REFINED FREQUENCY ESTIMATES FOR STABILITY DOMAINS OF SYNCHRONIZATION SYSTEMS

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Article history:
Received 09.08.2022, Accepted 25.09.2022

Abstract

In this paper we examine stability of Lur'e-type systems arising as a feedback superpositions of infinite-dimensional linear blocks, described by integro-differential Volterra equations, and periodic nonlinearities. Such systems have multiple equilibria, so traditional methods of stability investigation, defined for systems with single equilibrium are of no good here. In the paper traditional Popov method of a priori integral indices is combined with two special techniques: Leonov's nonlocal reduction method and the Bakaev-Guzh procedure. As a result new frequency–algebraic stability criteria are established, yielding tightened estimates of stability domains in the space of the system's parameters.

Key words

Lagrange stability, gradient-like behavior, frequency-domain methods

1 Introduction

Many nonlinear systems with periodic nonlinearities may be obtained by considered as a feedback interconnection of a general linear time-invariant block and a periodic nonlinearity. Such systems arise as models of synchronization circuits [Margaris, 2003; Best, 2003; Leonov et al., 2015]. Because of these applications, a term "'synchronization system" has been coined to denote systems with periodic nonlinearities. Other applications include electrical machines [Stoker, 1950] and vibration units [Blekhman, 1988].

Synchronization systems are featured by multiple equilibria (both stable and unstable ones). An important problem related to their dynamics is whether every solution converges to a certain equilibrium, as exemplified by the model of a mathematical pendulum. Such a property of the system is referred to as the *gradient-like behavior*. In phase-locked loops, the gradient-like behavior corresponds to asymptotic vanishing of the phase error (phase locking). In this work, we extend the criteria of gradient-like behavior established in the prior publications [Leonov et al., 1996; Smirnova and Proskurnikov, 2019; Smirnova and Proskurnikov, 2020].

Standard methods of stability analysis used for systems with a unique equilibrium are inapplicable to synchronization systems that are featured by multi-stability effects. Two important techniques proposed for examination of such systems (in combination with the traditional Popov and Lyapunov methods of stability theory) in the literature are the Leonov *nonlocal reduction* method and the Bakaev-Guzh procedure. Leonov's method [Leonov, 1984; Leonov and Smirnova, 1988; Leonov et al., 1996] employ trajectories of a special low-order "comparison system" (usually, mathematical pendulum) in order to design special Lyapunov functions (in the case of a finite-dimensional synchronization system) or special integral quadratic constraints (in case of general Volterra equation). The Bakaev-Guzh procedure [Bakaev and Guzh, 1965; Perkin et al., 2012; Smirnova and Proskurnikov, 2019] is a special trick that allows one to consider only nonlinearities with zero average value over the period.

Each of these two techniques has its own advantages and can be combined with the Popov method of a priory integral indices [Rasvan, 2006]; such a combination leads to efficient frequency-algebraic stability criteria [Leonov et al., 1996; Smirnova and Proskurnikov, 2019]. The resulting criteria have been tested on special synchronization systems [Smirnova and Proskurnikov, 2019; Smirnova and Proskurnikov, 2021; Smirnova et al., 2021; Smirnova et al., 2022]; the estimates of stability domains in the space of system's parameters appear to be rather tight. These becomes even tighter if one combines the two approaches; such "universal" stability criteria, employing both the Leonov and the Bakaev-Guzh techniques have been derived in our recent paper [Smirnova and Proskurnikov, 2020].

More specific: what is the difference with previous work? It is insufficient to say that we have an extension. In this paper we further develop the methods of [Smirnova and Proskurnikov, 2020] and obtain the generalization of frequency-algebraic criteria for Lagrange stability (every solution of the system is bounded), which together with dichotomy theorems (every bounded solution converges) provides the gradient-like behavior. The resulting stability criterion is applied to a phase-locked loop with a proportional integrating filter. It is shown that the criterion gives rather good estimate for genuine stability domain.

2 Problem setup

Consider the infinite-dimensional synchronization system described by integro-differential Volterra equation

$$\dot\sigma(t) = b(t) + \rho\varphi(\sigma(t-h)) - \int_0^t \gamma(t-\tau)\varphi(\sigma(\tau))\,d\tau. \tag{1}$$

Here $h \geq 0$; $\varphi : \mathbb{R} \to \mathbb{R}$; $\gamma, b : [0, +\infty) \to \mathbb{R}$. The solution of (1) is uniquely determined by the initial condition

$$\sigma(t)|_{t\in[-h,0]} = \sigma_0(t) \in \mathbb{C}[-h, 0] \tag{2}$$

with

$$\sigma(0+0) = \sigma_0(0). \tag{3}$$

We adopt the following assumptions.

A1) The function $b(t)$ is continuous, the function $\gamma(\cdot)$ is piece–wise continuous and $b(t) \to 0$ as $t \to \infty$.

