ON INVARIANTS BY FEEDBACK OF A FAMILY OF LINEAR DYNAMICAL SYSTEMS

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Abstract
In this paper we try the problem to provide a set of invariants for pointwise feedback equivalence of linear systems. This relation has sense in the case of rings of real continuous functions defined on a compact topological space.

We study, in general, the n-dimensional case.

Key words

1 Introduction
Let consider the family systems:

\[ \Sigma(\lambda) = \begin{cases} \dot{x} = A(\lambda)x(t) + B(\lambda)u(t) \\ y(t) = C(\lambda)x(t) \end{cases} \]

(1)

where \(A(\lambda), B(\lambda)\) and \(C(\lambda)\) are matrices with elements in the set of continuous functions of a compact topological space \(\Lambda\) in \(\mathbb{R}\), denoted by \(C(\Lambda, \mathbb{R})\).

It is well known that if the matrices have constant coefficients then there is a canonical form for \(\Sigma\) (Brunovsky’s canonical form [Brunovsky, 1970]).

2 Feedback classification problem
Throughout this paper \(R\) denotes a commutative ring with unit element. We consider an \(m\)-input, \(n\)-dimensional linear dynamical system \(\Sigma = (A,B)\) over \(R\), where \(A\) and \(B\) are \(n \times n\) and \(n \times m\) matrices with entries in \(R\) respectively. Let’s assume the system is reachable (i.e. the columns of the \(n \times nm\) block matrix \(A+B = (B, AB, \ldots, A^{n-1}B)\) generates \(R^n\)). \(\Sigma' = (A', B')\) is feedback equivalent to \(\Sigma\) when \(\Sigma\) can be transformed to \(\Sigma'\) by one element of the feedback group \(\mathbb{F}_{nm}(R)\) and we will note this by \(\Sigma \sim \Sigma'\). For the reader’s convenience we recall that \(\mathbb{F}_{nm}(R)\) is the generated group by the following three types of transformations:

(1) \(A \rightarrow A' = PAP^{-1}, B \rightarrow B' = PB\) for some invertible matrix \(P\). The transformation is a consequence of a change of base in \(R^n\), the state module.

(2) \(A \rightarrow A, B \rightarrow B' = BQ\) for some invertible matrix \(Q\). The transformation is a consequence of change of base in \(R^n\), the input module.

(3) \(A \rightarrow A' = A + BK, B \rightarrow B\) for some \(m \times n\) matrix \(K\) which is called a feedback matrix.

The feedback classification problem is what is known as a wild problem and is open in the general case. However in some cases it is posible to obtain a solution. When \(R\) is a field the problem is known as classical case and a classical result of Brunovsky [Brunovsky, 1970] characterizes the class of equivalence of \(\Sigma\) by the action of the feedback group as follows.

Theorem 1. Let \(\Sigma = (A,B)\) be a reachable linear dynamical system of size \((m,n)\) (i.e. \(m\)-input, \(n\)-dimensional) over a field \(R = K\). Then there exist positive integers \(\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_s\) uniquely determined by \(\Sigma\) with \(n = \kappa_1 + \kappa_2 + \cdots + \kappa_s\) such that \(\Sigma\) is feedback equivalent to the system \(\Sigma_{\kappa} = (A_{\kappa}, B_{\kappa})\) where \(A_{\kappa}\) is the block matrix

\[
A_{\kappa} = \begin{pmatrix}
A_{\kappa_1} & 0 & \cdots & 0 \\
0 & A_{\kappa_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{\kappa_s}
\end{pmatrix},
\]

with \(A_{\kappa}\) the \(\kappa_i \times \kappa_i\) matrix

\[
A_{\kappa_i} = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]
and

\[
B_{\kappa} = \begin{pmatrix}
\cdots & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 1
\end{pmatrix}
\]

\[
\mathcal{N}_{i}^{\Sigma} = N_{i-1}^{\Sigma} + \mathbb{F} \mathcal{N}_{i}^{\Sigma}
\]

The integers \(\kappa = \{\kappa_{1}, \kappa_{2}, \ldots, \kappa_{s}\}\) are called the Kronecker indices of \(\Sigma\). They are a complete set of invariants for \(\Sigma\) by the action of the feedback group.

