LMI APPROACH FOR SOLVING PERIODIC MATRIX RICCATI EQUATION

Sergei V. Gusev [*] **Anton S. Shiriaev** [**]
Leonid B. Freidovich [**]

[*] *Department of Mathematics and Mechanics, St. Petersburg State University, St. Petersburg, Russia*
[**] *Department of Applied Physics and Electronics, Umeå University, Umeå, Sweden*

Abstract: The paper presents a new method for numerical solution of matrix Riccati equation with periodic coefficients. The method is based on approximation of stabilizing solution of the Riccati equation by trigonometric polynomials.

Keywords: Riccati equation, Periodic, Numerical solution, Yakubovich Lemma, Linear matrix inequality

1. INTRODUCTION

To follow several feedback control design techniques it is required to solve a continuous-time periodic matrix differential Riccati equations. The examples are the linear-quadratic regulator optimal control problem for linear periodic systems (Yakubovich, 1986) and stabilization of periodic motions of underactuated mechanical systems (Shiriaev *et al.*, 2005).

The straightforward approach to the solution of the problem is based on different methods of numerical solution of ordinary differential equations (Johansson *et al.*, 2007) A disadvantage of this approach is a necessity to solve some unstable ordinary matrix differential equations.

Here we propose an alternative (spectral) approach for solving the periodic Riccati equations. The main idea is to reformulate the problem in terms of convex optimization in the space of Fourier expansions of solutions. It turns out that the obtained infinite-dimensional convex problem can be approximated by the sequence of finite-dimensional problems with growing dimensions. Using the generalization of Yakubovich lemma recently proposed in (Gusev, 2006), the latter problems are reduced to standard linear matrix inequalities (LMI) optimization.

A numerical solution for a simple benchmark problem, presented below, shows advantage of the proposed approach over the standard one.

2. EXTREMAL PROPERTY OF STABILIZING SOLUTION OF RICCATI EQUATION

Consider the matrix Riccati equation

$$\mathcal{R}(H(.), t) = 0 \ \forall t \geq 0, \tag{1}$$

where the Riccati operator is defined as

$$\mathcal{R}(H(.), t) = \dot{H}(t) + A^\top(t)H(t) + H(t)A(t) - H(t)B(t)R(t)^{-1}B^\top(t)H(t) + Q(t) \tag{2}$$

and $n\times n$ matrix-function $A(t)$ and $Q(t)=Q^\top(t)\geq 0$, $n\times m$ matrix function $B(t)$, and $m\times m$ matrix function $R(t)=R^\top(t)>0$ are continuous for $t\in[0,+\infty)$ and periodic with a given period $T>0$.

The solution $H_+(.)$ of (1) is called stabilizing if any solution $X(.)$ of the associated with this equation closed-loop linear system

$$\dot{X}(t)=\Big(A(t)-B(t)R^{-1}(t)B^\top(t)H_+(t)\Big)X(t)$$

belongs to $L_2([0,+\infty))$.

Our approach to be presented shortly is based on the following property of the stabilizing solution of the Riccati equation.

Theorem 1. Suppose that the function $A(.),B(.),Q(.),R(.),R^{-1}(.)$ are bonded and for all $t\geq 0: Q(t)>0$ and $R(t)>0$. Let $H_+(.)$ be a stabilizing solution of (1). Then, any bounded matrix $H(.)$, satisfying the Riccati inequality

$$\mathcal{R}(H(.),t)\geq 0 \qquad \forall t\geq 0, \tag{3}$$

defined by (2), satisfies also the inequality

$$H(t)\leq H_+(t) \qquad \forall t\geq 0.$$

The extremal property of stabilizing solution was first discovered by Willems (1971) for the algebraic Riccati equation. It was then generalized to periodic differential Riccati equation in (Bittanti *et al.*, 1989).

For a family of matrices $W_j=W_j^\top(t)>0$ and a set of numbers $t_j\geq 0$, $j=1,\ldots,l$, let us define the functional

$$J(H(.))=\sum_{j=1}^{l}\mathrm{tr}\Big(H(t_j)\,W_j\Big). \tag{4}$$

It follows from Theorem 1 that a maximum of the functional (4) over the set of bounded matrices $H(.)$, satisfying (3), is achieved on stabilizing solution $H_+(.)$.

Using this property we can reduce the numerical computation of $H_+(.)$ to maximization of linear functional over convex set of matrices $H(.)$, satisfying inequality (3). This is infinite dimensional convex optimization problem. To solve it we construct the sequence of approximating finite dimensional problems.