A2) The linear part of (1) is stable:

$$\gamma(t)e^{rt},\ b(t)e^{rt} \in L_2[0, +\infty) \quad (r > 0). \tag{4}$$

A3) The function $\varphi(\sigma)$ is $\mathbb{C}^1$-smooth and Δ–periodic. It has two simple zeros $0 \leq \sigma_1 < \sigma_2 < \Delta$ with $\varphi'(\sigma_1) > 0$, $\varphi'(\sigma_2) < 0$. Without loss of generality we assume that

$$\int_0^\Delta \varphi(\zeta)\,d\zeta \leq 0. \tag{5}$$

Equation (1) describes a special case of Lur'e system, which is interconnection of the infinite-dimensional linear system and the periodic nonlinearity. Such system is often called *synchronization system*.

The most significant asymptotic property of any synchronization system is its gradient–like behavior. System (1) is said to be *gradient–like* if every its solution converges to an equilibrium:

$$\dot\sigma(t) \xrightarrow[t\to\infty]{} 0, \quad \sigma(t) \xrightarrow[t\to\infty]{} \sigma_{eq}, \quad \varphi(\sigma_{eq}) = 0. \tag{6}$$

To guarantee the gradient–like behavior of synchronization system (1) there exists a number of theorems formulated in terms of the transfer function of its linear part from the input $\varphi(\sigma)$ to the output $(-\dot\sigma)$:

$$K(p) = -\rho e^{-ph} + \int_0^\infty \gamma(t)e^{-pt}\,dt \ \ (p \in \mathbb{C}) \tag{7}$$

(see [Smirnova and Proskurnikov, 2019; Smirnova and Proskurnikov, 2020] and references therein). These theorems have the form of frequency-algebraic criteria, containing frequency inequalities with a number of varying parameters and algebraic restrictions on these parameters.

The frequency-algebraic criteria provide the estimates of stability domains for synchronization systems. The goal of this paper is to obtain new criteria which give the opportunity to refine such estimates.

3 Lagrange stability

In this paper we divide the stability investigation of synchronization systems into two parts. First we shall establish the conditions for *Lagrange stability* of synchronization system, i.e. for the boundedness of its every solution. Then we shall go on with the frequency-algebraic conditions of gradient-like behavior for bounded solutions of (1). We exploit here the Popov method of a priori integral indices [Rasvan, 2006] traditionally used for Volterra equations. To prove Lagrange stability we use the nonlocal reduction technique [Leonov et al., 1996] which prescribes to inject into Popov functionals trajectories of Lagrange stable comparison system of low order.

We shall need a second oder comparison system

$$\begin{aligned} &\dot z = -az - \varphi(\sigma) \quad (a > 0),\\ &\dot\sigma = z, \end{aligned} \tag{8}$$

which has been exhaustively investigated (see [Leonov et al., 1996] and references therein). Equation (8) has Lyapunov stable equilibria $(0, \sigma_1 + \Delta k)$ and saddle-point equilibria $(0, \sigma_2 + \Delta k)$ $(k = 0, \pm 1, \ldots)$. It has a bifurcation value a_{cr} such that if $a > a_{cr}$ every solution of (8) converges to some equilibrium.

In this case the first order equation

$$F(\sigma)\frac{dF}{d\sigma} + aF(\sigma) + \varphi(\sigma) = 0 \quad (F(\sigma) = \dot\sigma = z), \tag{9}$$

associated with (8), has solutions $F_k(\sigma)$ $(k \in \mathbb{Z})$ such that

$$\begin{gathered} F_k(\sigma_2 + \Delta k) = 0, \quad F_k(\sigma) \neq 0 \quad \forall \sigma \neq \sigma_2 + \Delta k,\\ F_k(\sigma) \xrightarrow[\sigma\to\mp\infty]{} \pm\infty. \end{gathered} \tag{10}$$

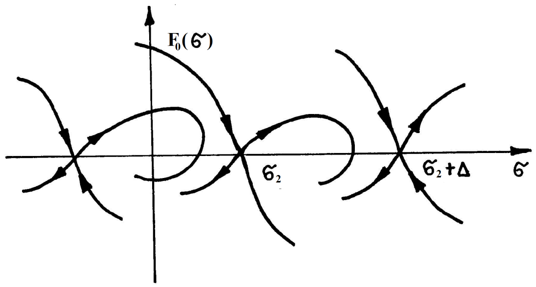

Figure 1. The separatrices of a saddle and the solution $F_0(\sigma)$

The solution $F_k(\sigma)$ is produced by two separatrices which "go in" the point $(\sigma_2 + \Delta k,\, 0)$ (see Fig. 1).

