DISTURBANCE DECOUPLING FOR SINGULAR SYSTEMS BY PROPORTIONAL AND DERIVATIVE FEEDBACK AND PROPORTIONAL AND DERIVATIVE OUTPUT INJECTION

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Abstract
We study the disturbance decoupling problem for linear time invariant singular systems. We give necessary and sufficient conditions for the existence of a solution to the disturbance decoupling problem with or without stability via a proportional and derivative feedback and proportional and derivative output injection that also makes the resulting closed-loop system regular and/or of index at most one. All results are based on canonical reduced forms that can be computed using a complete system of invariants that can be implemented in a numerically stable way.

Key words
Singular Systems, equivalence relation, disturbance decoupling.

1 Introduction

We consider linear and time-invariant continuous singular systems of the form

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bu(t) + Gg(t), \; x(t_0) = x_0, \; t \geq 0 \\ y(t) = Cx(t), \end{cases} \tag{1}$$

(1) where $E, A \in M_n(C)$, $B \in M_{n\times m}(C)$, $C \in M_{p\times n}(C)$, $G \in M_{n\times q}(C)$ and $\dot{x} = dx/dt$. The term $g(t)$, $t \geq 0$, represents a disturbance, which may represent modeling or measuring errors, noise, or higher order terms in linearization. Singular systems arise naturally in circuits design, mechanical multibody systems and a large variety of the applications (see [5] and [6], for example), and they have been studied under different points of view. The problem of constructing feedbacks and/or output injections that suppress this disturbance in the sense that g(t) does not affect the input-output behavior of the system is analyzed. In the case of standard state space systems the disturbance decoupling problem has been largely studied (see [1],[7],[8] for example), This problem for singular systems has also been studied (see [2], [4] for example). In this paper we study the disturbance decoupling problem for singular systems that can be stated as follows: Find necessary and sufficient conditions under which we can choose state and derivative feedback as well state and derivative output injection such that, the matrix pencil $(E + BF_E^B + F_E^C C, A + BF_A^B + F_A^C C)$ is regular of index at most one and

$$C(s(E+BF_E^B+F_E^C C)-(A+BF_A^B+F_A^C C))^{-1}G = 0.$$

We assume without loss of generality that matrices B, G are full column rank and C is full row rank, i.e., $\operatorname{rank} B = m$, $\operatorname{rank} G = q$, $\operatorname{rank} C = p$. If this is not the case, then this can be easily achieved, by removing the nullspaces and appropriate renaming of variables.

2 Notations

In the sequel we will use the following notations.

- I_n denotes the n-order identity matrix,
- N denotes a nilpotent matrix in its reduced form $N = \operatorname{diag}(N_1, \ldots, N_t)$, $N_i = \begin{pmatrix} 0 & I_{n_i-1} \\ 0 & 0 \end{pmatrix} \in M_{n_i}(C)$,
- J denotes the Jordan matrix $J = \operatorname{diag}(J_1, \ldots, J_t)$, $J_i = \operatorname{diag}(J_{i_1}, \ldots, J_{i_s})$, $J_{i_j} = \lambda_i I_{i_j} + N$,
- L denotes the diagonal matrix $L = \operatorname{diag}(L_1, \ldots, L_q)$, where $L_j = \begin{pmatrix} I_{n_j} & 0 \end{pmatrix} \in M_{n_j\times(n_j+1)}(C)$,
- R denotes the diagonal matrix $R = \operatorname{diag}(R_1, \ldots, R_p)$, where $R_j = \begin{pmatrix} 0 & I_{n_j} \end{pmatrix} \in M_{n_j\times(n_j+1)}(C)$.

We represent systems of the form (1) as quadruples of matrices (E, A, B, C) in the case of disturbance do not appear or it is not considered and a quintuples of matrices (E, A, B, C, G) otherwise.

