# METHODS OF ANALYSIS OF DYNAMIC SYSTEMS WITH VARIOUS DISSIPATION IN DYNAMICS OF A RIGID BODY

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**Abstract**
Due to its complexity, the problem of the motion of a rigid body in an unbounded medium requires the introduction of certain simplifying restrictions. The main aim in this connection is to introduce hypotheses that would make it possible to study the motion of the rigid body separately from the motion of the medium in which the body is embedded. On the one hand, a similar approach was realized in the classical Kirchhoff problem on the motion of a body in an unbounded ideal incompressible fluid that undergoes an irrotational motion and is at rest at infinity. On the other hand, it is obvious that the above-mentioned Kirchhoff problem does not exhaust the possibilities of this kind of simulation.

**Key words**
rigid body, resisting medium, integrability, various dissipation

## 1 Introduction

In this activity, we consider the possibility of transferring the results of the dynamics of the plane-parallel motion of a homogeneous axisymmetric rigid body interacting with a uniform flow of a resisting medium through its forward circular face to the case of three-dimensional motion. In contrast to the preceding works, the medium action on the rigid body is simulated with the inclusion of the effects of the so-called rotary derivatives of the moment of hydroaerodynamic forces with respect to the components of the angular velocity of the body itself.

On the basis of certain hypotheses, the main one of which is the quasi-stationarity hypothesis, a three–dimensional dynamic model of the medium action on the body was developed. In this connection, the possibility arises to formalize the model assumptions and derive a complete system of equations.

In what follows, we will address some typical representatives of the classes of the medium action functions under consideration, namely, the Chaplygin functions.

We will use Chaplygin's result as a reference point. Chaplygin calculated the medium action functions for an infinitely long plate in plane–parallel motion in the oncoming flow following the jet flow laws. In this case, the distance between the drag application point (center of pressure) and the plate center is proportional to the sine of the angle of attack, while the Newtonian drag coefficient is proportional to its cosine.

Moreover, integrable cases in the dynamics of the three-dimensional motion of a rigid body were also found for other model problems in the early works of the present author. For the Chaplygin medium action functions, the systems had a complete set of transcendental first integrals, which could be expressed in terms of a finite combination of elementary functions. In this case, transcendence is understood in the sense of the theory of functions of a complex variable (that is, their continuations to the complex plane have essentially singular points).

The present-day state-of-the-art of modeling the motion of a rigid body in a medium rests upon the model constraints imposed on the problem and uses a developing mathematical technique. Thus, the Chaplygin problem of the free fall of a massive body with plane symmetry in an unbounded volume (spatial region) of an ideal fluid was considered. In these monographs, the properties of trajectories were examined. The properties of Bernoulli surfaces of constant curvature for compressible continuous media were characterized.

## 2 Studies of Professor N. E. Zhukovskii

Zhukovskii was one of the pioneering scientists who examined various problems of the dynamics of a mass point in a medium, namely, the gravity drop of bodies, the motion of a body thrown off at an angle with the horizon, the motion of a pendulum, etc. Along with integration of equations of motion, Zhukovskii improved the model describing the interaction of bodies with a resisting medium. He assumed that the kinetic energy of a falling body goes partly to the generation of vortex motions of air and partly to overcoming the molecular forces of air adhesion to the moving body. The drag depends not only on the velocities of motion of the points of a body but also on the shape of this body. If the velocity is small, then it would be accurate enough to take the drag to be proportional to the first power of the velocity. At high velocities, the drag is proportional to the squared velocity.

Although the qualitative technique of the theory of differential equations was not so extensively used on the eve of the 20th century as it is now, researchers carried out a complete analysis of problems of motion in a medium not only for mass points but for rigid bodies as well. As a rule, these problems were purely model in their nature.

Zhukovskii also studied experiments on the self-rotation of plates falling in air. Here, one has to take into account such effects of the media as the drag and lifting forces. Precisely the aerodynamic characteristics of a plate were used to model the flight of a bird.

