SET-VALUED DYNAMICS IN PROBLEMS OF MATHEMATICAL THEORY OF CONTROL PROCESSES AND STATE ESTIMATION

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Abstract

We consider some approaches to study the dynamics and properties of set-valued states of differential control systems with uncertainties in initial data. It is assumed that the dynamical system has a special structure, in which nonlinear terms in the right-hand sides of related differential equations are quadratic in state coordinates. The model of uncertainty considered here is deterministic, with set-membership description of uncertain items which are taken to be unknown but bounded with given bounds. We construct external and internal ellipsoidal estimates of reachable sets of nonlinear control system and find differential equations of proposed ellipsoidal estimates of reachable sets of nonlinear control system. The results obtained for quadratic system nonlinearities are extended to other types of control systems under uncertainty. Numerical simulation results are also given.

Key words

Control problems, uncertainty, differential inclusion, estimation, ellipsoidal calculus, impulsive control.

1 Introduction

The topics of this paper come from the control theory for systems with unknown but bounded uncertainties related to the case of set-membership description of uncertainty [Bertsekas and Rhodes, 1971; Krasovskii and Subbotin, 1974; Kurzhanski, 1977; Milanese, Norton, Piet-Lahanier and Walter, 1996; Milanese and Vicino, 1991; Schweppe, 1973; Witsenhausen, 1968].

The motivation to consider a set-membership approach is that in the traditional formulation the characterization of parameter uncertainties requires assumptions on mean, variances or probability density function of errors. But very often in many applied areas ranged from engineering problems in physics to economics as well as to ecological modeling it occurs that a stochastic nature of the error sequence is questionable, for instance, in case of limited data or after some non-linear transformation of the data, the presumed stochastic characterization is not always valid. Hence, as an alternative to a stochastic characterization a so-called bounded-error characterization, also called set-membership approach, has been proposed in the last decades.

For models with linear dynamics under such set-membership uncertainty there are two constructive approaches which allow finding effective system state estimates. The first one is based on ellipsoidal calculus [Chernousko, 1994; Kurzhanski and Valyi, 1997; Kurzhanski and Varaiya, 2000; Polyak, Nazin, Durieu and Walter, 2004; Chernousko and Ovseevich, 2004] and the second one uses the interval analysis [Milanese, Norton, Piet-Lahanier and Walter, 1996; Kostousova and Kurzhanski, 1996; Walter and Pronzato, 1997].

However, in many applied problems including physical, ecological or economical applications the models are mostly nonlinear in their parameters (e.g., [Apreutesei, 2009; Ceccarelli, Di Marco, Garulli, and Giannitrapani, 2004]). Then, the set of feasible system states is usually non-convex or even non-connected. Nevertheless, non-linear set-membership approaches are able to give guaranteed inner or outer approximations for certain types of models. Hence, the key issue in non-linear set-membership estimation is to find suitable techniques, which are easy to interpret and which produce inner or outer bounds for the set of unknown system states without being too computationally demanding. Some approaches to the non-linear set-membership estimation problem are given by [Dontchev and Lempio, 1992; Panasyuk, 1990; Veliov, 1992; Wolenski, 1990; Chahma, 2003; Häckl, 1996] using a discrete approximation methods.

In this paper the modified state estimation approaches which use the special quadratic structure of nonlinearity of studied control system and use also the advantages of ellipsoidal calculus [Kurzhanski, 1977;

Chernousko, 1994] are presented. We develop here new ellipsoidal techniques related to constructing external and internal set-valued estimates of reachable sets and trajectory tubes of nonlinear systems.

The paper is organized as follows. After introducing some notations and standard definitions in the next section, the main problem is formulated is Section 3. Ellipsoidal external and internal (with respect to inclusion of sets) estimates are developed in Section 4. Differential equations describing the parameters of estimating ellipsoids are presented in Section 5. Extensions to impulsive control systems are discussed in Section 6. The examples to illustrate the theory are presented in Section 7. Finally, some concluding remarks are given.

2 Preliminaries

In this section we introduce some notations, standard definitions and necessary techniques related to considered problems.

2.1 Notation and Definitions

We start with the following basic notations. Let R^n be the n–dimensional Euclidean space and $x'y$ be the usual inner product of $x, y \in R^n$ with the prime as a transpose, with $\| x \| = (x'x)^{1/2}$.