**Proof.** See [Brunovsky, 1970].

Information over the feedback classification theorem can be found in [Kalman, 1972; Wonham and Morse, 1972]. \(\Sigma_{\kappa} = (A_{\kappa}, B_{\kappa})\) is called Brunovsky’s form associated to \(\kappa\). In general, if \(R\) is an arbitrary commutative ring and \(\Sigma = (A, B)\) is an \(m\)-input \(n\)-dimensional system over \(R\), \(\Sigma\) is not feedback equivalent to a system \(\Sigma_{\kappa} = (A_{\kappa}, B_{\kappa})\) when \(A_{\kappa}\) and \(B_{\kappa}\) are matrices as in the above theorem. An example it is shown in [Hermida-Alonso, Pérez and Sánchez-Giralda, 1995].

### 3 Sets of invariants over a ring

Let \(\Sigma = (A, B)\) be a linear dynamical system of size \((m, n)\) over \(R\). We introduce some notation. \(N_{i}^{\Sigma}\) denotes the submodule of \(R\) generated by the columns of the \(n \times \text{im}\) matrix

\[
(A \ast B)_{i} = (B, AB, \ldots, A^{i-1}B)
\]

and we denote by \(M_{i}^{\Sigma}\) the submodule

\[
M_{i}^{\Sigma} = R^{n}/N_{i}^{\Sigma}
\]

for \(1 \leq i \leq n\).

The following result contains the main properties of these modules.

**Proposition 1.** Let \(\Sigma = (A, B)\) be a linear dynamical system of size \((m, n)\) over a ring \(R\). Then

(i) \((0) \subseteq N_{0}^{\Sigma} \subseteq N_{1}^{\Sigma} \subseteq \cdots \subseteq N_{n}^{\Sigma}\).

(ii) The canonical homomorphism

\[
\varphi_{i} : N_{i}^{\Sigma}/N_{i-1}^{\Sigma} \rightarrow N_{i+1}^{\Sigma}/N_{i}^{\Sigma}
\]

is surjective for \(1 \leq i \leq n-1\).

(iii) If \(\Sigma\) is feedback equivalent to \(\Sigma'\) then \(N_{i}^{\Sigma}\) and \(M_{i}^{\Sigma}\) are isomorphic to \(N_{i}^{\Sigma'}\) and \(M_{i}^{\Sigma'}\) respectively, for \(1 \leq i \leq n\).

(iv) If \(\Sigma\) is a reachable system of simple input \(n\)-dimensional then the modules \(\{N_{i}^{\Sigma}\}_{1 \leq i \leq n}\) and \(\{M_{i}^{\Sigma}\}_{1 \leq i \leq n}\) are free.

(v) If \(\Sigma\) is a Brunovsky system then the modules \(\{N_{i}^{\Sigma}\}_{1 \leq i \leq n}\) and \(\{M_{i}^{\Sigma}\}_{1 \leq i \leq n}\) are free.

**Proof.** See [Hermida-Alonso, Pérez and Sánchez-Giralda, 1996].

As a consequence when \(R = \mathbb{R}\) we have the following result.

**Corollary 1.** Let \(\Sigma = (A, B)\) be a reachable linear dynamical system of size \((m, n)\) over \(\mathbb{R}\). Then the feedback equivalence class of \(\Sigma\) is characterized for each one of the following sets:

(i) The Kronecker’s indices \(\{\kappa_{i}\}_{1 \leq i \leq s}\).

(ii) \(\{\text{rank}_{\mathbb{R}} N_{i}^{\Sigma}\}_{1 \leq i \leq s}\).

(iii) \(\{\text{rank}_{\mathbb{R}} N_{i}^{\Sigma}/N_{i-1}^{\Sigma}\}_{1 \leq i \leq n}\).

