ON THE APPLICATION OF A LINEAR PROGRAMMING METHOD TO THE EVALUATION OF THE ENTROPY OF A SYMBOLIC IMAGE

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Abstract

Entropy is one of the most important characteristics of the behavior of a dynamical system. For direct calculation both topological and metric entropy involves problems, the elaboration of numerical methods of their estimation is of considerable importance in applications. We use the concept of symbolic image, which is a finite approximation of a dynamical system. Symbolic image is constructed as an oriented graph for a mapping f and a fixed covering of its phase space. The vertices of the graph correspond to the cells of the covering and edges show the existence of nonempty intersections of the covering cells with their images. We use a method of linear programming, which allows us to construct an invariant measure on the graph and thereby to specify a stationary process. The estimation of the entropy of such a process gives a bound for the entropy of f.

Key words

dynamical system, symbolic image, entropy, stationary process.

1 Introduction

This paper is dedicated to an elaboration of a numerical method of the estimation of the entropy of a dynamical system using the notion of symbolic image. It was introduced by G.S. Osipenko [Osipenko, 1983] and became one of main tools for the investigation of dynamical system by symbolic analysis methods. The advantage of such a method is that many problems (localization of periodic orbits and invariant sets, estimation of Lyapunov exponents, estimation of topological entropy) may be solved using well known algorithms for directed graphs.

The estimation of the topological entropy of a dynamical system may be obtained by application of subdivision technique and construction of a topological Markov chain. The chain is considered as a directed graph, the arcs are labeled according to the fixed partition, which leads to the construction of a sofic shift. Then a subshift of finite type is produced via a standard techique [Lind, Marcus, 1995]. The entropy of this subshift is estimated by the logarithm of the maximal eigenvalue of its adjacency matrix. Such a method was implemented in [Froyland, Junge and Ochs, 2001] and [Osipenko, Ampilova, 2005]. Following this line of attack, we estimate a "topological entropy of a symbolic image" which gives an upper bound for the topological entropy of the system.

We consider a method of the estimation of " metric entropy of a symbolic image" based on the construction of an invariant measure of an directed graph.

An algorithm of the construction of such a measure using prime cycles was designed and implemented in [Ampilova, 2007]. In a prime cycle with l edges the value $1/l$ is assigned to every edge. A coefficient (weight) is designated to every prime cycle, being the sum of weights equals to one. The measure of an edge belonging to more than one cycle is defined as the sum of the measures which the edge has in every cycle. If an edge does not belong to any cycle, its measure is zero. The measure of a vertex is the sum of measures of outcoming (or incoming) edges. This method, while clear, has an evident disadvantage: the number of prime cycles may be very significant and the algorithm becomes time-consuming. An optimization may lead to cycle missing. Hence, the measure is not assigned to all edges of the graph.

The proposed method is aimed at the construction of an invariant measure, such that to assign a value to every edge. To solve the problem we apply a linear programming technique. It allows us to construct a stationary process on the graph (with a given accuracy), using a method of the sequential balance of the vertices measures. L.M. Bregman proved the convergence of the method in [Bregman, 1967]. The entropy computed with regard to the measure is an estimation of the stationary process entropy. Numerical experiments show that this value less than the entropy of corresponding topological Markov chain.

The paper is organized as follows: next section is dedicated to the notion of symbolic image. In sections 3-4 definitions of Markov chain, stationary processes on a graph and their entropy are given. Section 5 describes the algorithm of the construction of the invariant measure. Finally in sections 6 and 7 we give the data of numerical experiments and summarize our results.

2 Symbolic image of a dynamical system

Let ϕ be a discrete dynamical system generated by a homeomorphism f on a compact $M \in R^n$. Symbolic image of a dynamical system f [Osipenko, 1983] is an oriented graph G, constructed in accordance with a covering $\{M_i\}, i = 1, \ldots, k$ of M by closed sets, being vertices correspond to the covering cells and the existence of the edge (i, j) means that $f(M_i) \cap M_j \neq \emptyset$. The symbolic image is a finite approximation of the system f.