Denote compR^n to be the variety of all compact subsets $A \subseteq R^n$ and convR^n to be the variety of all compact convex subsets $A \subseteq R^n$. We denote as $B(a, r)$ the ball in R^n, $B(a, r) = \{x \in R^n : \| x - a \| \le r\}$, I is the identity $n \times n$–matrix.

Denote by $E(a, Q)$ the ellipsoid in R^n, $E(a, Q) = \{x \in R^n : (x - a)'Q^{-1}(x - a) \le 1\}$ with a center $a \in R^n$ and a symmetric positive definite $n \times n$–matrix Q, for any $n \times n$–matrix $M = \{m_{ij}\}$ denote $Tr(M) = \sum_{i=1}^{i=n} m_{ii}$.

Let $h(A, B) = \max\{h^+(A, B),\ h^-(A, B)\}$ be the Hausdorff distance for $A,\ B \subset R^n$, with $h^+(A, B)$ and $h^-(A, B)$ being the Hausdorff semidistances between A and B, $h^+(A, B) = \sup\{d(x, B) \mid x \in A\}$, $h^-(A, B) = h^+(B, A)$, $d(x, A) = \inf\{\|x - y \| \mid y \in A\}$.

Consider the control system described by the ordinary differential equation

$$\dot{x} = f(t, x, u(t)),\ \ t \in [t_0, T] \tag{1}$$

with function $f : T \times R^n \times R^n \to R^m$ measurable in t and continuous in other variables. Here x stands for the state vector, t stands for time and control $u(\cdot)$ is a measurable function satisfying the constraints

$$u(\cdot) \in U = \{u(\cdot) : u(t) \in U_0,\ \ t \in [t_0, T]\} \tag{2}$$

where $U_0 \in \text{comp}R^m$.

Let us assume that the initial condition $x(t_0)$ to the system (1) is unknown but bounded

$$x(t_0) = x_0,\ \ x_0 \in X_0 \in \text{comp}R^n. \tag{3}$$

Let absolutely continuous function $x(t) = x(t, u(\cdot), t_0, x_0)$ be a solution to (1) with initial state x_0 satisfying (3) and with control function $u(t)$ satisfying (2). The differential system (1)–(3) is studied here in the framework of the theory of uncertain dynamical systems (differential inclusions [Aubin and Frankowska, 1990; Filippov, 1988]) through the techniques of trajectory tubes [Kurzhanski and Filippova, 1993]:

$$\begin{aligned} X(\cdot) = \bigcup \{ \ x(\cdot) = x(\cdot, u(\cdot), t_0, x_0) \mid \\ x_0 \in X_0,\ u(\cdot) \in U \ \}. \end{aligned} \tag{4}$$

2.2 Funnel Equations

Let us mention here the well-known result [Filippov, 1988] from the theory of differential inclusion. It consists in the fact that the trajectory tube $X(\cdot)$ coincides with the set of all solutions $\{x(\cdot) = x(\cdot, t_0, x_0)\}$ to the following differential inclusions

$$\dot{x} \in F(t, x),\ \ t \in [t_0, T], \tag{5}$$

with the initial state similar to (3)

$$x(t_0) = x_0,\ \ x_0 \in X_0, \tag{6}$$

where

$$F(t, x) = \bigcup \{ \ f(t, x, u) \ \mid \ u \in U_0 \ \}.$$

So we will use further the same notation $X(\cdot)$ for both trajectory tubes either for the control system (1)–(3) or for the differential inclusion (5)–(6).

One approach that we will use here refers to the theory of evolution equations of the funnel type [Kurzhanski and Filippova, 1993; Panasyuk, 1990; Wolenski, 1990]. Note first that we will consider the Caratheodory–type solutions $x(\cdot)$ for (5)–(6), i.e. absolutely continuous functions $x(t)$ that satisfy the inclusion (5) almost everywhere on $[t_0, T]$ (we will use further the abbreviation "for a.e. $t \in [t_0, T]$"). Assume that all solutions $\{x(t) = x(t, t_0, x_0) \mid x_0 \in X_0\}$ are extendable up to the instant T that is possible under some additional assumptions [Filippov, 1988].

Let us consider the funnel equation [Kurzhanski and Filippova, 1993; Panasyuk, 1990; Wolenski, 1990] related to the system (5)–(6)

$$\lim_{\sigma \to +0} \sigma^{-1} h(X(t + \sigma), \bigcup_{x \in X(t)} (x + \sigma F(t, x))) = 0, \tag{7}$$

$$X(t_0) = X_0,\ \ \ t \in [t_0, T]. \tag{8}$$

Under above mentioned assumptions the following theorem is true.