Introduce the constants

$$\mu_1 \triangleq \inf_{\sigma\in[0,\Delta)} \varphi'(\sigma);\ \mu_2 \triangleq \sup_{\sigma\in[0,\Delta)} \varphi'(\sigma)\,(\mu_1\mu_2 < 0) \tag{11}$$

and the function

$$\Phi(\sigma) \triangleq \sqrt{(1-\alpha_1^{-1}\varphi'(\sigma))(1-\alpha_2^{-1}\varphi'(\sigma))} \tag{12}$$

with $\alpha_1 \le \mu_1, \alpha_2 \ge \mu_2$.

Theorem 1. *Suppose there exist* $\varepsilon > 0, \tau \ge 0, \delta > 0, \varkappa \in [0,1], \lambda \in (0, \frac{r}{2})$ $\alpha_1 \le \mu_1, \alpha_2 \ge \mu_2$ *such that the following conditions are true:*

1)

$$\begin{gathered}\pi(\omega,\lambda) \triangleq Re\{K(\imath\omega-\lambda) - \tau(K(\imath\omega-\lambda)+ \\ +\alpha_1^{-1}(\imath\omega-\lambda))^*(K(\imath\omega-\lambda)+\alpha_2^{-1}(\imath\omega-\lambda))\}- \\ -\varepsilon|K(\imath\omega-\lambda)|^2 - \delta \ge 0,\ \forall\omega\ge 0,\end{gathered} \tag{13}$$

where the symbol $(^*)$ *means the complex conjugation;*

2)

$$4\lambda\varepsilon\delta > (1-\varkappa)^2\nu^2\lambda + a_{cr}^2\varkappa\delta \tag{14}$$

with

$$\nu = \nu(\tau_1) \triangleq \frac{\displaystyle\int_0^\Delta \varphi(\sigma)d\sigma}{\displaystyle\int_0^\Delta |\varphi(\sigma)|\sqrt{1+\frac{\tau_1}{\varepsilon}\Phi^2(\sigma)}\,d\sigma}, \tag{15}$$

where $\tau_1 \in [0,\tau]$ *and is such that*

$$|\nu(0)|\sqrt{1+\frac{\tau_1}{\varepsilon}\max_{\sigma\in[0,\Delta)}\Phi^2(\sigma)} \le 1. \tag{16}$$

Then (1) *is Lagrange stable.*

Proof. We use the standard scheme of Popov's method [Smirnova and Proskurnikov, 2019]. Let $\sigma(t)$ be an arbitrary solution of (1), $\eta(t) = \varphi(\sigma(t))$. Determine the functions $(T > 0)$:

$$v(t) \triangleq \begin{cases} 0, & \text{if } t<0, \\ t, & \text{if } t\in[0,1], \\ 1, & \text{if } t>1; \end{cases} \tag{17}$$

$$\eta_T(t) \triangleq \begin{cases} v(t)\eta(t), & \text{if } t\le T, \\ 0, & \text{if } t>T; \end{cases} \tag{18}$$

$$\zeta_T(t) \triangleq \rho\eta_T(t-h) - \textstyle\int_0^t \gamma(t-\tau)\eta_T(\tau)\,d\tau. \tag{19}$$

Functions η_T are continuous only if $\sigma(T) = \sigma_i + \Delta k$ where $k\in\mathbb{Z}$, $\sigma_i \in [0,\Delta)$ $(i=1,2)$ is a zero of $\varphi(\sigma)$. Otherwise they have a gap for $t = T$. We consider only continuous η_T. For these it is obvious that

$$\eta_T, \dot\eta_T, \zeta_T \in L_1[0,+\infty)\cap L_2[0,+\infty). \tag{20}$$

Note that for $t\in[0,T]$

$$\zeta_T(t) = \dot\sigma(t) + b_T(t), \tag{21}$$

where

$$b_T(t)e^{rt} \in L_2[0,+\infty). \tag{22}$$

Let $[f]^\mu(t) \triangleq f(t)e^{\mu t}\quad(\mu\in\mathbb{R})$.

Then

$$[\zeta_T]^\lambda(t) = \rho e^{\lambda h}[\eta_T]^\lambda(t-h) - \int_0^t [\gamma]^\lambda(t-\tau)[\eta_T]^\lambda(\tau)\,d\tau. \tag{23}$$

Denote the set of all $\sigma_2 + \Delta k$ $(k\in Z)$ by S. Let

$$\Sigma \triangleq \{T : T>0,\ \sigma(T)\in S\}. \tag{24}$$

If Σ is bounded then the function $\sigma(t)$ is bounded as well.