Zhukovskii conjectured the existence of a dynamic equilibrium of a "bird's body" relative to the center of mass. In this equilibrium, the angle made by the center–of–mass velocity vector with the plane of a plated wing (the angle of attack) is a control parameter, i.e., it can be assigned arbitrarily. This assumption is equivalent to the assumption on decoupled motion of a body wherein the characteristic time of motion relative to the center of mass is substantially smaller than the characteristic time of motion of the center of mass itself.

The study of the motion of a body in a medium when its translational motion is coupled with the rotational motion is also of great interest. The Kirchoff problem mentioned above does not exhaust the list of all possible cases of this sort.

## 3 Semiconservative systems

Dynamic systems investigated are the dynamic systems with variable dissipation with zero mean value (over the angle of attack, in our case). This means that the integral over a period of the angle of attack from the divergence of its right-hand side is equal to zero [this integral is responsible for the phase volume variation (after the corresponding reduction of the system)]. In this sense, the system is "*semiconservative*."

In contrast to the preceding works, the medium action on the rigid body is simulated with the inclusion of the effects of the so-called rotary derivatives of the moment of hydrodynamic forces with respect to the components of the angular velocity of the body itself.

## 4 A dynamically symmetric body under the action of the newtonian drag and a controlling force

On the basis of certain hypotheses, the main one of which is the quasi–stationarity hypothesis, a three–dimensional dynamic model of the medium action on

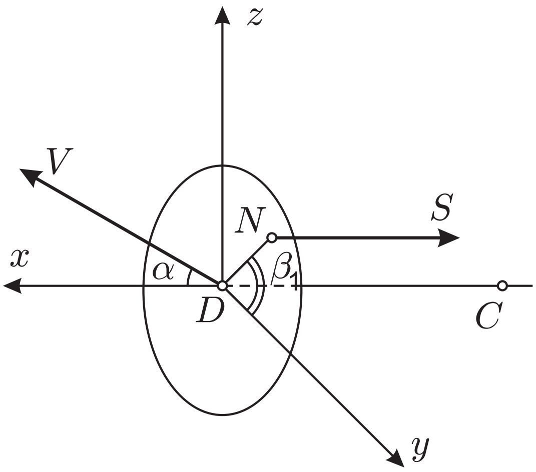


Figure 1.

the body was developed.

And so, under the assumption that the interaction obeys the jet flow laws, the interaction force S is normal to the disk, while the application point $N$ of the force **S** is determined by at least one parameter, namely, the angle of attack. Thus, we have

$$DN = R(\alpha, \dots) \quad (1)$$

We will take the magnitude of the Newtonian drag **S** in the form

$$S = s_1 v^2 \mathbf{e}_x \quad (2)$$

where the drag coefficient $s1$ is a function of $\alpha$ only:

$$s_1 = s_1(\alpha) \quad (3)$$

(fig. 1). At the same time, we will separate a class of problems related to the medium action on a body in which the controlling force acting along the geometrical symmetry axis ensures the realization of the classes of motions of interest under certain conditions (imposed constraints). Precisely the controlling force is a reaction of the constraints imposed. In this study, the controlling force always ensures the fulfillment of the condition

$$|\mathbf{v}| = v = \text{const} \quad (4)$$

In a body–fitted coordinate system, with one of the coordinate axes aligned with the axis of symmetry and two other axes lying in the plane of the disk, the tensor of inertia is diagonal

$$\text{diag}\{I_1, I_2, I_2\} \quad (5)$$

We will consider the spherical coordinates

$$(v, \alpha, \beta_1) \tag{6}$$

of the end of the velocity vector **v** of the point $D$ relative to the flow. Expressing the quantities (6) in terms of the cyclic kinematic variables and velocities via nonintegrable relations, we will consider them as quasivelocities supplementing them by the components