It depends on the covering and may be specified by the following parameters: d — diameter of the covering, which is the largest of diameters of M_i; q — upper bound of the symbolic image, which is the largest diameter of $f(M_i)$; r — lower bound of the symbolic image, which is the minimum of the distances between $f(M_i)$ and M_j, if $f(M_i) \cap M_j = \emptyset$. Being a relationship between the parameters and an value ε is given, there is a correspondence between the ε-orbits of the system and paths on G. [Osipenko, 2007] The construction of a sequence of symbolic images corresponding to a sequential subdivision of the set M results in obtaining sequential approximations of the system dynamics.

It is known that any invariant measure μ of the system f may induce a stationary process (an invariant measure ν) on its symbolic image, providing that $\nu_{ij} = \mu(f(M(i)) \cap M(j))$, where $\sum_{ij} \nu_{ij} = 1$. [Osipenko, Krupin, Bezruchko, Petrenko and Kapitanov, 2007] We consider a method of construction of an invariant measure on the graph and calculate the entropy of the stationary process. It allows us to estimate the entropy of a symbolic image and in some way [Osipenko, Ampilova, 2005] the entropy of the initial system.

3 Markov chain on a graph

Consider a graph $G = (V, E)$ and a finite set S of states of a process. We assign a probability (measure) to every state. The vertices of G are the states of S with positive probability, $V = \{I \in S, \mu(I) \geq 0\}$. The edges of G are the transitions from one state to another that have positive conditional probability, $E = \{(I, J), \mu(J|I) \geq 0\}$.

Let $i(e)$ be the beginning of an edge e. Denote by $\mu(i(e))$ a probability to be at the beginning of e, $\mu(e|i(e))$ the conditional probability of e.

Definition 1. *A Markov chain μ on a graph $G = (V, E)$ is an assignment of probabilities $\mu(I) \geq 0, I \in S$ and conditional probabilities $\mu(e|i(e)) \geq 0$ such that $\sum_{I \in S} \mu(I) = 1; \sum_{e \in E_I} \mu(e|I) = 1 \, \forall I \in S$, where E_I denotes the set of edges outcoming from I.*

The probability of an edge e :

$$\mu(e) = \mu(i(e))\mu(e|i(e)). \quad (1)$$

The probability of a path $e_1 e_2 \ldots e_n$:

$$\mu(i(e_1))\mu(e_1|i(e_1))\mu(e_2|i(e_2)) \ldots \mu(e_n|i(e_n)). \quad (2)$$

(It should be noted that such a definition means that the process is Markovian.)

It follows from (1) that

$$\sum_{e \in E_{i(e)}} \mu(e) = \mu(i(e)) \sum_{e \in E_{i(e)}} \mu(e|i(e)) = \mu(i(e)),$$

i.e. the probability to be in a vertex equals to the sum of probabilities of outcoming edges.

The initial state probabilities are assembled into a row vector $\mathbf{p}$ — initial state distribution, indexed by the states of G, such that $p_I = \mu(I)$. The conditional probabilities $\mu(e|I)$ form a stohastic square matrix defined by

$$P_{IJ} = \sum_{e \in E_I^J} \mu(e|I),$$

where E_I^J denotes the set of edges outcoming from I and incoming to J.

4 The entropy of a stationary process on a graph

Let for a Markov chain on $G = (V, E)$ a vector $\mathbf{p}$ and a matrix P be defined.

Definition 2. *The Markov chain is said to be stationary if the equality*

$$\mathbf{p}P = \mathbf{p} \quad (3)$$

holds.

Hence $p_I = \sum_{J=1}^{n} p_J P_{JI}$ and stationarity means that for any state I the probability to be in it equals to the sum of all incoming edges probabilities. Taking into account the definition of a Markov chain, it easy to understand that the stationarity condition means the following: for any vertex the sum of incoming edges probabilities equals to the sum of outcoming ones.

The entropy of a stationary Markov chain is naturally defined in the more generall setting of stationary processes. Let A be an alphabet and X be a set of sequences of symbols from A. Let F be a set of blocks which are not allowed in X (forbidden blocks). Denote by X_F the set of sequences which do not contain forbidden blocks. A subset X_1 of X is called shift space over A if it coincides with X_F. In what follows we assume that we deal with a shift space. A stationary process over an alphabet A is an assignment μ to blocks over A such that

$$\sum_{a\in A}\mu(a)=1 \tag{4}$$

and

$$\mu(w)=\sum_{a\in A}\mu(wa)=\sum_{a\in A}\mu(aw) \tag{5}$$

for any block w over A.