Suppose Σ is not bounded. For $T\in\Sigma$ consider the functionals

$$\begin{gathered}R_T \triangleq \int_0^\infty \{[\eta_T]^\lambda[\zeta_T]^\lambda + \varepsilon([\zeta_T]^\lambda)^2 + \delta([\eta_T]^\lambda)^2 + \\ +\tau([\zeta_T]^\lambda - \alpha_1^{-1}\eta_{T,\lambda})([\zeta_T]^\lambda - \alpha_2^{-1}\eta_{T,\lambda})\}\,dt\end{gathered} \tag{25}$$

where

$$\eta_{T,\lambda}(t) \triangleq \frac{d}{dt}([\eta_T]^\lambda) - \lambda[\eta_T]^\lambda \quad (t\ne 0,T). \tag{26}$$

Notice that

$$\mathfrak{F}([\zeta_T]^\lambda)(\imath\omega) = -K(\imath\omega-\lambda)\mathfrak{F}([\eta_T]^\lambda)(\imath\omega), \tag{27}$$

$$\mathfrak{F}(\frac{d}{dt}[\eta_T]^\lambda) = \imath\omega\mathfrak{F}[\eta_T]^\lambda(\imath\omega), \tag{28}$$

where $\mathfrak{F}(f)(\imath\omega)$ stands for Fourier–transform of function f.

Then in virtue of Plancherel theorem one has

$$R_T = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} \pi(\omega,\lambda)|\mathfrak{F}([\eta_T]^\lambda)(\imath\omega)|^2\, d\omega. \tag{29}$$

It follows from (13) that

$$R_T \le 0 \quad \forall T \in \Sigma. \tag{30}$$

On the other hand

$$R_T \ge R_{1T} \triangleq \int\limits_0^T \{\eta v\zeta_T + \varepsilon\zeta_T^2 + \delta(v\eta)^2 + \\ +\tau(\zeta_T - \alpha_1^{-1}\frac{d}{dt}(v\eta))(\zeta_T - \alpha_2^{-1}\frac{d}{dt}(v\eta))\}e^{2\lambda t}\, dt. \tag{31}$$

In virtue of (21), (22) one has

$$R_{1T} = I_T + I_{1T} \quad (T \in \Sigma), \tag{32}$$

where

$$I_T = \int\limits_0^T \{\delta(\varphi(\sigma(t)))^2 + \varepsilon\dot\sigma^2(t) + \dot\sigma(t)\varphi(\sigma(t)) + \\ +\tau\dot\sigma^2(t)\Phi^2(\sigma(t)\}e^{2\lambda t}\, dt \tag{33}$$

and the integrals I_{1T} are uniformly bounded:

$$|I_{1T}| \le C \quad (T \in \Sigma), \tag{34}$$

where C does not depend on T.

Then relations (32), (30) and (34) imply that

$$I_T \le C \quad (T \in \Sigma), \tag{35}$$

where C does not depend on T.

Consider condition 2) of Theorem 1.

Let $\varkappa > 0$. Denoting

$$\varepsilon_2 \triangleq \frac{(1-\varkappa)^2}{4\delta}\nu^2, \quad \varepsilon_1 \triangleq \varepsilon - \varepsilon_2, \tag{36}$$

one obtains from (14) that

$$4\lambda\varepsilon_1 > a_{cr}^2\varkappa. \tag{37}$$

According to (36) one has

$$I_T \ge J_{1T} + J_{2T}, \tag{38}$$

where

$$J_{1T} \triangleq \int\limits_0^T \{(1-\varkappa)\varphi(\sigma(t))\dot\sigma(t) + \tau_1\dot\sigma^2(t)\Phi^2(\sigma(t)) + \\ \varepsilon_2\dot\sigma^2(t) + \delta\varphi^2(\sigma(t))\}e^{2\lambda t}\, dt, \tag{39}$$

$$J_{2T} \triangleq \int\limits_0^T \{\varkappa\varphi(\sigma(t))\dot\sigma(t) + \varepsilon_1\dot\sigma^2(t)\}e^{2\lambda t}\, dt. \tag{40}$$

Consider the functional J_{1T}. In order to get uniform estimate we apply the Bakaev–Guzh procedure [Leonov et al., 1996] which singles out from $\varphi(\sigma)$ the function with the zero mean integral value on the period. We introduce:

$$P(\sigma) = \sqrt{1 + \frac{\tau_1}{\varepsilon}\Phi^2(\sigma)}, \tag{41}$$

$$\Psi(\sigma) = \varphi(\sigma) - \nu|\varphi(\sigma)|P(\sigma). \tag{42}$$

As $\varepsilon_2 < \varepsilon$ we obtain from (39)

$$J_{1T} > \int\limits_0^T \{(1-\varkappa)\nu|\varphi(\sigma(t))|\dot\sigma(t)P(\sigma(t)) + \\ +\varepsilon_2 P^2(\sigma(t))\dot\sigma^2(t) + \delta\varphi^2(\sigma(t))\}e^{2\lambda t}\, dt + \\ +(1-\varkappa)\int\limits_0^T \Psi(\sigma(t))\dot\sigma(t)e^{2\lambda t}\, dt. \tag{43}$$