$$(\Omega_x, \Omega_y, \Omega_z) \tag{7}$$

## 5 Equations of motion in the case of zero spin of the rigid body about the longitudinal axis

In what follows, we will investigate in more detail the case of zero spin of the rigid body about its longitudinal axis. In this case, the following condition is fulfilled

$$\Omega_{x_0} = 0 \tag{8}$$

Then the independent dynamic part of the equations of motion in the four–dimensional phase space has the form

$$\begin{gathered}\dot{\alpha} v \cos\alpha \cos\beta_1 - \dot{\beta}_1 v \sin\alpha \sin\beta_1 + \\ + \Omega_z v \cos\alpha - \sigma \dot{\Omega}_z = 0\end{gathered} \tag{9}$$

$$\begin{gathered}\dot{\alpha} v \cos\alpha \sin\beta_1 + \dot{\beta}_1 v \sin\alpha \cos\beta_1 - \\ - \Omega_y v \cos\alpha + \sigma \dot{\Omega}_y = 0\end{gathered} \tag{10}$$

$$I_2 \dot{\Omega}_y = -z_N s(\alpha) v^2, \; I_2 \dot{\Omega}_z = y_N s(\alpha) v^2 \tag{11}$$

$$\begin{gathered}\sigma = CD, \; N = (0, y_N, z_N) \\ s(\alpha) = s_1(\alpha) \mathrm{sign} \cos\alpha\end{gathered} \tag{12}$$

System (9)–(11) involves the medium action functions $y_N$, $z_N$, and $s$. For their qualitative determination, we will use the experimental information on the jet flow properties.

## 6 Chaplygin functions of medium action

In what follows, we will address some typical "representatives" of the classes of the medium action functions under consideration, namely, the Chaplygin functions.

We will use Chaplygin's result as a reference point. Chaplygin calculated the medium action functions for an infinitely long plate in plane–parallel motion in the oncoming flow following the jet flow laws. In this case, the distance between the drag application point (center of pressure) and the plate center is proportional to the sine of the angle of attack, while the Newtonian drag coefficient is proportional to its cosine.

Thus, in what follows, we will restrict ourselves to the investigation of our system for the following medium action functions ($A, B, h > 0$)

$$\begin{gathered}y_N = A \sin\alpha \cos\beta_1 + h\Omega_z / v \\ z_N = A \sin\alpha \sin\beta_1 - h\Omega_y / v \\ s(\alpha) = B \cos\alpha\end{gathered} \tag{13}$$

where the coefficient $h$ occurs in terms that are proportional to the rotary derivatives of the moment of hydrodynamic forces with respect to the components of the angular velocity of the rigid body.

## 7 A system with variable dissipation with zero mean value and an analytical right side

Projecting then the angular velocities onto movable axes, which are unfitted to the body, so that

$$\begin{gathered}z_1 = \Omega_y \cos\beta_1 + \Omega_z \sin\beta_1 \\ z_2 = -\Omega_y \sin\beta_1 + \Omega_z \cos\beta_1\end{gathered} \tag{14}$$

and introducing dimensionless variables, parameters and new differentiality in accordance with the formulas

$$\begin{gathered}w_k, \; k = 1, 2, \; h_1 = hB, \; \sigma h_1 / I_2 = H_1 \\ \beta = \sigma AB / I_2, \; \sigma z_k = v w_k, \; \alpha' = v\dot{\alpha}/\sigma, \; \ldots\end{gathered} \tag{15}$$

we obtain the fourth–order dynamic system

$$\alpha' = -(1 + H_1) w_2 + \beta \sin\alpha \tag{16}$$

$$\begin{gathered}w_2' = \beta \sin\alpha \cos\alpha - (1 + H_1) w_1^2 \mathrm{tg}\alpha - \\ - H_1 w_2 \cos\alpha\end{gathered} \tag{17}$$

$$w_1' = (1 + H_1) w_1 w_2 \mathrm{tg}\alpha - H_1 w_1 \cos\alpha \tag{18}$$

$$\beta_1' = (1 + H_1) w_1 \mathrm{tg}\alpha \tag{19}$$

which incorporates an independent third–order subsystem given by first three equations.