The shift space over A defined by forbidding the blocks with zero probability is said to be the support of a stationary process. A stationary process on a shift space X is one whose support is contained in X.

A stationary process on $G=(V,E)$ is defined to be a stationary process on the set $A=E$ of edges of G whose support is in X_G, where X_G denotes the edge shift (the elements of edge shift correspond to the admissible paths on G). A stationary Markov chain on a graph G is a stationary process on G. (It should be noted that the stationarity means the invariance of the measure μ with regard to the shift map σ.)

Given a stationary process μ, for each n, assemble the probabilities of the paths of length n into a probability vector $p_\mu^{(n)}$. The entropy of μ is defined as

$$h(\mu)=\lim_{n\to\infty}\frac{1}{n}h(p_\mu^{(n)}). \tag{6}$$

It is well known [Petersen, 1989] that the limit exists and the entropy may be computed by the formula

$$h(\mu)=-\sum_{e\in E(G)}\mu(i(e))\mu(e|i(e))\log(\mu(e|i(e))), \tag{7}$$

which may be rewritten in the equivalent form

$$\begin{aligned} h(\mu)=&\sum_{i\in V}\mu(i(e))\log(\mu(i(e)))\\ &-\sum_{e\in E}\mu(e)\log(\mu(e)). \end{aligned} \tag{8}$$

Let G be an irreducible graph. Then the entropy of every stationary process μ on G satisfies the inequality [Lind, Marcus, 1995]

$$h(\mu)\le\log\lambda, \tag{9}$$

where λ is the maximal eigenvalue of the adjacency matrix of G. If G is an arbitrary graph then in (9) λ is the maximal value of eigenvalues of all irreducible components of G.

5 Construction of an invariant measure μ

Assign probabilities to all edges of the graph G arbitrary. Denote by $P=\{p_{ij}\}, i,j=1,\ldots,m$, the matrix formed by these values. Our goal is to transform P in such a way to obtain a stationary process on G. This problem may be formulated as a part of the following linear programming task.

Maximize the function $\sum_{i,j}x_{ij}\ln\frac{p_{ij}}{x_{ij}}$ on conditions

$$\sum_{j=1}^m x_{ij}=a_i,\sum_{i=1}^m x_{ij}=b_j,x_{ij}\ge 0;$$

$$\sum_{i=1}^m a_i=\sum_{j=1}^m b_j;a_i,b_j>0;p_{ij}\ge 0,\sum_{i,j}x_{ij}=1.$$

For a vertex i the above conditions mean that the sum of outcoming edges is equal to a_i and the sum of incoming ones is b_i. A method of solution was proposed by G.V.Sheleihovsky, its convergence was proved by L.M.Bregman [Bregman, 1967]. It is based on a method of balance of vertices such that to satisfy the stationarity condition accurate within a given ε. Let $n\in V$ and

$$beg(n)=\{e\in E,e=(n,j),j\in V\},$$

$$end(n)=\{e\in E,e=(i,n),i\in V\}.$$

Assign measures to all edges of G. As the normalization step may be fulfilled at the and of operating period, we assume $\mu(e)=1,\forall e\in E$.

For each vertex n calculate its balans

$$q(n)=|\sum_{e\in beg(n)}\mu(e)-\sum_{e\in end(n)}\mu(e)|.$$

Construct a priority queue Q of the vertices of G, being a vertex n with the greatest $q(n)$ has the greatest priority. So, we assign the greatest priority to the most unbalanced vertex.

In the cycle: select the next vertex n from Q.

If $q(n)<\varepsilon$, then complete the processing of n and go out from the cycle. (In view of the priority of Q such an inequality holds for all remaining elements.)

Else calculate

$out=\sum_{e\in beg(n)}\mu(e)$

$in=\sum_{e\in end(n)}\mu(e)$

$\forall e\in end(n)\,\mu(e):=\mu(e)*\sqrt{\frac{out}{in}}$.