**Proof.** See [Hermida-Alonso, Pérez and Sánchez-Giralda, 1996].

Let \(M = (a_{ij})\) be an \(n \times m\) matrix with entries in \(R\) and let \(i\) be a nonnegative integer. The \(i\)-th determinantal ideal of \(M\), denoted by \(U_{i}(M)\), is the ideal of \(R\) generated by all the \(i \times i\) minors of \(M\). By construction we have

\[
R = U_{0}(M) \supseteq U_{1}(M) \supseteq \cdots \supseteq U_{n}(M) \supseteq \cdots
\]

and \(U_{i}(M) = 0\) for \(i > \min \{m, n\}\). The rank of \(M\), denoted by \(\text{rank}_{R}(M)\), is the largest \(i\) such that \(U_{i}(M) \neq 0\). Then \(\Sigma\) is reachable if and only if \(U_{n}(A + B) = R\).

**Proposition 2.** Let \(R = K\) be a field and \(\Sigma = (A, B)\) a reachable linear dynamical system of size \((m, n)\) over \(R\). We put \(\sigma_{i}^{\Sigma} = \dim_{K} M_{i}^{\Sigma}\) for \(1 \leq i \leq n\). Then \(\{\sigma_{i}^{\Sigma}\}_{1 \leq i \leq n}\) is a complete set of invariants of the class of equivalence of \(\Sigma\) (i.e., \(\Sigma\) is feedback equivalent to \(\Sigma'\) if and only if \(\sigma_{i}^{\Sigma} = \sigma_{i}^{\Sigma'}\) for all \(i\) with \(1 \leq i \leq n\)).

**Proof.** See [Carriego, Hermida-Alonso and Sánchez-Giralda, 1998].

Denote by

\[
\psi_{M} : R^{m} \longrightarrow R^{n}
\]

the homomorphism of free \(R\)-modules defined respect to the standard basis by the matrix \(M\). Consider \(\text{Coker}\psi_{M}\) and the following exact sequence
Since the system $\Sigma(\lambda)$ is a system over the field $\mathbb{R}$ it is necessary to know what a set of invariants. In this line, we give the following result. But first we recall some definitions. Let $n$ a positive integer and $\kappa = \{\kappa_1, \kappa_2, \ldots, \kappa_s\}$ a partition of $n$ ($n = \kappa_1 + \kappa_2 + \cdots + \kappa_s$ with $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_s$). We’ll call conjugate partition of $\kappa$ to the partition $\eta = \{n_1, n_2, \ldots, n_p\}$ (with $n_1 \geq n_2 \geq \cdots \geq n_p$) of $n$ where $n_i$ is the number of $\kappa_j$ bigger or equal than $i$. We denote $P_n$ the set of partitions of $n$. The application $\kappa \to \eta$ is a bijection in the set of partitions of $n$, see [Biggs, 2002].

**Theorem 3.** Let $\Sigma = (A, B)$ and $\Sigma' = (A', B')$ be two reachable linear dynamical systems of size $(m, n)$ over $R = C(\Lambda, \mathbb{R})$. For $\lambda_0 \in \Lambda$, the following conditions are equivalent.

(i) $\Sigma(\lambda_0) \sim \Sigma_\kappa$ where $\Sigma_\kappa = (A_\kappa, B_\kappa)$ is the Brunovsky’s linear form associated to the Kronecker’s indices $\kappa = \{\kappa_1, \kappa_2, \ldots, \kappa_s\}$.

(ii) Let $\eta = \{n_1, n_2, \ldots, n_p\}$ be the conjugate partition of $\kappa$. Then

$$\dim_{\mathbb{R}}(N^{\Sigma(\lambda_0)}_i) = n_1 + n_2 + \cdots + n_i; \quad 1 \leq i \leq p.$$ 

**Proof.** See [García-Fernández, 2005].

In order to prove our main result we need to remark usual notation of ideal of zeros of a function. Let $a$ be an ideal of $R = C(\Lambda, \mathbb{R})$. We denote by $Z(a)$ the set

$$Z(a) = \{\lambda \in \Lambda / f(\lambda) = 0 \text{ for all } f \in a\}.$$ 

**Lemma 2.** Let be $a$ and $b$ two finite generated ideals of $C(\Lambda, \mathbb{R})$. Then

(i) There is $a \subseteq a$ with $Z(a) = Z(a)$.