For $\beta = H_1$, the divergence of the right-hand side of system (16)–(18), or (16)–(19), is identically zero after the change of variables

$$w_1^* = \ln |w_1| \tag{20}$$

this property makes it possible to consider these systems as conservative.

**Theorem 1.** *System (16)–(19) possesses a complete set of first integrals being elementary transcendental functions of their phase variables. Two of them form a complete set of the first integrals of system (16)–(18).*

Indeed, we will associate system (16)–(18) of order three with the second-order non–autonomous system

$$\frac{dw_2}{d\alpha} = \frac{\beta \sin\alpha\cos\alpha - (1+H_1)w_1^2 \mathrm{tg}\alpha - H_1 w_2 \cos\alpha}{-(1+H_1)w_2 + \beta\sin\alpha} \tag{21}$$

$$\frac{dw_2}{d\alpha} = \frac{(1+H_1)w_1 w_2 \mathrm{tg}\alpha - H_1 w_1 \cos\alpha}{-(1+H_1)w_2 + \beta\sin\alpha} \tag{22}$$

Applying the substitution

$$\tau = \sin\alpha \tag{23}$$

we transform the last system to the algebraic form

$$\frac{dw_2}{d\tau} = \frac{\beta\tau - (1+H_1)/\tau - H_1 w_2}{-(1+H_1)w_2 + \beta\tau} \tag{24}$$

$$\frac{dw_2}{d\tau} = \frac{(1+H_1)w_1 w_2/\tau - H_1 w_1}{-(1+H_1)w_2 + \beta\tau} \tag{25}$$

and making then the change

$$w_k = u_k \tau,\ k = 1, 2 \tag{26}$$

characteristic of homogeneous systems, we will associate the system specified by Eqs. (24) and (25) with the non–autonomous differential equation

$$\frac{du_2}{du_1} = \frac{\beta + (1+H_1)(u_2^2 - u_1^2) - (H_1+\beta)u_2}{2(1+H_1)u_1 u_2 - (H_1+\beta)u_1} \tag{27}$$

which has a first integral of the form

$$\frac{(1+H_1)u_2^2 - (H_1+\beta)u_2 + (1+H_1)u_1^2 + \beta}{u_1} = C_1 \tag{28}$$

In other words, our system investigated has a first integral of the form

$$\frac{(1+H_1)w_2^2 - (H_1+\beta)w_2\sin\alpha + (1+H_1)w_1^2 + \beta\sin^2\alpha}{w_1 \sin\alpha} = C_1 \tag{29}$$

As noted above, at

$$\beta = H_1 \tag{30}$$

the dynamic system investigated is conservative. Indeed, Eq. (29) is transformed to the invariant relation

$$\frac{w_2^2 + (1+\beta)w_1^2 + \beta[w_2 - \sin\alpha]^2}{w_1\sin\alpha} = C_1 \tag{31}$$

Moreover, it is easy to verify that both the numerator and the denominator of Eq. (31) at $\beta = H_1$ are the first integrals of system investigated

$$\begin{aligned} &w_2^2 + (1+\beta)w_1^2 + \beta[w_2 - \sin\alpha]^2 = C_1^* \\ &w_1 \sin\alpha = C_2^* \end{aligned} \tag{32}$$

For

$$\beta \neq H_1 \tag{33}$$

investigated system is no longer conservative and neither the numerator nor the denominator of the invariant relation (29) is the first integral. This fact can not necessarily be verified analytically, because investigated system has attractive and repulsive limiting sets, which preclude the existence of the complete set of even continuous first integrals for the system under consideration.