3. FINITE-DIMENSIONAL OPTIMIZATION PROBLEM

Suppose that matrices $A(.)$, $B(.)$, $R(.)$, and $Q(.)$ are periodic, bounded, and moreover, can be written in the form

$$\begin{aligned} A(t)&=\sum_{j=-k}^{k}e^{\mathbf{i}j\omega t}A_j, \quad B(t)=\sum_{j=-k}^{k}e^{\mathbf{i}j\omega t}B_j,\\ Q(t)&=\sum_{j=-k}^{k}e^{\mathbf{i}j\omega t}Q_j, \quad R(t)=\sum_{j=-k}^{k}e^{\mathbf{i}j\omega t}R_j \end{aligned} \tag{5}$$

where $\mathbf{i}=\sqrt{-1}$, and the $n\times n$ complex matrices A_j, B_j, Q_j, and R_j are such that $A_{-j}=\bar{A}_j$, $B_{-j}=\bar{B}_j$, $Q_{-j}=\bar{Q}_j$, $R_{-j}=\bar{R}_j$ for $j=0,\ldots,k$.

Let us consider the problem of minimization of the cost function J, defined by (4), over the set of matrices

$$H(t)=\sum_{j=-k}^{k}e^{\mathbf{i}j\omega t}H_j$$

with $H_{-j}=\bar{H}_j, j=0,\ldots,k$, which satisfy the inequality (3).

The quadratic inequality (3) is equivalent to the parameter-dependent LMI

$$\begin{pmatrix}\dot{H}(t)+H(t)A(t)+A^\top(t)H(t)+Q(t) & H(t)B(t)\\ B^\top(t)H(t) & R(t)\end{pmatrix}\geq 0 \quad \forall t\geq 0. \tag{6}$$

Denote the right hand side of (6) by $\mathcal{S}(H(.),t)$. Let $\hat{H}=(H_0,H_1,\ldots,H_k)$ be the matrix of coefficients of $H(t)$. Then

$$\mathcal{S}(H(.),t)=\sum_{j=-2k}^{2k}e^{\mathbf{i}j\omega t}\mathcal{S}_j(\hat{H}),$$

where each $\mathcal{S}_j(\hat{H})$ is linear in $\hat{H}$.

Define $(2k+1)\times(2k+1)$ matrices K_j as follows: $K_{j,\alpha,\beta}=\begin{cases}1, \text{ if } \alpha=2k+1-j, \beta=2k+1\\ 0 \text{ otherwise,}\end{cases}$ $j=0,\ldots 2k+1$, $K_j=K_{-j}^\top$, $j=-1,\ldots,-2k-1$. Let $M=(I_{2k},0)$ and $N=(0,I_{2k})$, where I_d here and below denotes the identity matrix of dimension d.

It follows from Yakubovich lemma for matrix frequency domain inequalities (Gusev, 2006) that $H(.)$ satisfies (6) if and only if there exists a complex $2k(m+n)\times 2k(m+n)$ matrix $F=\bar{F}^\top$ such that

$$\sum_{j=-2k-1}^{2k+1}\mathcal{S}_j(\hat{H})\otimes K_j\geq (I_{m+n}\otimes N)^\top F(I_{m+n}\otimes N)-(I_{m+n}\otimes M)^\top F(I_{m+n}\otimes M). \tag{7}$$

Let us define the linear functional

$$L(\hat{H})=\sum_{\alpha=1}^{l}\mathrm{tr}\left(W_\alpha\sum_{j=-k}^{k}e^{\mathbf{i}j\omega t_\alpha}H_j\right).$$

It follows from definition that $L(\hat{H})=J(H(.))$.

The auxiliary finite-dimensional problem is to maximize this functional over the set of matrices $\{\hat{H}, F\}$ that satisfy (7). This is a standard LMI optimization problem.

4. APPROXIMATION FOR THE SOLUTION OF THE RICCATI EQUATION

Consider the continuous-time differential matrix Riccati equation (1) and suppose that the matrices $A(.)$, $B(.)$, $R(.)$, and $Q(.)$ are continuous and periodic with period $T = 2\pi/\omega$.

Let $A(t) = \sum_{j=-\infty}^{+\infty} e^{\mathbf{i}j\omega t} A_j$, $\quad B(t) = \sum_{j=-\infty}^{+\infty} e^{\mathbf{i}j\omega t} B_j$, $Q(t) = \sum_{j=-\infty}^{+\infty} e^{\mathbf{i}j\omega t} Q_j$, $\quad R(t) = \sum_{j=-\infty}^{+\infty} e^{\mathbf{i}j\omega t} R_j$ be the Fourier expansions of $A(.)$, $B(.)$, $R(.)$, and $Q(.)$, correspondingly.