Theorem 1 ([Panasyuk, 1990]). *The nonempty compact valued function $X(t) = X(t, t_0, X_0)$ is the unique solution to the evolution equation (7)-(8).*

We would like to underline here that the funnel equation (7)-(8) can be interpreted as a set-valued analogy of an ordinary differential equation (of the first order) describing the dynamics of set-valued system state in the space of compact subsets of R^n.

Discrete approximations for differential inclusions through a set-valued analogy of well-known Euler's method were developed in [Dontchev and Lempio, 1992; Veliov, 1992]. Funnel equations for differential inclusions with state constraints were studied in [Kurzhanski and Filippova, 1993], the analogies of funnel equations for impulsive control systems were given in [Filippova, 2004; Filippova, 2005].

3 Problem Statement

One of the main problems of the theory of uncertain systems consists in describing and estimating the trajectory tube $X(\cdot)$ of the nonlinear system (1)- (3). The point of special interest is to find the t – cross-section $X(t)$ of $X(\cdot)$ which is actually the attainability domain (reachable set) of the control system (1)–(3) at the instant t. The set $X(t)$ may be considered also as the set-valued estimate of the unknown state $x(t)$ of the uncertain dynamical system if we will treat the control functions in (1)- (3) as unknown but bounded disturbances.

It should be noted that the exact description of reachable sets $X(t)$ of a control system is a very difficult problem even in the case of linear dynamics. The estimation theory and related algorithms basing on ideas of construction outer and inner set-valued estimates of reachable sets have been developed in [Kurzhanski and Valyi, 1997; Chernousko, 1994; Kurzhanski and Varaiya, 2000] for linear control systems.

The main problem of this research is to construct external and internal set-valued estimates of reachable sets $X(t)$ of the nonlinear system (1)–(3). The approaches presented here use the techniques of ellipsoidal calculus developed for linear control systems. It should be noted that ellipsoidal approximations of trajectory tubes may be chosen in various ways and several minimization criteria are well-known. We consider here the ellipsoidal techniques related to construction of estimates with optimal volume (details of this approach and motivations for linear control systems may be found in [Chernousko, 1994; Kurzhanski and Valyi, 1997]).

4 Ellipsoidal Estimates

In [Filippova, 2010] we presented techniques of constructing the external and internal ellipsoidal estimates of trajectory tubes $X(\cdot, t_0, X_0)$ based on the combination of ellipsoidal calculus [Chernousko, 1994; Kurzhanski and Valyi, 1997] and the techniques of evolution funnel equations of type (7)-(8). We considered there the differential inclusion of the following type

$$\dot{x} \in Ax + \tilde{f}(x)d + P(t), \; x_0 \in X_0, \; t_0 \le t \le T, \quad (9)$$

where $x \in R^n$, $\|x\| \le K$, $X_0 = E(a_0, Q_0)$, $P(t) = E(\hat{a}, \hat{Q})$, $d, a_0, \hat{a}$ are given n-vectors, a scalar function $\tilde{f}(x)$ has a form $\tilde{f}(x) = x'Bx$, matrices B, Q_0 and $\hat{Q}$ are symmetric and positive definite.

Theorem 2 ([Filippova, 2009]). *Assume that $X_0 = E(a, k^2B^{-1})$ (with some $k > 0$), then the following inclusion is true for all $\sigma > 0$*

$$\begin{gathered} X(t_0 + \sigma) \subseteq E(a^+(\sigma), Q^+(\sigma)) + \\ o(\sigma)B(0, 1), \; \lim_{\sigma \to +0} \sigma^{-1} o(\sigma) = 0. \end{gathered} \quad (10)$$

Here the ellipsoid $E(a^+(\sigma), Q^+(\sigma))$ is defined by the following relations

$$\begin{gathered} a^+(\sigma) = a(\sigma) + \sigma\hat{a}, \\ Q^+(\sigma) = (p^{-1} + 1)Q(\sigma) + (p + 1)\sigma^2\hat{Q}, \end{gathered} \quad (11)$$

where p is the unique positive root of the equation

$$\sum_{i=1}^{n} \frac{1}{p + \lambda_i} = \frac{n}{p(p + 1)},$$

with $\lambda_i \ge 0$ being the roots of the algebraic equation $|Q(\sigma) - \lambda\sigma^2\hat{Q}| = 0$, and

$$\begin{gathered} a(\sigma) = a + \sigma(Aa + a'Ba \cdot d + k^2 d), \\ Q(\sigma) = k^2(I + \sigma R)B^{-1}(I + \sigma R)', \\ R = A + 2da'B. \end{gathered} \quad (12)$$

Numerical algorithms basing on Theorem 2 and producing the discrete–time external ellipsoidal tube $E^+(t)$ are given in [Filippova, 2009].