3 Reduced Form

We recall that, given a singular system (not necessarily square) using standard transformations in state, input and output spaces $x(t) = Px_1(t)$, $u(t) = Ru_1(t)$, $y_1(t) = Sy(t)$, premultiplication by an invertible matrix $QE\dot{x}(t) = QAx(t) + Qu(t)$ making feedback $u(t) = u_1(t) - Vx(t)$ and derivative feedback $u(t) = u_1(t) - U\dot{x}(t)$ as well as output injection $u(t) = u_1(t) - Wy(t)$ and derivative output injection $u(t) = u_1(t) - Z\dot{y}(t)$, it is possible to reduced to $E_r\dot{x}_1(t) = A_r x_1(t) + B_r u_1(t) + G_1$, $y_1 = C_r x(t)$ where

$$E_r = \begin{pmatrix} I_1 &&&&&&& \\ & I_2 &&&&&& \\ && I_3 &&&&& \\ &&& I_4 &&&& \\ &&&& N_1 &&& \\ &&&&& L_1 && \\ &&&&&& L_2^t & \\ &&&&&&& 0 \end{pmatrix}$$

$$A_r = \begin{pmatrix} N_2 &&&&&&& \\ & N_3 &&&&&& \\ && N_4 &&&&& \\ &&& J &&&& \\ &&&& I_5 &&& \\ &&&&& R_1 && \\ &&&&&& R_2^t & \\ &&&&&&& 0 \end{pmatrix}$$

$$B_r = \begin{pmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_3 \end{pmatrix}$$

$$C_r = \begin{pmatrix} C_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

or

$$E_r = \begin{pmatrix} I_1 &&&&&&& \\ & I_2 &&&&&& \\ && I_3 &&&&& \\ &&& I_4 &&&& \\ &&&& N_1 &&& \\ &&&&& L_1 && \\ &&&&&& L_2^t & 0 \end{pmatrix}$$

$$A_r = \begin{pmatrix} N_2 &&&&&&& \\ & N_3 &&&&&& \\ && N_4 &&&&& \\ &&& J &&&& \\ &&&& I_5 &&& \\ &&&&& R_1 && \\ &&&&&& R_2^t & 0 \end{pmatrix}$$

$$B_r = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$C_r = \begin{pmatrix} C_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_3 \end{pmatrix}$$

and

i) (I_1, N_2, B_1, C_1) is a n_1 size completely controllable and observable system.
ii) (I_2, N_3, B_2) is a n_2 size completely controllable non observable system.
iii) (I_3, N_4, C_2) is a n_3 size completely observable non controllable system.
iv) (I_4, J) is a n_4 size system having only finite zeroes.
v) (N_1, I_5) is a n_5 size system having only non transferable infinite zeroes.
vi) (L_1, R_1), completely singular systems of n_6 rows.
vii) (L_2^t, R_2^t) completely singular systems of n_7 rows respectively.
viii) $B_3 = \text{diag}(I_{n_{8_1}}, 0_{n_{8_2}})$ or $C_3 = \text{diag}(I_{n_{9_1}}, 0_{n_{9_2}})$.

$(\sum_{i=1}^{7} n_i + n_{8_1} + n_{8_2} = n)$ or $(\sum_{i=1}^{7} n_i + n_{9_1} + n_{9_2} = n)$.

The regular part of the system is maximal among all possible reductions of the system decomposing it in a regular part and a singular part.

Remark 1. *Not all parts i),..., viii), necessarily appears in the decomposition of a system.*

The proof is based in the following proposition.