The additional first integral for the investigated system of order three may be obtained from the quadrature

$$\begin{aligned} &\int \frac{d\tau}{\tau} = \\ &= \int \frac{[\beta - (1+H_1)u_2]du_2}{\beta - (H_1+\beta)u_2 + (1+H_1)[u_2^2 - U(u_1, C_1)]} \end{aligned} \tag{34}$$

where

$$U(u_1, C_1) = \frac{1}{2(1+H_1)}\{C_1 \pm \sqrt{C_1^2 - 4D_1}\} \tag{35}$$

$$D_1 = (1+H_1)u_2^2 - (H_1+\beta)u_2 + \beta \tag{36}$$

The general structural form of the additional first integral for the investigated system of order three is as follows

$$\Phi_1(w_1, w_2, \sin\alpha) = C_2 \tag{37}$$

In view of Eq. (19) the additional first integral for the fourth-order system investigated is obtained from the solution of the equation

$$\frac{du_1}{d\beta_1} + \frac{\beta - (1+H_1)u_2}{1+H_1} = u_2 - \frac{H_1}{1+H_1} \tag{38}$$

which leads to the invariant relation

$$\begin{aligned} &\sin^2\{2(1+H_1)^2(\beta + C_3)\} = \\ &= \frac{(2(1+H_1)w_1 - 2C_1\sin\alpha)^2}{[(H_1+\beta)^2 - 4\beta(1+H_1) + C_1^2]\sin^2\alpha} \end{aligned} \tag{39}$$

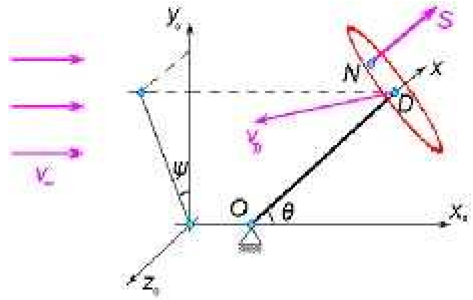


Figure 2.

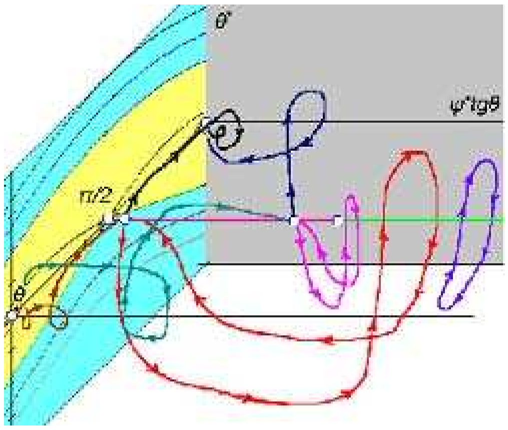


Figure 3.

## 8 Three–dimensional pendulum in an oncoming flow

By analogy with a free body, we will consider the problem of the motion of a three–dimensional pendulum in an oncoming uniform flow for the following case: the flow acts only on a circular disk fixed rigidly at its center, perpendicular to a sting that, in turn, is fixed by its other end on a spherical hinge. The model of the medium action on the disk is the same as above. The pendulum moves without its own spin. As before, the effects of the rotary derivatives of the moment of hydrodynamic forces with respect to the angular velocity of the rigid body are taken into account using the Chaplygin functions of the medium action [Shamolin, M. V. (1993) Classification...], [Shamolin, M. V. (1993) Application...], [Shamolin, M. V. (1993) Existence...].

If $\theta,\ \psi$ are the angles determining the position of the three–dimensional pendulum on a sphere $\mathbb{S}^2$ (fig. 2) then the equations of motion of the system on the tangent bundle $T^*\mathbb{S}^2$ of the two–dimensional sphere can be written in the form

$$\begin{aligned}&\ddot{\theta}+(\beta-H_1)\dot{\theta}\cos\theta+\beta\sin\theta\cos\theta-\\&-\dot{\psi}^2\mathrm{tg}\theta=0\end{aligned} \tag{40}$$

$$\begin{aligned}&\ddot{\psi}+(\beta-H_1)\dot{\psi}\cos\theta+\\&+\dot{\theta}\dot{\psi}\left(\tfrac{1+\cos^2\theta}{\cos\theta\sin\theta}\right)=0\end{aligned} \tag{41}$$