Theorem 3 ([Filippova, 2010]). *Assume that $X_0 = E(a, k^2B^{-1})$ (with some $k > 0$), then the following inclusion is true for all $\sigma > 0$*

$$\begin{gathered} E(a^-(\sigma), Q^-(\sigma)) \subseteq X(t_0 + \sigma) + \\ o(\sigma)B(0, 1), \; \lim_{\sigma \to +0} \sigma^{-1} o(\sigma) = 0. \end{gathered} \quad (13)$$

Here the ellipsoid $E(a^-(\sigma), Q^-(\sigma))$ is defined by formulas

$$\begin{gathered} a^-(\sigma) = a(\sigma) + \sigma\hat{a}, \\ Q^-(\sigma) = Q(\sigma) + \sigma^2\hat{Q} + 2\sigma \times \\ Q(\sigma)^{1/2}(Q(\sigma)^{-1/2}\hat{Q}Q(\sigma)^{-1/2})^{1/2}Q(\sigma)^{1/2}, \end{gathered} \quad (14)$$

with $a(\sigma),\ Q(\sigma)$ satisfying the equations

$$\begin{aligned} a(\sigma) &= a + \sigma(Aa + a'Ba \cdot d + k^2 d), \\ Q(\sigma) &= k^2(I + \sigma R)B^{-1}(I + \sigma R)', \\ R &= A + 2da'B. \end{aligned} \tag{15}$$

Numerical algorithms basing on Theorem 3 and producing the discrete–time internal ellipsoidal tube $E^-(t)$ are given in [Filippova, 2010].

5 Differential Equations of Set-valued States

In this section we move from discrete to continuous case and find differential equations of proposed external and internal ellipsoidal estimates of reachable sets of the nonlinear system (9).

Let k_0^- and k_0^+ be such that the following inclusions hold true

$$\begin{aligned} &E(a_0, (k_0^-)^2 B^{-1}) \subseteq E(a_0, Q_0), \\ &E(a_0, Q_0) \subseteq E(a_0, (k_0^+)^2 B^{-1}). \end{aligned} \tag{16}$$

We assume that k_0^- is maximal and k_0^+ is minimal for which the inclusions (16) are true.

Theorem 4. *The inclusion is true for any $t \in [t_0, T]$*

$$X(t; t_0, X_0) \subseteq E(a^+(t), r^+(t)B^{-1}), \tag{17}$$

where functions $a^+(t),\ r^+(t)$ are the solutions of the following system of ordinary differential equations

$$\begin{aligned} \dot{a}^+(t) &= Aa^+(t) + ((a^+(t))'Ba^+(t) + \\ &\quad r^+(t))d + \hat{a},\ t_0 \le t \le T, \\ \dot{r}^+(t) &= \max_{\|l\|=1} \{ l' (2r^+(t)(B^{1/2}AB^{-1/2} + \\ &\quad 2B^{1/2}d\,(a^+(t))'B^{1/2} + q^{-1}(r^+(t)) \times \\ &\quad B^{1/2}\hat{Q}B^{1/2})l\} + q(r^+(t))r^+(t), \\ q(r) &= ((nr)^{-1} Tr(B\hat{Q}))^{1/2}, \end{aligned} \tag{18}$$

with initial state

$$a^+(t_0) = a_0,\ \ r^+(t_0) = (k_0^+)^2. \tag{19}$$

Proof. The proof of this theorem follows from Theorem 2 and may be easily derived from it by related limit procedures.