The first addend in the right–hand part of (43) is non-negative in virtue of (36). Consider the second addend:

$$\int\limits_0^T \Psi(\sigma(t))\dot\sigma(t)e^{2\lambda t}\, dt = e^{2\lambda T}\int\limits_{\sigma(T_0)}^{\sigma(T)} \Psi(\zeta)\, d\zeta, \tag{44}$$

where $T_0 \in [0, T]$. The functions $\Psi(\sigma)$ and $\varphi(\sigma)$ have the same zeros. It is obvious that

$$\int_0^\Delta \Psi(\zeta)\, d\zeta = 0. \tag{45}$$

It follows from (5), (16), and (45) that

$$\int_{\sigma(T_0)}^{\sigma(T)} \Psi(\zeta)\, d\zeta \ge 0 \quad (T \in \Sigma). \tag{46}$$

As a result we conclude that

$$J_{1T} > 0 \quad (T \in \Sigma). \tag{47}$$

Then we have from (35) and (38) that

$$J_{2T} < C \quad (T \in \Sigma), \tag{48}$$

where C does not depend on T.

We shall apply the nonlocal reduction technique now. Consider the equation

$$F(\sigma)\frac{dF(\sigma)}{d\sigma}+2\sqrt{\frac{\lambda\varepsilon_1}{\varkappa}}F(\sigma)+\varphi(\sigma)=0. \quad (49)$$

It follows from (37) that (49) has solutions $F_k(\sigma)$ with the properties (10). Note that $\hat{F}_k=\sqrt{\frac{\varkappa}{2}}F_k$ is a solution of the equation

$$\hat{F}(\sigma)\hat{F}'(\sigma)+\sqrt{2\lambda\varepsilon_1}\hat{F}+\frac{\varkappa}{2}\varphi(\sigma)=0. \quad (50)$$

Inject $\hat{F}_k$ into J_{2T}:

$$J_{2T}=\int_0^T\{\varkappa\varphi(\sigma(t))\dot\sigma(t)+\varepsilon_1\dot\sigma^2(t)+ \\ +2\hat{F}_k'(\sigma(t))\hat{F}_k(\sigma(t))\dot\sigma(t)+2\lambda\hat{F}_k^2(\sigma(t))\}e^{2\lambda t}\,dt- \\ -\hat{F}_k^2(\sigma(T))e^{2\lambda T}+\hat{F}_k^2(\sigma(0)). \quad (51)$$

Transform J_{2T}:

$$J_{2T}=J_{0T}-\hat{F}_k^2(\sigma(T)))e^{2\lambda T}+\hat{F}_k^2(\sigma(0)), \quad (52)$$

where

$$J_{0T}=\int_0^T\Bigg\{\bigg[\sqrt{\varepsilon_1}\dot\sigma(t)+\frac{1}{2\sqrt{\varepsilon_1}}\Big(\varkappa\varphi(\sigma(t))+ \\ +2\hat{F}_k(\sigma(t))\hat{F}_k'(\sigma(t))\Big)\bigg]^2-\tfrac{1}{4\varepsilon_1}\Big(\varkappa\varphi(\sigma(t))+ \\ +2\hat{F}_k(\sigma(t))\hat{F}_k'(\sigma(t))\Big)^2+2\lambda\hat{F}_k^2(\sigma(t))\Bigg\}e^{2\lambda t}\,dt. \quad (53)$$

Notice that

$$J_{0T}\geq\int_0^T\Big\{2\lambda\hat{F}_k^2(\sigma(t))-\frac{1}{4\varepsilon_1}(\varkappa\varphi(\sigma(t))+ \\ +2\hat{F}_k(\sigma(t))\hat{F}_k'(\sigma(t)))^2\Big\}e^{2\lambda t}\,dt= \\ =\int_0^T\Bigg\{\bigg[\sqrt{2\lambda}\,\hat{F}_k(\sigma(t))-\frac{\varkappa}{2\sqrt{\varepsilon_1}}\varphi(\sigma(t))- \\ -\frac{1}{\sqrt{\varepsilon_1}}\,\hat{F}_k(\sigma(t))\hat{F}_k'(\sigma(t))\bigg]\bigg[\sqrt{2\lambda}\hat{F}_k(\sigma(t))+ \\ +\frac{\varkappa}{2\sqrt{\varepsilon_1}}\,\varphi(\sigma(t))+\frac{1}{\sqrt{\varepsilon_1}}\hat{F}_k(\sigma(t))\hat{F}_k'(\sigma(t))\bigg]\Bigg\}e^{2\lambda t}\,dt. \quad (54)$$

Since $\hat{F}_k$ is a solution of (50) the integral in right–hand part of (54) is equal to zero. So

$$J_{2T}\geq-\hat{F}_k^2(\sigma(T))e^{2\lambda T}+\hat{F}_k^2(\sigma(0)). \quad (55)$$

Then it follows from (55) and (48) that

$$\hat{F}_k^2(\sigma(t))e^{2\lambda t}\geq\hat{F}_k^2(\sigma(0))-C \quad (56)$$

for $\sigma(t)=\sigma_2+\Delta l$ $(l,k\in Z)$.

