ON INVARIANTS BY FEEDBACK OF A FAMILY OF LINEAR DYNAMICAL SYSTEMS

R. Marta García Fernández
Department Mathematics
University of León
Spain
marta.garcia@unileon.es

Miguel V. Carriegos
Department Mathematics
University of León
Spain
miguel.carriegos@unileon.es

Abstract
In this paper we try the problem to provide a set of invariants for pointwise feedback equivalence of linear systems. This relation has sense in the case of rings of real continuous functions defined on a compact topological space.
We study, in general, the n-dimensional case.

Key words
Feedback classification. Systems over commutative rings. Controllability.

1 Introduction

Let consider the family systems:

$$\Sigma(\lambda) = \begin{cases} \dot{x} = A(\lambda)x(t) + B(\lambda)u(t) \\ y(t) = C(\lambda)x(t) \end{cases} \tag{1}$$

where $A(\lambda), B(\lambda)$ and $C(\lambda)$ are matrices with elements in the set of continuous functions of a compact topological space Λ in $\mathbb{R}$, denoted by $C(\Lambda, \mathbb{R})$.

It is well known that if the matrices have constant coefficients then there is a canonical form for Σ (Brunovsky's canonical form [Brunovsky, 1970]).

2 Feedback classification problem

Throughout this paper R denotes a commutative ring with unit element. We consider an m-input, n-dimensional linear dynamical system $\Sigma = (A, B)$ over R, where A and B are $n \times n$ and $n \times m$ matrices with entries in R respectively. Let's assume the system is reachable (i. e. the columns of the $n \times nm$ block matrix $A * B = (B, AB, \ldots, A^{n-1}B)$ generates R^n). $\Sigma' = (A', B')$ is feedback equivalent to Σ when Σ can be transformed to Σ' by one element of the feedback group $\mathbb{F}_{nm}(R)$ and we will note this by $\Sigma \sim \Sigma$. For the reader's convenience we recall that $\mathbb{F}_{nm}(R)$ is the generated group by the following three types of transformations:

(1) $A \longrightarrow A' = PAP^{-1}$, $B \longrightarrow B' = PB$ for some invertible matrix P. The transformation is a consequence of a change of base in R^n, the state module.
(2) $A \longrightarrow A$, $B \longrightarrow B' = BQ$ for some invertible matrix Q. The transformation is a consequence of change of base in R^m, the input module.
(3) $A \longrightarrow A' = A + BK$, $B \longrightarrow B$ for some $m \times n$ matrix K which is called a feedback matrix.

The feedback classification problem is what is known as a wild problem and is open in the general case. However in some cases it is posible to obtain a solution. When R is a field the problem is known as classical case and a classical result of Brunovsky [Brunovsky, 1970] characterizes the class of equivalence of Σ by the action of the feedback group as follows.

Theorem 1. *Let $\Sigma = (A, B)$ be a reachable linear dynamical system of size (m, n) (i.e. m-input, n-dimensional) over a field $R = K$. Then there exist positive integers $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_s$ uniquely determined by Σ with $n = \kappa_1 + \kappa_2 + \cdots + \kappa_s$, such that Σ is feedback equivalent to the system $\Sigma_\kappa = (A_\kappa, B_\kappa)$ where A_κ is the block matrix*

$$A_\kappa = \begin{pmatrix} A_{\kappa_1} & 0 & \cdots & 0 \\ 0 & A_{\kappa_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{\kappa_s} \end{pmatrix},$$

with A_{κ_i} the $\kappa_i \times \kappa_i$ matrix

$$A_{\kappa_i} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$
B_\kappa = \left(\begin{array}{c|c|c|c|ccc}
\multicolumn{4}{c}{\overbrace{\hspace{6em}}^{s}} & \multicolumn{3}{c}{\overbrace{\hspace{4em}}^{m-s}} \\
0 & & & & 0 & \cdots & 0 \\
\vdots & \mathit{0} & \cdots & \mathit{0} & & & \\
1 & & & & 0 & \cdots & 0 \\ \hline
 & 0 & & & 0 & \cdots & 0 \\
\mathit{0} & \vdots & \cdots & \mathit{0} & & & \\
 & 1 & & & 0 & \cdots & 0 \\ \hline
\vdots & \vdots & \ddots & \vdots & & \vdots & \\ \hline
 & & & 0 & 0 & \cdots & 0 \\
\mathit{0} & \mathit{0} & \cdots & \vdots & & & \\
 & & & 1 & 0 & \cdots & 0
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \kappa_1 \\
\left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \kappa_2 \\
\\
\left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \kappa_s
\end{array}
$$