$\forall e \in beg(n)\, \mu(e) := \mu(e) * \sqrt{\frac{in}{out}}$.

If some of values $out, in, \sqrt{\frac{out}{in}}$ is too large (or small), we fulfill the normalization.

Fulfill the normalization. The algorithm is completed.

To obtain the value of the entropy according to the constructed measure we use formula (8).

6 Implementation

To provide the efficiency of the algorithm we have to save both forth and back directions of the edges, which results in the representation of the graph with using two hash-tables. Priority queue has been implemented using Fibonacci trees [Cormen, Leiserson and Rivest, 2001].

Example 1. *Estimate the entropy of a symbolic image for Henon map [Henon, 1976].*

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 1 - ax^2 + by \\ x \end{pmatrix}, \qquad (10)$$

for $a = 1.4$, $b = 0.3$. *We consider area* $D = [-10, 10] \times [-10, 10]$ *and use both linear and punctual methods [Petrenko, 2006] to construct a symbolic image. The initial partition consists from 9 cells. On each step every cell is subdivided in 4 cells. In the table the entropy, the number of nodes and the estimation of the entropy by (9) are given.*

Subdivision	*Nodes*	*Entropy*	$\log \lambda$
$9 \times 2^{2\times6}$	*231*	*1.384079*	*1.386763*

Example 2. *Consider logistic map* $f(x) = ax(1 - x), x \in [0, 1]$, *for* $a = 4$ *and* $a = 3.569$. *The results are given in the table.*

Subdivision	*Nodes*	*Entropy*	$\log \lambda$
$9 \times 2^{2\times6}$	*162*	*0.916999*	*0.940517*
$9 \times 2^{2\times12}$	*1 032*	*0.729919*	*0.731549*
$9 \times 2^{2\times11}$	*672*	*0.756997*	*0.779899*

7 Conlusion

This paper provides a numerical characteristic of a dynamical system, namely metric entropy of its symbolic image. This value is computed as the entropy of a stationary process on a graph with regard to an invariant measure. Such a measure is assigned to all edges of the graph using a linear programming technique. The most important goal of our future work is to provide a relationship between the metric entropy of a dynamical system and this characteristic.

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References

Ampilova, N. B. (2007) On the construction of an invariant measure of a symbolic image, *Int.Congress Nonlinear dynamical analysis-2007*, June 4-8, p.359.

Bregman, L.M. (1967) The proof of convergence of the Sheleihovsky method for the problem with transport constraints (in Russian), *Journal of computational mathematics and mathematical physics*,v.7, no.1, pp.147-156.

Cormen, T., Leiserson, C., Rivest, R., (2001) *Introduction to algorithms* (in Russian),M.

Froyland, G., Junge, O., Ochs, G. (2001) Rigorous computation of topological entropy with respect to a finite partition, *Physica D*, 154(1-2), pp.68-84.

Henon, M. (1976) A two-dimensional mapping with a strange attractor. *Comm. Math.Phys.*, v.50, pp.69-77.

Lind, D., Marcus B.(1995) *An introduction to symbolic dynamics and coding*, New York.

Osipenko, G.S.(1983) On symbolic image of a dynamical system.(in Russian) In *Boundary problems*, Perm, pp.101-105.

Osipenko, G.S, Ampilova, N.B. (2005) *Introduction to symbolic analysis of dynamical systems (in Russian)*, SPbGU.

Osipenko, G.S. (2007) *Dynamical systems, Graphs, and Algorithms*, Lect.Notes in Math, 1889, Springer.

Osipenko, G.S., Krupin, A.V., Bezruchko, A.A., Petrenko E.I., Kapitanov, A.Y. (2007) On a construction of invariant measures of dynamical systems, *E-Journal Differential equations and control processes*, http://www.neva.ru/journal 4.

Petersen, K. (1989) *Ergodic Theory*, Cambridge Univ.Press.Cambridge.

Petrenko, E.I. (2006) Design and implementation of the algorithms of construction of a symbolic image. *E-Journal Differential equations and control processes*, http://www.neva.ru/journal 3.