Theorem 5. *The inclusion is true for any $t \in [t_0, T]$*

$$E(a^-(t), r^-(t)B^{-1}) \subseteq X(t; t_0, X_0), \tag{20}$$

where functions $a^-(t),\ r^-(t)$ are the solutions of the following system of ordinary differential equations

$$\begin{aligned} \dot{a}^-(t) &= Aa^-(t) + ((a^-(t))'Ba^-(t) + \\ &\quad r^-(t))d + \hat{a},\ t_0 \le t \le T, \\ \dot{r}^-(t) &= 2 \min_{\|l\|=1} \{ l' (r^-(t)(B^{1/2}AB^{-1/2} + \\ &\quad 2B^{1/2}d(a^-(t))'B^{1/2}) + \\ &\quad (r^-(t))^{1/2}(B^{1/2}\hat{Q}B^{1/2})^{1/2})l\}, \end{aligned} \tag{21}$$

with

$$a^-(t_0) = a_0,\ \ r^-(t_0) = (k_0^-)^2. \tag{22}$$

Proof. The proof of this theorem follows from Theorem 3 and is similar to the proof of Theorem 4.

6 Possible Generalizations

The techniques of ellipsoidal calculus may be developed for estimating trajectory tubes of the following nonlinear impulsive control systems with uncertainty in initial data

$$\begin{aligned} dx(t) &= \big(Ax + \tilde{f}(x)d + u(t)\big)dt + Gdv(t), \\ &x \in R^n,\ t_0 \le t \le T, \end{aligned} \tag{23}$$

with unknown but bounded initial state

$$x(t_0 - 0) = x_0, \quad x_0 \in X_0 \in compR^n. \tag{24}$$

Here, A is a constant $n \times n$-matrix; the vectors $d, G \in R^n$; $u(t)$ is a classical (measurable) control function with constraints

$$u(t) \in U_0,\ \ U_0 \in compR^m,$$

and $v(t)$ is an increasing right-continuous scalar function of bounded variation defined on $[t_0, T]$ and satisfying the constraint (the parameter $\mu > 0$ is given)

$$\mathrm{Var}_{t \in [t_0, T]}\, v(t) \le \mu.$$

We assume as before that $\tilde{f}(x) = x'Bx$ where B is a symmetric positive definite $n \times n$-matrix.

Let $x(\cdot) = x(\cdot; t_0, x_0, u(\cdot), v(\cdot))$ be a solution of system (23) on the interval $[t_0, T]$ for admissible $x_0 \in X_0$ and controls $u(\cdot),\ v(\cdot)$. Let U be the class of admissible measurable controls $u(\cdot)$, and let V be the class of admissible controls–measures $v(\cdot)$. The trajectory tube of system (23) from the initial state $\{t_0, X_0\}$ is denoted as

$$\begin{aligned} X(\cdot) = X(\cdot; t_0, X_0) = \bigcup \big\{ &x(\cdot; t_0, x_0, u(\cdot), v(\cdot)) \mid \\ &x_0 \in X_0,\ u(\cdot) \in U,\ v(\cdot) \in V \big\}. \end{aligned}$$

Note that the cross-section $X(t) = X(t; t_0, X_0)$ of the trajectory tube $X(\cdot)$ at time $t \in [t_0, T]$ coincides with the reachable set of system (23) at this time t from the initial state $\{t_0, X_0\}$.

It should be noted that one of the principal points of interest of the theory of control under uncertainty conditions is to study the set of all solutions to (1)-(2) that satisfy an additional restriction on the state vector (the so-called "viability" constraint [Aubin and Frankowska, 1990]):

$$x(t) \in Y(t), \quad t \in [t_0, T], \tag{25}$$

where $Y(\cdot)$ is a known set-valued function. The constraint (25) may be induced by state constraints defined by a plant model or by the measurement equation [Kurzhanski, 1977]

$$y(t) = G(t)x(t) + w(t), \ w \in Q(t) \in compR^p,$$

where y is the measurement vector, $G(t)$ - a matrix function, w - the unknown but bounded "noise". The problem consists now in describing the set $X(\cdot)$ of all solutions to the system (23),(25) which is called also as the viable trajectory tube [Kurzhanski and Filippova, 1993].

Basing on the techniques of approximation of the discontinuous generalized trajectory tubes by the solutions of usual differential systems without measure terms we study the properties of trajectory tubes and reachable sets of the impulsive control system under uncertainty. Using a special discontinuous change of time [Rishel, 1965], we transform the impulsive system under consideration into an ordinary differential inclusion that no longer contains generalized functions.

Hence we may apply approaches described in previous sections and derive new estimates for set-valued states of impulsive dynamical systems studied under uncertainty and nonlinearity assumptions.