The integers $\kappa = \{\kappa_1, \kappa_2, \ldots, \kappa_s\}$ are called the Kronecker indices of Σ. They are a complete set of invariants for Σ by the action of the feedback group.

Proof. See [Brunovsky, 1970].

Information over the feedback classification theorem can be found in [Kalman, 1972; Wonham and Morse, 1972]. $\Sigma_\kappa = (A_\kappa, B_\kappa)$ is called Brunovsky's form associated to κ. In general, if R is an arbitrary commutative ring and $\Sigma = (A, B)$ is an m-input n-dimensional system over R, Σ is not feedback equivalent to a system $\Sigma_\kappa = (A_\kappa, B_\kappa)$ when A_κ and B_κ are matrices as in the above theorem. An example it is shown in [Hermida-Alonso, Pérez and Sánchez-Giralda, 1995].

3 Sets of invariants over a ring

Let $\Sigma = (A, B)$ be a linear dynamical system of size (m, n) over R. We introduce some notation. N_i^Σ denotes the submodule of R generated by the columns of the $n \times im$ matrix

$$(A * B)_i = (B, AB, \ldots, A^{i-1}B)$$

and we denote by M_i^Σ the submodule

$$M_i^\Sigma = R^n / N_i^\Sigma$$

for $1 \leq i \leq n$.

The following result contains the main properties of these modules.

Proposition 1. *Let $\Sigma = (A, B)$ be a linear dynamical system of size (m, n) over a ring R. Then*

(i) $(0) \subseteq N_0^\Sigma \subseteq N_1^\Sigma \subseteq \ldots \subseteq N_n^\Sigma$.

(ii) The canonical homomorphism

$$\varphi_i : N_i^\Sigma / N_{i-1}^\Sigma \rightarrow N_{i+1}^\Sigma / N_i^\Sigma$$

$$\underline{x} + N_{i-1}^\Sigma \rightarrow F\underline{x} + N_i^\Sigma$$

is surjective for $1 \leq i \leq n - 1$.

(iii) If Σ is feedback equivalent to Σ' then N_i^Σ and M_i^Σ are isomorphic to $N_i^{\Sigma'}$ and $M_i^{\Sigma'}$ respectly, for $1 \leq i \leq n$.

(iv) If Σ is a reachable system of simple input n-dimensional then the modules $\left\{N_i^\Sigma\right\}_{1\leq i\leq n}$ and $\left\{M_i^\Sigma\right\}_{1\leq i\leq n}$ are free.

(v) If Σ is a brunovsky system then the modules $\left\{N_i^\Sigma\right\}_{1\leq i\leq n}$ and $\left\{M_i^\Sigma\right\}_{1\leq i\leq n}$ are free.

Proof. See [Hermida-Alonso, Pérez and Sánchez-Giralda, 1996].

As consequence when $R = \mathbb{R}$ we have the following result.

Corollary 1. *Let $\Sigma = (A, B)$ be a reachable linear dynamical system of size (m, n) over $\mathbb{R}$. Then the feedback equivalence class of Σ is characterized for each one of the following sets:*

(i) The Kronecker's indices $\{\kappa_i\}_{1\leq i\leq s}$.

(ii) $\left\{rank_{\mathbb{R}} N_i^\Sigma\right\}_{1\leq i\leq n}$.

(iii) $\left\{rank_{\mathbb{R}} N_i^\Sigma / N_{i-1}^\Sigma\right\}_{1\leq i\leq n}$.