Let us exploit the properties (10) of functions $F_k(\sigma)$ and

$$\hat{F}_{\pm k_0}^2(\sigma(0))>C. \quad (58)$$

The inequalities (56) and (58) imply that

$$F_{k_0}(\sigma_2+\Delta l)\neq 0 \quad (l\in\mathbb{Z}). \quad (59)$$

Thus

$$\sigma_2-\Delta k_0<\sigma(t)<\sigma_2+\Delta k_0. \quad (60)$$

Consider now the case $\varkappa=0$. The inequality (14) takes the form

$$4\varepsilon\delta>\nu^2. \quad (61)$$

It this case we return to (33) and use the following relations:

$$I_T\geq\int_0^T\Big\{\varphi(\sigma(t))\dot\sigma(t)+\tau_1\dot\sigma^2(t)\Phi^2\sigma(t)+ \\ +\varepsilon\dot\sigma^2(t)+\delta\varphi^2(\sigma(t))\Big\}e^{2\lambda t}\,dt= \\ =\int_0^T\Big\{\varepsilon P^2(\sigma(t))\dot\sigma^2(t)+\delta\varphi^2(\sigma(t))+ \\ +\nu|\varphi(\sigma(t))|\dot\sigma(t)P(\sigma(t))\Big\}e^{2\lambda t}\,dt+ \\ +\int_0^T\Psi(\sigma(t))\dot\sigma(t)e^{2\lambda t}\,dt. \quad (62)$$

The estimates (35) and (46) imply that

$$\int_0^T\{\varepsilon P^2(\sigma(t))\dot\sigma^2(t)+\delta\varphi^2(\sigma(t))+ \\ +\nu|\varphi(\sigma(t))|P(\sigma(t))\}e^{2\lambda t}\,dt\leq C \quad (T\in\Sigma), \quad (63)$$

where C does not depend on T. In virtue of (61) it follows from (63) that for $T\in\Sigma$

$$\int_0^T\varphi^2(\sigma(t))e^{2\lambda t}\,dt<C_0 \quad (64)$$

where C_0 does not depend on T. Since Σ is not bounded the estimate (64) is valid for all $T>1$ which implies that

$$\varphi(\sigma(t))\to 0 \quad \text{as} \quad t\to+\infty, \quad (65)$$

whence [Leonov et al., 1996]

$$\sigma(t)\to q \quad \text{as} \quad t\to+\infty, \quad (66)$$

where $\varphi(q)=0$.

Theorem 1 is proved. □

Theorem 1 is a generalization of a frequency-algebraic criterion demonstrated in [Smirnova and Proskurnikov, 2020]

4 Gradient–like behavior

The Lagrange stability does not guarantee the convergence of solutions. That is why it is often considered together with the *dichotomy property*: every solution of (1) is either unbounded or converges.

In particular the following dichotomy criterion is valid.

Theorem 2. *[Leonov et al., 1996] Suppose for* $\varepsilon > 0, \delta > 0, \tau \geq 0$ *the frequency condition* $\pi(\omega, 0) \geq 0, \forall\omega \geq 0$ *is true.*

Then every bounded solution of (1) *converges.*

Meanwhile for certain values of varying parameters the conditions of Theorem 1 guarantee the gradient–like behavior.

Theorem 3. *Suppose all the conditions of Theorem 1 are fulfilled in the case of* $\tau = 0$.

Then equation (1) *is gradient–like.*

Proof. Since $\tau = 0$ the functionals (25) transform into

$$R_{1T} \triangleq \int\limits_0^{+\infty} \{[\eta_T]^\lambda(t)[\zeta_T]^\lambda(t) + \\ +\varepsilon([\zeta_T]^\lambda(t))^2 + \delta([\eta_T]^\lambda)^2\}\, dt \qquad (67)$$

where functions η_T and ζ_T are defined by (18) and (19). As R_{1T} do not contain the derivatives $\dot\eta_T$ we can use any η_T with $T > 0$. In virtue of Plancherel theorem and (13) with $\tau = 0$ we establish that

$$R_{1T} \leq 0, \quad \forall T > 0. \qquad (68)$$

It follows from the relations (18), (19), (21) that

$$R_{1T} \geq J_{3T} + J_{4T} \qquad (69)$$

where

$$J_{3T} = \int\limits_0^T \{\eta(t)\dot\sigma(t) + \varepsilon\dot\sigma^2(t) + \delta\eta^2(t)\}e^{2\lambda t}\, dt \qquad (70)$$

and J_{4T} is uniformly bounded. Then (68) and (69) imply the estimates

$$J_{3T} \leq C_1, \qquad (71)$$

where C_1 does not depend on T.