Details of this approach applied for some special classes of impulsive control systems in linear and nonlinear cases may be found also in [Filippova, 2004; Filippova, 2005; Vzdornova and Filippova, 2007].

7 Example

Consider the following control system

$$\begin{cases} \dot{x}_1 = 2x_1 + u_1, \\ \dot{x}_2 = 2x_2 + x_1^2 + x_2^2 + u_2, \end{cases} \tag{26}$$
$$x_0 \in X_0, \ 0 \le t \le T.$$

Here we take $t_0 = 0$, $T = 0.4$, $X_0 = B(0, 1)$ and put $U_0 = B(0, r)$ with $r = 0.01$ in the control constraint (2). We have $A = 2I$, $B = I$, $d_1 = 0$, $d_2 = 1$, $P(t) = B(0, r)$ for the system (26) written in the form (9).

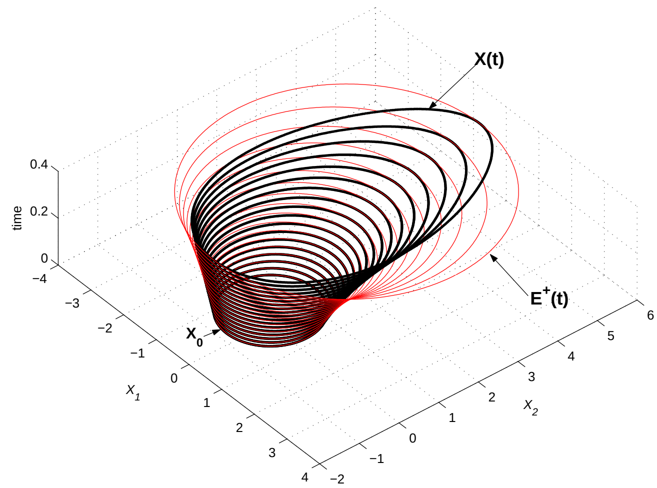

Figure 1. External ellipsoidal estimating tube $E^+(t)$ for $X(t)$.

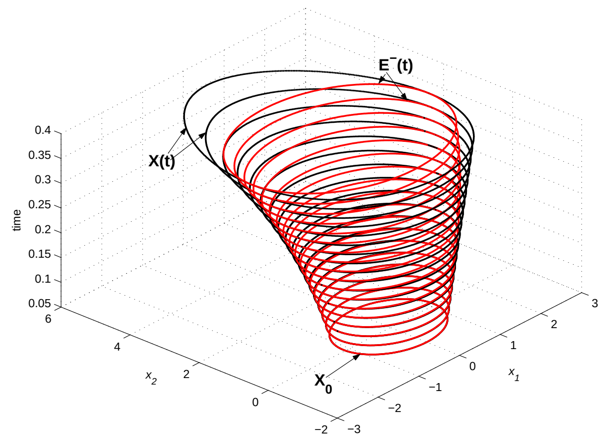

Figure 2. Internal ellipsoidal estimating tube $E^-(t)$ for $X(t)$.

External ellipsoidal tube $E^+(t) = E(a^+(t), Q^+(t))$ found by Theorem 4 and the trajectory tube $X(t)$ are shown at Figure 1. We see at the Figure 1 that the exact reachable set $X(t)$ lies inside the ellipsoidal estimate $E^+(t)$ which touches it at some points so the set $E^+(t)$ really presents the tight external estimate of $X(t)$.

Internal ellipsoidal tube $E^-(t) = E(a^-(t), Q^-(t))$ found by Theorem 5 for the trajectory tube $X(t)$ is shown at Figure 2. Figure 2 shows also that the ellipsoidal estimates $E^-(t)$ lie inside the set $X(t)$ and touch it at some points so these estimates are internal indeed and are exact in some sense.

8 Conclusion

The paper deals with the problems of control and state estimation for a dynamical control system described by differential inclusions with unknown but bounded initial state. The solution to the differential system is studied through the techniques of trajectory tubes with their cross-sections $X(t)$ being the reachable sets at instant t to control system.

Basing on the well-known results of ellipsoidal calculus developed for linear uncertain systems we present the modified state estimation approaches which use the special nonlinear structure of the control system and simplify calculations. Examples and numerical results related to procedures of set-valued approximations of trajectory tubes and reachable sets were also presented.

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