For a natural number k let us: denote by $A^{(k)}(t)$, $B^{(k)}(t)$, $R^{(k)}(t)$, $Q^{(k)}(t)$ the partial sums (5) for these expansions, solve the auxiliary optimization problem for these matrices, denote by $\hat{H}^{(k)} = (H_0^{(k)}, \ldots, H_k^{(k)})$ a solution of this problem, define the periodic function $H_+^{(k)}(t) = \sum_{j=-k}^{k} e^{\mathbf{i}j\omega t} H_j^{(k)}$.

The following result can be proved.

Theorem 2. Suppose that: the matrix functions $A(t)$, $B(t)$, $R(t)$, and $Q(t)$ are continuous and periodic with period T; the Riccati equation (1) has a continuous periodic stabilizing solution $H_+(.)$; there is a continuous periodic matrix $H_o(.)$ that satisfies the strict Riccati inequality $\mathcal{R}(H_o(.), t) > 0$ for all $t \in [0, T]$. Then, $\lim_{k\to\infty} H_+^{(k)}(t) = H_+(t)$ for all $t \in [0, T]$.

5. AN EXAMPLE

Let us consider the following benchmark problem. Let $A \in \mathbb{R}^{2\times 2}, B \in \mathbb{R}^{1\times 2}$, $R > 0$, $Q = Q^\top > 0 \in \mathbb{R}^{2\times 2}$. Suppose that matrix H_+ is the stabilizing solution of the algebraic Riccati equation

$$A^\top H + HA - HBR^{-1}B^\top H + Q = 0.$$

Let $P(t) = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix}$ and define the matrix-functions

$$A(t) = \frac{dP(t)}{dt} P(t)^{-1} + P(t)AP(t)^{-1}, \quad B(t) = P(t)B,$$
$$Q(t) = (P(t)^{-1})^\top QP(t)^{-1}, R(t) = R.$$

Then, the matrix-function $H_+(t) = (P(t)^{-1})^\top H_+ P(t)^{-1}$ is a stabilizing solution of the periodic Riccati equation (1). The problem is to compute the approximation $\tilde{H}_+(.)$ of $H_+(.)$ using the sampled values $A(t_j), B(t_j), Q(t_j)$, $t_j = (j-1)\frac{T}{N}$, where $T = 2\pi/\omega$, $j = 1, \ldots, N$. The accuracy of approximation is measured by the relative error $e(\tilde{H}_+(.)) = \frac{1}{N}\sum_{j=1}^{N} \frac{\|\tilde{H}_+(t_j) - H_+(t_j)\|}{\|H_+(t_j)\|}$.

Two methods have been compared: the iterative numerical solution of the differential equation (1) presented in (Johansson *et al.*, 2007) and the proposed LMI approach. The parameters were taken as follows: $A = \begin{pmatrix} 1 & 0.5 \\ 3 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $Q = \begin{pmatrix} 0.01 & 0 \\ 0 & a \end{pmatrix}$, $R = 1$, $\omega = 2$ $(T = \pi)$, $N = 100$. The dependence of the relative error for various values of the parameter a has been investigated.

For the first method, we have $e \approx 10^{-15}$ for $1 \le a \le 33$ and $e = 9\cdot 10^{-2}$ for $a = 34$. The method fails for larger values of a.

For the LMI approach (k = 3) we have $e = 1.6\cdot 10^{-10}$ for $a = 1$, $e = 1.2\cdot 10^{-10}$ for $a = 34$, $e = 4.8\cdot 10^{-9}$ for $a = 10^3$, $e = 1.4\cdot 10^{-5}$ for $a = 10^6$. The method fails for larger values of a.

The experiments show that in the considered benchmark problem the LMI approach is more stable to the variation in the parameter. For the same time for small values of a the first method provides higher precision of the solution.

REFERENCES

Bittanti, S. P. Colaneri, and G. De Nicolao (1989). A Note on the Maximal Solution of the Periodic Riccati Equation, *IEEE Trans. on Autom. Control*, **34**, 1316–1319.

Gusev, S.V.(2006). Parameter-Dependent S-Procedure And Yakubovich Lemma, *arxiv:math/0612794*.

Johansson, S., B. Kågström, A. Shiriaev, and A. Varga(2007). Comparing one-shot and multi-shot methods for solving periodic Riccati differential equations. *Proc. of PSYCO'07.*

Shiriaev, A.S., J.W. Perram, and C. Canudas de Wit(2005). Constructive tool for orbital stabilization of underactuated nonlinear systems: virtual constraints approach. *IEEE Trans. on Autom. Control*, **50**, 1164–1176.

Willems, J.(1971). Least squares stationary optimal control and the algebraic Riccati equation, *IEEE Trans. on Autom. Control*, **16**, 621–634.

Yakubovich, V.A.(1986). Linear-quadratic optimization problem and frequency theorem for periodic systems. *Siberian Math. J.,* **27**, 181–200.