin{pmatrix} I \\ 0 \end{pmatrix}, \begin{pmatrix} I & 0 \end{pmatrix}\right)$

Suppose now $t = 1$, that is to say $J^{-1} = \begin{pmatrix} a & 1 & & & \\ & a & & & \\ & & \ddots & & \\ & & & a & 1 \\ & & & & a \end{pmatrix}$, and taking $F^B_A = \begin{pmatrix} -1 & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & & & \ddots & \ddots & & & & \vdots \\ 0 & 0 & 0 & \ldots & -1 & 1 & 0 & \ldots & 0 \end{pmatrix}$, $F^C_A = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ \vdots & & & \vdots \\ 1 & 0 & \ldots & 0 \end{pmatrix}$, and $F^C_E = 0$, $F^B_E = 0$. So $\det(s(E + BF^B_E + F^C_E) + A + BF^B_A + F^C_A) =$

$$\det\begin{pmatrix} 0 & 1 & 0 & 0 & & & & \\ 0 & 0 & 1 & 0 & & & & \\ 0 & & \ddots & \ddots & & & & \\ 0 & 0 & & & 1 & 0 & & \\ 0 & 0 & & & a+s & 1 & & \\ 0 & 0 & & & & a+s & 1 & \\ \vdots & & & & & & \ddots & \\ 0 & 0 & 0 & & & \ldots & a+s & 1 \\ 1 & 0 & 0 & & & \ldots & & a+s \end{pmatrix} = 1.$$

For $1 < t \leq r = \operatorname{rank} B_1 = \operatorname{rank} C_1$, the system (E, A, B, C) with $E = \begin{pmatrix} 0 & & & \\ & J_1 & & \\ & & \ddots & \\ & & & J_s \end{pmatrix}$,

$A = \begin{pmatrix} 0 & & & \\ & I_1 & & \\ & & \ddots & \\ & & & I_s \end{pmatrix}$ $(0 \in M_r(\mathbb{C})$, is equivalent to (E_1, A_1, B_1, C_1) with

$E_1 = \begin{pmatrix} 0_1 & & & & & \\ & I & & & & \\ & & \ddots & & & \\ & & & 0_1 & & \\ & & & & I & \\ & & & & & 0_{r-s} \end{pmatrix}$ $(0_i \in M_i(\mathbb{C})$, $A_1 = \begin{pmatrix} 0 & & & & & \\ & J_1^{-1} & & & & \\ & & \ddots & & & \\ & & & 0 & & \\ & & & & J_s^{-1} & \\ & & & & & I_{r-s} \end{pmatrix}$, $B_1 = \begin{pmatrix} 1 & \\ 0 & \\ \vdots & \\ 0 & \\ \vdots & \\ & 1 \\ & 0 \\ & \vdots \\ & 0 \\ & & 0 \end{pmatrix}$, $C_1 = \begin{pmatrix} 1\ 0 \dots 0 & & \\ & \dots & \\ & & 1\ 0 \dots 0 \\ & & & 0 \end{pmatrix}$. Then, it suffices to apply the case $s = 1$ □

For $t > r$ the result is not true, as we can see in the following example.

Example 3.1. *Let* $\left(\begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, (1\ 0\ 0)\right)$ *a repairable system,*

$$\det \begin{pmatrix} s(a_1 + b_1) + (c_1 + d_1 & sa_2 + c_2 & sa_3 + c_3 \\ sb_2 + d_2 & s & 0 \\ sb_3 + d_3 & 0 & s \end{pmatrix} \notin \mathbb{C}.$$

So, the system is not pbfoi.

3.2 Case R a principal ideal domain

On one hand, by proposition 3.1, it is clear that if we have an equivalent system to a system in the previous form, then we can construct a coprime factorization of the transfer matrix of the system. On the other hand, in principal ideal domains, it is no possible to reduce a system to a form like $\mathbb{C}$. So, in order to realize a first study over principal ideal domains, we consider systems $x^+(t) = Ax(t) + Bu(t)$, it is, we consider $C = 0$.

Proposition 3.2. *Let (A, B) be a system over a principal ideal domain. Then are equivalent conditions:*

1. *There exist F_E and F_A such that $P(s) = (sI_n - A + sBF_E + BF_A)$ is an unimodular matrix.*
2. *The system is repairable, it is, there exist F_A such that $A - BF_A$ is invertible. The equation $XE + NXA = BY$, with N nilpotent, has a solution (X, Y) with X invertible.*

Proof. First implication is direct by corollary **??** and proposition **??**. Reciprocally, we consider $F_E = (F_AXN - Y)X^{-1} \in M_{m\times n}(R)$, then $(I_n + BF_E)(-A + BF_A)^{-1}$ is nilpotent of order r: $((I_n + BF_E)(-A + BF_A)^{-1})^r = TN^rT^{-1} = 0$, where $T = ((-A + BF_A))X$. Furthermore, since $((I_n + BF_E)(-A + BF_A))^{r-1} \neq 0$, we define

$$Q_i = (-1)^i((-A+BF_A)^{-1}(I_n+BF_E))^i(-A+BF_A)^{-1},$$

for all $i = 0, 1, \dots, r-1$. So, we have $(I_n + BF_E)Q_{r-1} = 0$ and $Q_{r-1} \neq 0$. Finally, we consider polynomial matrix $Q(s) = \sum_{i=0}^{r-1} Q_is^i$ verifying $P(s)Q(s) = I_n$. Note that $r = \ell + 1$. □