The phase pattern of this system is the following (fig. 3) Here, $\beta$ and $H_1$ are dimensionless physical constants and the coefficient $H_1$ is, as before, proportional to the rotary derivatives of the moment of hydrodynamic forces with respect to the components of the angular velocity of the three–dimensional pendulum. The sting length is equivalent to the distance $\sigma$ and the constant velocity of the oncoming flow is equivalent to the constant parameter $v$. The angle of attack of the free body is equivalent to the angle $\theta$ of the pendulum deviation from the flow velocity vector and the angle $\beta_1$ is equivalent to the cyclic variable (angle) $\psi$.

**Theorem 2.** *System (16)–(18) is topologically equivalent to the system given by Eqs. (40) and (41).*

We note that system of order three for pendulum for the case $\cos\theta = 0$ can be defined using continuity. And the singularity $\sin\theta = 0$ has the kinematic sense [Shamolin, M. V. (1994) New...], [Shamolin, M. V. (1996) The definition...], [Shamolin, M. V. (1996) Periodic...], [Shamolin, M. V. (1996) A variety...].

## 9 Conclusion

As is obvious from the foregoing account, only one aspect of the problem of motion of rigid bodies in resisting media was considered in the past. Namely, the primary interest of researchers was to obtain particular trajectories that could be described (yet only approximately) in explicit form. In this process, the problem of a more precise modeling of the interaction of a body in a resisting medium was considered concurrently. We briefly describe the latter problem for bodies of simple shape.

A plane plate is the simplest body for which various distinctive features of motion in a medium can be considered. The phenomena that are related to the effect of associated masses (the classical Kirchoff problem) are demonstrated in the manual written by Lamb by using the example of the motion of a plate in a fluid (the study of this effect, as is known, was initiated by Thompson, Tait, and Kirchoff).

The Kirchoff problem, which was stated in the second half of the 19th century, laid the basis for the consideration of the second aspect of the problem, namely, the integrability of a particular nonlinear system of differential equations that describes this particular motion (the problem of existence of analytic (smooth, meromorphic) first integrals) [Shamolin, M. V. (1996) An introduction...].

Because of its complexity, various versions of the Kirchoff problem up to now almost always have been considered from the standpoint of the integrability problem. Only in several cases was a qualitative analysis of a number of trajectories carried out. In works of Kirchhoff, Clebsch, Steklov, Lyapunov, Chaplygin, Kharlamov, and other scientists, the existence conditions for the additional analytic first integral are stated. New approaches to this problem are being elaborated at present. Thus the theory of integrable cases is constructed (the construction of an L–A-pair). The conditions for the nonexistence of the first integral of Kirchoff equations are found.

We point to the third aspect of this problem, namely, the qualitative analysis of sets of differential equations describing a given motion (phase–space fibering, qualitative arrangement of phase trajectories, symmetries, etc.). Although all the problems listed above are closely related to the integrability problem, solutions of them are of independent significance. Moreover, the consideration of this aspect of the problem would foster the elaboration of techniques for the qualitative analysis.

The *innovative* character of this paper is seen from the following:

> *new integrable cases and families of phase portraits in the plane dynamics of rigid bodies* are discovered. Certain model cases of the motion of rigid bodies in a resisting medium are qualitatively studied and integrated. The first integrals of corresponding systems are found; these integrals are transcendental functions and functions that can be expressed in terms of elementary functions;
>
> *new integrable cases and families of multidimensional phase portraits in the spatial dynamics of rigid bodies* are found. The problem of the three–dimensional motion of a dynamically symmetrically fixed rigid body placed in the flow of an incoming medium is integrated in the sense of Jacobi [Shamolin, M. V. (1997) On the integrable...], [Shamolin, M. V. (1997) Spatial...], [Shamolin, M. V. (1998) On the integrability...], [Shamolin, M. V. (1998) A family...].

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