Proof. See [Hermida-Alonso, Pérez and Sánchez-Giralda, 1996].

Let $M = (a_{ij})$ be an $n \times m$ matrix with entries in R and let i be a nonnegative integer. The $i-$th determinantal ideal of M, denoted by $\mathcal{U}_i(M)$, is the ideal of R generated by all the $i \times i$ minors of M. By construction we have

$$R = \mathcal{U}_0(M) \supseteq \mathcal{U}_1(M) \supseteq \ldots \supseteq \mathcal{U}_i(M) \supseteq \ldots$$

and $\mathcal{U}_i(M) = 0$ for $i > \min\{m, n\}$. The rank of M, denoted by $rank_R(M)$, is the largest i such that $\mathcal{U}_i(M) \neq 0$. Then Σ is reachable if and only if $\mathcal{U}_n(A * B) = R$.

Proposition 2. *Let $R = K$ be a field and $\Sigma = (A, B)$ a reachable linear dynamical system of size (m, n) over R. We put $\sigma_i^\Sigma = dim_K M_i^\Sigma$ for $1 \leq i \leq n$. Then $\{\sigma_i^\Sigma\}_{1\leq i\leq n}$ is a complete set of invariants of the class of equivalence of Σ (i.e. Σ is feedback equivalent to Σ' if and only if $\sigma_i^\Sigma = \sigma_i^{\Sigma'}$ for all i with $1 \leq i \leq n$).*

Proof. See [Carriegos, Hermida-Alonso and Sánchez-Giralda, 1998].

Denote by

$$\psi_M : R^m \longrightarrow R^n$$

the homomorphism of free R-modules defined respect to the standard basis by the matrix M. Consider $Coker\psi_M$ and the following exact sequence

$$R^m \quad \xrightarrow{M} \quad R^n \quad \xrightarrow{\psi_M} \quad Coker\psi_M \quad \longrightarrow \quad 0.$$

We have the following property:

Lemma 1. *If $R = K$ is a field then*

$$dim_R(Coker\psi_M) = n - sup\{j : \mathcal{U}_j\left(M\right) \neq 0\}$$

Proof. See [Northcott, 1976], ch. 1,3.

By Proposition 2 and Lemma 1 feedback relation is characterized by the determinantal ideals of the matrix $A * B$ in the following form.

Corollary 2. *Let $\Sigma = (A, B)$ and $\Sigma^{'} = (A', B')$ be two linear dynamical systems of size (m, n) over R. If Σ is reachable, then following conditions are equivalent.*

(i) *Σ is feedback equivalent $\Sigma^{'}$.*
(ii) *$\mathcal{U}_j\left((A * B)_i\right) = \mathcal{U}_j\left((A^{'} * B^{'})_i\right)$ for $1 \leq i \leq n, 1 \leq j \leq n$ where*

$$(A * B)_i = (B, AB, \ldots, A^{i-1}B)$$
$$(A^{'} * B^{'})_i = \left(B', A'B', \ldots, A'^{i-1}B'\right).$$

Proof. See [Carriegos, Hermida-Alonso and Sánchez-Giralda, 1998].

The importance of these invariants is that they behave well for base change, something that it does not happen with the set $\left\{rank_R N_i^{\Sigma}\right\}_{1 \leq i \leq n}$.

4 Continuous families of linear systems

Let Λ a compact topological space and let $R = C\left(\Lambda, \mathbb{R}\right)$ be the continuous real functions defined over Λ. Let $\Sigma = (A, B)$ and $\Sigma^{'} = (A^{'}, B^{'})$ be two linear dynamical systems over $R = C\left(\Lambda, \mathbb{R}\right)$. We denote $\Sigma(\lambda) = (A(\lambda), B(\lambda))$ the valuation at λ of the Σ. We say Σ and $\Sigma^{'}$ are pointwise feedback equivalents if the systems $\Sigma(\lambda) = (A(\lambda), B(\lambda))$ and $\Sigma^{'}(\lambda) = (A^{'}(\lambda), B^{'}(\lambda))$ over $\mathbb{R}$ are feedback equivalents for all $\lambda \in \Lambda$.