Corollary 3.1. *Let (A, B) be a repairable system. If equation $XE + NXA = BY$, with N nilpotent, has a solution (X, Y) with X invertible, then there exist a coprime factorization of the transfer matrix associated to the system.*

Proof. By theorem **??** and proposition 3.2, $(N(s) = \sum_{i=0}^{l} N_is^i, D(s) = \sum_{i=0}^{l} N_is^i)$ with $N_0 = XC$, $N_i = (-1)^iXN^iC$ for all $i = 1, \dots, \ell$, $D_0 = I_m - F_A(-A + BF_A)^{-1}B$, $D_1 = -YC$ and $D_{i+1} = (-1)^{i+1}YN^iC$ for all $i = 1, \dots, \ell$, where $C = X^{-1}(-A + BF_A)^{-1}B$, is a coprime factorization of the transfer matrix associated to the system (A, B). □

Remark 3.1. *We can write a procedure with Input (A, B) n-dimensional m-input reachable system, and Output $(N(s), D(s))$ coprime matrix fraction description of the transfer matrix of the system. In particular, $H(s) = (sI_n - A + sBF_E + BF_A)^{-1}B$ is a polynomial transfer matrix.*

`Step 1.` – *Give canonical form* $(A_1, B_1) = (P^{-1}AP + P^{-1}BF, P^{-1}BQ)$.

`Step 2.` – *Find F' such that $A_1 + B_1F'$ is invertible.*

`Step 3.` – *Solve equation $A_1X_1N + X_1 = B_1Y_1$.*

`Step 4.` – *Calculate*

$X = PX_1$ *and* $Y = QY_1 - FX_1N$.

`Step 5.` – *Calculate*

$F_A = (F + QF')P^{-1}$ *and* $F_E = (F_AXN-)X^{-1}$.

`Step 6.` – *Return polynomial coeff. of $N(s)$ and $D(s)$*

$N_0 = XC, \quad N_i = (-1)^iXN^iC,$
$C = X^{-1}(-A + BF_A)^{-1}B$
$D_0 = I_m - F_A(-A + BF_A)^{-1}B, \quad D_1 = -YC,$
$D_{i+1} = (-1)^{i+1}YN^iC$

3.2.1 Single input reachable system

Theorem 3.2. *Let (A, B) be a single input reachable system. If N is nilpotent of order n, then there exist Y such that $AXN + X = BY$ equation has a solution (X, Y) with X invertible.*

Proof. First, by proposition 3.1, we can consider an equivalent canonical system.

$$(A_R, B_R) = \left(\begin{pmatrix} \underline{0}^t & 0 \\ \mathrm{I}_{n-1} & \underline{0} \end{pmatrix}, \begin{pmatrix} 1 \\ \underline{0} \end{pmatrix} \right)$$

Second, if N has nilpotent order $r < n$ then X is no invertible: $X = (B \, \dots \, (-1)^{r-1}A^{r-1}B \; (-1)^r A^r B \, \dots \, (-1)^{n-1}A^{n-1}B)\; (Y \, \dots \, YN^{r-1} \; 0 \, \dots \, 0)^t = (B \, \dots \, (-1)^{r-1}A^{r-1}B)\; (Y \, \dots YN^{r-1})^t$, so

$$X = \begin{pmatrix} 1 \dots & 0 \\ \ddots & \\ 0 \dots & (-1)^{r-1} \\ \underline{0} \dots & \underline{0} \end{pmatrix} \begin{pmatrix} Y \\ \vdots \\ YN^{r-1} \end{pmatrix}$$

is no invertible. Hence, we suppose N of order n and reduced triangular form (see [**?**]), $N = (a_{ij})$ with $a_{ij} = 0 \; \forall j \leq i$. In this case

$$X = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & -1 & & \\ & & \ddots & \\ & & & (-1)^{n-1} \end{pmatrix} \cdot$$

$$\begin{pmatrix} y_1 & y_2 & y_3 & \dots & y_n \\ 0 & a_{12}y_1 & a_{13}y_1 + a_{23}y_2 & \dots & \sum_{i=1}^{n-1} a_{in}y_i \\ & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & \prod_{i=1}^{n-1} a_{ii+1}y_1 \end{pmatrix} .$$

Since N is of order n, $a_{ii+1} \neq 0$ for all $i = 1, \dots n-1$. so, we can consider Y such that $y_1 \neq 0$. □

Corollary 3.2. *Let (A, B) be a single input reachable system. Then (A, B) is a pfboi-system.*

Proof. We suppose (A, B) reduced canonical system. If we consider $F_A = \begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix}$ and $F_E = (F_A XN - Y)X^{-1}$, then $A + BF_A$ and $P(s) = (sI_n - A + sBF_E + BF_A)$ are invertible matrices. □

□

4 Conclusions

The goal of this paper is the study of the coprime factorization of the transfer matrix of a singular linear system (E, A, B), throughout repairable property and solutions of a particular equation $XE - NXA = Z$. In particular, repairable property has been study over principal ideal domains (see [M. Carriegos, 1999]) and stable rings (see [J.A. Hermida-Alonso, M.M. López-Cabeceira and M.T. Trobajo, 2005]). Currently, we are developing our study over no single input systems over principal ideal domains.

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