Consider the functionals

$$J_{5T} = \int\limits_0^T \Big\{\eta(t)\dot\sigma(t) + \varepsilon\dot\sigma^2(t) + \delta\eta^2(t)\Big\}\, dt \ \ (T > 0). \qquad (72)$$

We have

$$J_{5T} = \int\limits_0^T \Big\{\big(\eta(t)\dot\sigma(t) + \varepsilon\dot\sigma^2(t) + \delta\eta^2(t)\big)e^{2\lambda t}\Big\}e^{-2\lambda t}\, dt. \qquad (73)$$

Note that

$$J_{5T} = J_{3\hat{T}} \qquad (74)$$

with $\hat{T} \in [0, T]$. Then

$$J_{5T} < C_1, \ \ \forall T > 0. \qquad (75)$$

It is obvious that

$$J_{5T} = \int\limits_{\sigma(0)}^{\sigma(T)} \varphi(\sigma)\, d\sigma + \varepsilon\int\limits_0^T \dot\sigma^2(t)\, dt + \delta\int\limits_0^T \eta^2(t)\, dt \qquad (76)$$

It follows from Theorem 1 that $\sigma(t)$ is bounded. Then we have from (75) that

$$\dot\sigma(t), \varphi(\sigma(t)) \in L_2[0, +\infty). \qquad (77)$$

It is easy to show [Smirnova and Proskurnikov, 2019] that (77) entails (6). □

Theorem 4. *Let* $h = 0$ *and the functions* $\gamma(t), b(t)$ *have piece–wise continuous derivatives with*

$$\dot b(t)e^{rt}, \dot\gamma(t)e^{rt} \in L_2[0, +\infty). \qquad (78)$$

Suppose the conditions of Theorem 1 are fulfilled and besides

$$\alpha_1^{-1}\alpha_2^{-1} = 0, \qquad (79)$$

$$\rho(\alpha_2^{-1} + \alpha_1^{-1}) \leq 0. \qquad (80)$$

Then system (1) *is gradient–like.*

Proof. Thanks to condition (79) we can get rid of $\dot\eta(t)$ in functionals (25) and use the functions $\eta(t)$ and $\zeta(t)$ for all $T > 0$.

Let $\alpha_1^{-1} = 0$. Determine the functions $\bar\zeta_T(t) = \zeta_T(t) - \rho\eta_T(t)$ and consider the functionals

$$R_{2T} \triangleq \int\limits_0^{+\infty} \{(1 + \lambda\tau\alpha_2^{-1})[\eta_T]^\lambda(t)[\zeta_T]^\lambda(t) + \\ +(\tau + \varepsilon)([\zeta_T]^\lambda(t))^2 + \delta([\eta_T]^\lambda(t))^2 + \\ +\tau\alpha_2^{-1}(\frac{d}{dt}[\bar\zeta_T]^\lambda(t))[\eta_T]^\lambda(t)\}\, dt \qquad (81)$$

Note that

$$\mathfrak{F}(\frac{d}{dt}[\bar\zeta_T]^\lambda(t))(\imath\omega) = -\imath\omega(K(\imath\omega - \lambda) + \rho)\mathfrak{F}([\eta_T]^\lambda)(\imath\omega). \qquad (82)$$

Then in view of Plancherel theorem we have

$$R_{2T} = -\frac{1}{2\pi}\int\limits_{\infty}^{+\infty} Re\{(1+\lambda\tau\alpha_2^{-1})K(\imath\omega-\lambda)- \\ -\delta-(\tau+\varepsilon)|K(\imath\omega-\lambda)|^2+ \\ +\imath\omega\tau\alpha_2^{-1}K(\imath\omega-\lambda)\}|\mathfrak{F}([\eta_T]^\lambda)(\imath\omega)|^2\,d\omega \quad (83)$$

whence in virtue of frequency-domain condition (13) it follows that

$$R_{2T} \le 0. \quad (84)$$

On the other hand

$$R_{2T} \ge J_{6T} - \tau\alpha_2^{-1}\rho\int\limits_0^T \frac{d([\eta_T]^\lambda(t))}{dt}[\eta_T]^\lambda(t)\,dt + J_{7T} \quad (85)$$

where

$$J_{6T} = \int\limits_0^T \{\bar{\varkappa}\eta(t)\dot\sigma(t)+\varepsilon\dot\sigma^2(t)+\delta\eta^2(t)\}e^{2\lambda t}\,dt+ \\ +\tau\int\limits_0^T(\dot\sigma^2(t)+(\alpha_2^{-1}+\alpha_1^{-1})\ddot\sigma(t)\eta(t))e^{2\lambda t}\,dt, \quad (86)$$

with $\bar{\varkappa} = 1+2\lambda\tau(\alpha_2^{-1}+\alpha_1^{-1})$ and J_{7T} is uniformly bounded. The inequalities (84) and (85) imply

$$J_{6T} \le C_2, \quad (87)$$

where C_2 does not depend on T.

The case $\alpha_2^{-1} = 0$ can be treated in the same way as the previous one.