Reachability in the ring $R = C\left(\Lambda, \mathbb{R}\right)$ is shown in the following result.

Theorem 2. *Let Λ a compact topological space and $\Sigma = (A, B)$ be a reachable linear dynamical system of size (m, n) over $R = C\left(\Lambda, \mathbb{R}\right)$. Then the following conditions are equivalents.*

(i) *Σ is reachable over $C\left(\Lambda, \mathbb{R}\right)$.*
(ii) *$\Sigma(\lambda)$is reachable over $\mathbb{R}$, for all $\lambda \in \Lambda$.*

Proof. See [García-Fernández, 2005] p. 56.

Since the system $\Sigma(\lambda)$ is a system s over the field $\mathbb{R}$ is necessary to know what a set of invariants. In this line, we give the following result. But first we recall some definitions. Let n a positive integer and $\kappa = \{\kappa_1, \kappa_2, \ldots, \kappa_s\}$ a partition of n ($n = \kappa_1 + \kappa_2 + \cdots + \kappa_s$ with $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_s$). We'll call *conjugate partition* of κ to the partition $\eta = \{n_1, n_2, \ldots, n_p\}$ (with $n_1 \geq n_2 \geq \cdots \geq n_p$) of n where n_i, is the number of κ_j bigger or equal than i. We denote $\mathcal{P}_n$ the set of partitions of n. The application $\kappa \to \eta$ is a biyection in the set of partitions of n, see [Biggs, 2002].

Theorem 3. *Let $\Sigma = (A, B)$ and $\Sigma^{'} = (A^{'}, B^{'})$ be two reachable linear dynamical systems of size (m, n) over $R = C\left(\Lambda, \mathbb{R}\right)$. For $\lambda_0 \in \Lambda$, the following conditions are equivalent.*

(i) *$\Sigma(\lambda_0) \sim \Sigma_\kappa$ where $\Sigma_\kappa = (A_\kappa, B_\kappa)$ is the Brunovsky's linear form associated to the Kronecker's indices $\kappa = \{\kappa_1, \kappa_2, \ldots, \kappa_s\}$.*
(ii) *Let $\eta = \{n_1, n_2, \ldots, n_p\}$ be the conjugate partition of κ. Then*

$$\dim_{\mathbb{R}}\left(N_i^{\Sigma(\lambda_0)}\right) = n_1 + n_2 + \ldots + n_i; \quad 1 \leq i \leq p.$$

Proof. See [García-Fernández, 2005].

In order to prove our main result we need to remark usual notation of ideal of zeros of a function. Let $\mathfrak{a}$ be an ideal of $R = C\left(\Lambda, \mathbb{R}\right)$. We denote by $Z(\mathfrak{a})$ the set

$$Z(\mathfrak{a}) = \{\lambda \in \Lambda \;/\; f(\lambda) = 0 \quad \textit{for all} \;\; f \in \mathfrak{a}\}.$$

Lemma 2. *Let be $\mathfrak{a}$ and $\mathfrak{b}$ two finite generated ideals of $C\left(\Lambda, \mathbb{R}\right)$. Then*

(i) *There is $a \in \mathfrak{a}$ with*

$$Z(\mathfrak{a}) = Z(a).$$

(ii) *If $\mathfrak{a} \subseteq \mathfrak{b}$ then we can choose $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$ with $a = \lambda b$ where*

$$Z(\mathfrak{a}) = Z(a) \supseteq Z(b) = Z(\mathfrak{b}).$$

Proof. See [García-Fernández, 2005] p. 54.

If $R = C\left(\Lambda, \mathbb{R}\right)$ is a ring continuous functions of a compact topological space Λ in $\mathbb{R}$, then the pointwise feedback relation is characterized by the invariant sets

$$\left\{Z\left(\mathcal{U}_j\left((A * B)_i\right)\right)\right\}_{1 \leq i \leq n, 1 \leq j \leq n}$$

in the following form.