(ii) If $a \subseteq b$ then we can choose $a \in a$ and $b \in b$ with $a = \lambda b$ where $Z(a) = Z(a) \supseteq Z(b) = Z(b)$.


If $R = C(\Lambda, \mathbb{R})$ is a ring continuous functions of a compact topological space $\Lambda$ in $\mathbb{R}$, then the pointwise feedback relation is characterized by the invariant sets

$$\{Z(U_j ((A * B)_{\lambda_j}))\}_{1 \leq i \leq n, 1 \leq j \leq n}$$

in the following form.

**Theorem 4.** Let be $\Sigma = (A, B)$ and $\Sigma' = (A', B')$ two reachable linear dynamical systems of size $(m, n)$ over $R = C(\Lambda, \mathbb{R})$. Then the following statements are equivalent.
for all $\lambda \in \Lambda$.

(ii) $Z(\mathcal{U}_j((A * B)_i)) = Z(\mathcal{U}_j((A' * B')_i))$ for $1 \leq i \leq n$, $1 \leq j \leq n$.

Proof. (i) $\Rightarrow$ (ii) As $\Sigma(\lambda) \sim \Sigma'(\lambda)$ for all $\lambda \in \Lambda$, we have by Corollary 2

$$\mathcal{U}_j((A(\lambda) * B(\lambda))_i) = \mathcal{U}_j((A'(\lambda) * B'(\lambda))_i)$$

for $1 \leq i \leq n$, $1 \leq j \leq n$ and for all $\lambda \in \Lambda$. It follows that

$$Z(\mathcal{U}_j((A * B)_i)) = Z(\mathcal{U}_j((A' * B')_i)).$$

(ii) $\Rightarrow$ (i) Conversely if

$$Z(\mathcal{U}_j((A * B)_i)) = Z(\mathcal{U}_j((A' * B')_i))$$

for $1 \leq i \leq n$, $1 \leq j \leq n$ we have

$$\text{rank}_R(\mathcal{U}_j((A(\lambda) * B(\lambda))_i)) =$$

$$\text{rank}_R(\mathcal{U}_j((A'(\lambda) * B'(\lambda))_i))$$

for $1 \leq i \leq n$ and for all $\lambda \in \Lambda$ or equivalent

$$\dim_R N_1^\Sigma(\lambda) = \dim_R N_1^{\Sigma'(\lambda)}$$

for $1 \leq i \leq n$ and for all $\lambda \in \Lambda$, by Corollary 1.

$$\dim_R N_1^\Sigma(\lambda) = \dim_R N_1^{\Sigma'(\lambda)}$$

is a complete set of invariants for the feedback equivalence over $R$, then

$$\Sigma(\lambda) \sim \Sigma'(\lambda)$$

for all $\lambda$ of $\Lambda$.

For general reading on the subject, see [Carriegos, Hermida-Alonso and Sánchez-Giralda, 1998].

5 Conclusion

The problem of obtain a set of invariants for poinwise feedback equivalence over $R = C(\Lambda, \mathbb{R})$ has been considered. Some questions must be the subject of future research.

Question 1. The question of reducing the number of invariants

$$\{Z(\mathcal{U}_j((A * B)_i))\}_{1 \leq i \leq n, 1 \leq j \leq n}$$

Question 2. Given a set of closed sets over a compact $\Lambda$, $F_1 \supseteq F_2 \supseteq \ldots \supseteq F_s$

Is it possible to find a system $\Sigma$ over $C(\Lambda, \mathbb{R})$ where

$$Z(\mathcal{U}_j((A * B)^{\Sigma})_i)) = F_k?$$

Question 3. To extend results to the ring $C^k(\Lambda, \mathbb{R})$ where $\Lambda$ is a differentiable manifold.

Question 4. To extend results to the ring of holomorphic functions $H(\Omega)$ where $\Omega \subseteq \mathbb{C}$.

References