Proposition 1. *Two quadruples of matrices (E_i, A_i, B_i, C_i) are equivalent under equivalence relation considered if and only if the matrix pencils* $\lambda \begin{pmatrix} E_i & B_i & 0 \\ C_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} A_i & 0 & B_i \\ 0 & 0 & 0 \\ C_i & 0 & 0 \end{pmatrix}$ *are strictly equivalent.*

Based on reduced form, the system (1), is reduced to the following independent subsystems.

$$\begin{cases} \dot{x}_1 = N_2 x_1 + B_1 u_1 + G_1 g_1 \\ y_1 = C_1 x_1 \end{cases} \tag{2}$$

$$\{\dot{x}_2 = N_3x_2 + B_2u_2 + G_2g_2 \tag{3}$$

$$\begin{cases} \dot{x}_3 = N_4x_3 + G_3g_3 \\ y_3 = C_2x_3 \end{cases} \tag{4}$$

$$\{\dot{x}_4 = Jx_4 + G_4g_4 \tag{5}$$

$$\{N_1\dot{x}_5 = x_5 + G_5g_5 \tag{6}$$

$$\{L_1\dot{x}_6 = R_1x_6 + G_6g_6 \tag{7}$$

$$\{L_2^t\dot{x}_7 = R_2^tx_7 + G_7g_7 \tag{8}$$

$$\{B_3u_3 = 0 \quad \text{or} \quad \{C_3x_8 = 0. \tag{9}$$

Systems from (2) to (6) are regular and (7), (8) are completely singular and there are not feedbacks, derivative feedbacks, output injections and derivative output injections regularizing partially or totally the systems (7) and (8).

4 The disturbance decoupling problem

In this section we will use the reduced form for the system in order to analyze the disturbance decoupling problem.

Proposition 2. *Consider a system of the form (1). The system can be regularized by means a state and derivative feedback as well state a derivative output injection with index at most one if and only if the reduced form does not contain parts vi), vii), and viii), and if it contains v), the nilpotent matrix N_1 is the zero matrix.*

Proof. It suffices to observe that a system is regularisable if and only if the reduced form is regularisable and the index of the system is the index of matrix N_1.

Theorem 1. *Consider a system of the form (1). The system can be regularized by means a state and derivative feedback as well state a derivative output injection with index at most one if and only if*

i) $r_1 - r_0 \geq n$,
ii) $s_k \leq 2(r_B - t)$.
iii) $l_k \leq 2(r_C - t)$,

where
- $r_0 = \text{rank} \begin{pmatrix} E & B \\ C & 0 \end{pmatrix}$
- $r_1 = \text{rank} \begin{pmatrix} E & B & & \\ C & 0 & & \\ A & 0 & E & B \\ & & C & 0 \end{pmatrix}$
- *s_k is the number of column minimal indices of the pencil* $\lambda \begin{pmatrix} E & B & 0 \\ C & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ C & 0 & 0 \end{pmatrix}$
- $r_B = \text{rank}\, B$
- *l_k is the number of row minimal indices of the pencil* $\lambda \begin{pmatrix} E & B & 0 \\ C & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ C & 0 & 0 \end{pmatrix}$
- $r_C = \text{rank}\, C$
- $t = r_n - r_{n-1} - n$
- $r_\ell = \text{rank}\, M_\ell$

$$M_\ell = \begin{pmatrix} E & B & & & & & & \\ C & 0 & & & & & & \\ A & 0 & E & B & & & & \\ & & C & 0 & & & & \\ & & A & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & E & B & & \\ & & & & & C & 0 & & \\ & & & & & A & 0 & E & B \\ & & & & & & & C & 0 \end{pmatrix}$$

$\in M_{(\ell+1)(n+p)\times(\ell+1)(n+m)}(C)$.

Proof. It suffices to observe that the subsystem controllable and observable joint with subsystem (N_1, I_5), correspond to the infinite zeros of the pencil associate. Controllable non observable subsystem correspond to the column singular part of the pencil and observable non controllable subsystem correspond to the row singular part of the pencil.

Using quadruples in its reduced form, extending the equivalence to the quintuples of matrices (i.e. $QG = \overline{G}$) and taking into account [2], lemma 2.4, we have the following proposition.