Theorem 4. *Let be$\Sigma = (A, B)$ and $\Sigma^{'} = (A^{'}, B^{'})$ two reachable linear dynamical systems of size (m, n) over $R = C\left(\Lambda, \mathbb{R}\right)$. Then the following statements are equivalent.*

(i) $\Sigma(\lambda)$ *is feedback equivalent to* $\Sigma^{'}(\lambda)$ *for all* $\lambda \in \Lambda$.
(ii) $Z\left(\mathcal{U}_j\left((A*B)_i\right)\right) = Z\left(\mathcal{U}_j\left((A'*B')_i\right)\right)$ *for* $1 \leq i \leq n, 1 \leq j \leq n$.

Proof. (i) $\Rightarrow$ (ii) As $\Sigma(\lambda) \sim \Sigma^{'}(\lambda)$ for all $\lambda \in \Lambda$, we have by Corollary 2

$$\mathcal{U}_j\left(\left(A\left(\lambda\right) * B\left(\lambda\right)\right)_i\right) = \mathcal{U}_j\left(\left(A^{'}\left(\lambda\right) * B^{'}\left(\lambda\right)\right)_i\right)$$

for $1 \leq i \leq n,\ 1 \leq j \leq n$ and for all $\lambda \in \Lambda$. It follows that

$$Z(\mathcal{U}_j\left((A*B)_i\right)) = Z\left(\mathcal{U}_j\left((A'*B')_i\right)\right).$$

(ii) $\Rightarrow$(i) Conversely if

$$Z(\mathcal{U}_j\left((A*B)_i\right)) = Z\left(\mathcal{U}_j\left((A'*B')_i\right)\right)$$

for $1 \leq i \leq n, 1 \leq j \leq n$ we have

$$\text{rank}_{\mathbb{R}}\left(\mathcal{U}_j\left((A(\lambda)*B(\lambda))_i\right)\right) =$$

$$\text{rank}_{\mathbb{R}}\left(\mathcal{U}_j\left(\left(A^{'}(\lambda)*B^{'}(\lambda)\right)_i\right)\right)$$

for $1 \leq i \leq n$ and for all $\lambda \in \Lambda$ or equivalent

$$\dim_{\mathbb{R}}\left(N_i^{\Sigma(\lambda)}\right) = \dim_{\mathbb{R}}\left(N_i^{\Sigma^{'}(\lambda)}\right)$$

for $1 \leq i \leq n$ and for all $\lambda \in \Lambda$. by Corollary 1

$$\left\{\dim_{\mathbb{R}}\left(N_i^{\Sigma(\lambda)}\right)\right\}_{1 \leq i \leq n}$$

is a complete set of invariants for the feedback equivalence over $\mathbb{R}$, then

$$\Sigma(\lambda) \sim \Sigma^{'}(\lambda)$$

for all λ of Λ.

For general reading on the subject, see [Carriegos, Hermida-Alonso and Sánchez-Giralda, 1998].

5 Conclusion

The problem of obtain a set of invariants for poinwise feedback equivalence over $R = C(\Lambda, \mathbb{R})$ has been considered. Some questions must be the subject of future research.

Question 1. *The question of reducing the number of invariants*

$$\left\{Z\left(\mathcal{U}_j\left((A*B)_i\right)\right)\right\}_{1 \leq i \leq n, 1 \leq j \leq n}$$

Question 2. *Given a set of closed sets over a compact* Λ*,*

$$F_1 \supseteq F_2 \supseteq \ldots \supseteq F_s$$

Is it possible to find a system Σ *over* $C(\Lambda, \mathbb{R})$ *where*

$$Z(\mathcal{U}_j((A*B)_i^{\Sigma})) = F_k?$$

Question 3. *To extend results to the ring* $C^k(\Lambda, \mathbb{R})$ *where* Λ *is a differentiable manifold.*

Question 4. *To extend results to the ring of holomorphic functions* $H(\Omega)$ *where* $\Omega \subseteq \mathbb{C}$.

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