Proposition 3. *Assume* $\overline{G} = \begin{pmatrix} G_1 \\ \vdots \\ G_5 \end{pmatrix}$ *according to the subsystems (2), . . ., (6). Let $s \in C$ such that* $\det(sI_{n_1} - N_2) \neq 0$, $\det(sI_{n_2} - N_3) \neq 0$, $\det(sI_{n_3} - N_4) \neq 0$, $\det(sI_n - J) \neq 0$ *and* $\det(sN_1I_{n_5}) \neq 0$, *(it exists because of regularity of the subsystems (2),..., (6)). Then*

i) $C_1(sI_{n_1} - N_2)^{-1}G_1 = 0$ *if and only if* $\text{rank} \begin{pmatrix} sI_{n_1} - N_2 & G_1 \\ C_1 & 0 \end{pmatrix} = n_1$,
ii) $(sI_{n_2} - N_3)^{-1}G_2 = 0$ *if and only if* $G_2 = 0$

iii) $C_1(sI_{n_3} - N_4)^{-1}G_3 = 0$ *if and only if* rank $\begin{pmatrix} sI_{n_3} - N_4 & G_3 \\ C_2 & 0 \end{pmatrix} = n_3$

iv) $(sI_{n_4} - J)^{-1}G_4 = 0$ *if and only if* $G_4 = 0$

v) $(sN_1 - I_{n_5})^{-1}G_5 = 0$ *if and only if* $G_5 = 0$

As a consequence we have.

Corollary 1. *Let* (E, A, B, C, G) *a quintuple of matrices in its reduced form, and we assume* $\overline{G} = \begin{pmatrix} G_1 \\ \vdots \\ G_5 \end{pmatrix}$ *according to the decomposition of the system. If* $G_2 = 0$, $G_4 = 0$, $G_5 = 0$, rank $\begin{pmatrix} sI_{n_1} - N_2 & G_1 \\ C_1 & 0 \end{pmatrix} = n_1$ *and* rank $\begin{pmatrix} sI_{n_3} - N_2 & G_3 \\ C_1 & 0 \end{pmatrix} = n_3$, *then the given system is trivially disturbance decoupled.*

The disturbance decoupling problem is called with stability if one imposes the additional constraint that the close-loop $(E+BF_E^B+F_E^C C)\dot{x}(t) = (A+BF_A^B+F_A^C C)x(t) + Bu(t) + Gg(t)$, $y(t) = Cx(t)$ system is stable. Remember that a singular system is stable if and only if the spectrum of the system lies in C^{-1}.

Proposition 4. *Given a singular system* (E, A, B, C). *There exist a proportional and derivative feedback as well a proportional and derivative output injection such that the close-loop system* $(E+BF_E^B+F_E^C C, A+BF_A^B + F_A^C C, B, C)$ *is stable (and we call stable under proportional and derivative feedback and proportional and derivative output injection) if and only if* rank $\begin{pmatrix} sE - A & B \\ C & 0 \end{pmatrix} = n$, $\forall s \in C^+$.

Proof. The spectrum of a system coincides with the spectrum of the associate pencil, and the spectrum is invariant under equivalence relation.

As a consequence we have.

Corollary 2. *Let* (E, A, B, C, G) *a quintuple of matrices in its reduced form, and we assume* $\overline{G} = \begin{pmatrix} G_1 \\ \vdots \\ G_5 \end{pmatrix}$ *according to the decomposition of the system. If* $G_2 = 0$, $G_4 = 0$, $G_5 = 0$, rank $\begin{pmatrix} sI_{n_1} - N_2 & G_1 \\ C_1 & 0 \end{pmatrix} = n_1$, rank $\begin{pmatrix} sI_{n_3} - N_4 & G_3 \\ C_1 & 0 \end{pmatrix} = n_3$ *and* $\sigma(J) \subset C^{-1}$. *Then the given system is trivially disturbance decoupled with stability.*

5 Conclusions

In this paper a qualitative description of the disturbance decoupling problem is considered and a necessary and sufficient condition for the existence of a proportional and derivative feedback as well a proportional and derivative output injection such that the close-loop system is regular with index at most one and for systems in its reduced form a condition for decoupling is presented.

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