Introduce the functional

$$J_{8T} \triangleq \int\limits_0^T \{\bar{\varkappa}\eta(t)\dot\sigma(t)+\varepsilon\dot\sigma^2(t)+\delta\eta^2(t)+ \\ +\tau(\dot\sigma^2(t)+(\alpha_1^{-1}+\alpha_2^{-1})\ddot\sigma(t)\eta(t))\}\,dt. \quad (88)$$

$$J_{8T} = J_{6\hat{T}} \quad (89)$$

where $\hat{T}\in[0,T]$. Then

$$J_{8T} < C_2 \quad \forall T>0. \quad (90)$$

From (88) we have

$$J_{8T} = \bar{\varkappa}\int\limits_{\sigma(0)}^{\sigma(T)}\varphi(\sigma)\,d\sigma+\varepsilon\int\limits_0^T\dot\sigma^2(t)\,dt+\delta\int\limits_0^T\dot\eta^2(t)\,dt+ \\ +\tau\int\limits_0^T(\dot\sigma^2(t)-(\alpha_1^{-1}+\alpha_2^{-1})\dot\sigma(t)\dot\eta(t))\,dt+ \\ +\tau(\alpha_1^{-1}+\alpha_2^{-1}))(\dot\sigma(T)\eta(T)-\dot\sigma(0)\eta(0)). \quad (91)$$

Notice that in virtue of (11)

$$\dot\sigma^2(t)-(\alpha_1^{-1}+\alpha_2^{-1})\dot\sigma(t)\dot\eta(t) = \\ = \dot\sigma^2(t)(1-(\alpha_1^{-1}+\alpha_2^{-1})\tfrac{d\varphi}{d\sigma}) \ge 0.$$

Since every solution $\sigma(t)$ is bounded on $[0,+\infty)$, the relations (91) and (90) imply that the relations (77) are valid. Thus the theorem is proved. □

5 Stability of the phase-locked loop with a proportional integrating filter

In this section, we use the results of the previous ones to obtain estimates of *pull-in* range for phase-locked loop (PLL) with a proportional integrating filter (PIF).

The minimal structure of a PLL circuit is shown in Fig. 2 and comprises the *phase detector* (comparator), the low-pass *loop filter* and the *voltage control oscillator* (VCO), which has to be synchronized with the reference oscillatory signal.

The pull-in range characterizes capturing capabilities of the PLL. It is defined mathematically [Leonov et al., 2015] as the deviation between the reference and controlled oscillators' frequencies, for which phase locking is guaranteed. Mathematically, phase locking is equivalent to gradient-like behavior of the system, i.e., convergence of each solution to one of the equilibria.

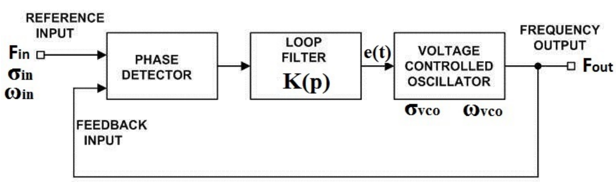

Figure 2. The minimal structure of a PLL circuit

We consider the PLL with PIF which is often described by the second order system

$$\begin{cases}\dot z = -\frac{1}{T}z-(1-m)\varphi, \\ \dot\sigma = z - Tm\varphi\end{cases} \quad (T>0,\ m\in(0,1)), \quad (92)$$

where $\varphi(\sigma)$ describes the phase detector.

Let

$$\varphi(\sigma) = \sin(\sigma)-\beta \quad (\beta\in(0,1)). \quad (93)$$

It is obvious that (92) is easily reduced to integro-differential equation (1) with

$$K(p) = T\frac{Tmp+1}{Tp+1}. \quad (94)$$

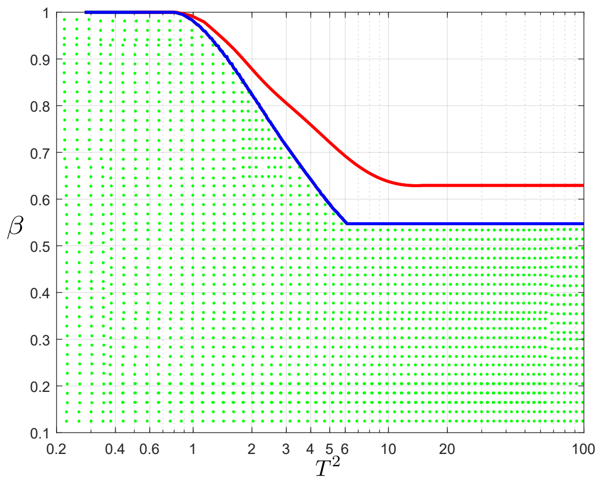

Figure 3. The stability domain of PLL with PIF (the area under the red (upper) line) and its estimate (the area under the blue (lower) line)

The stability domain of synchronization system (93), (94) in the plain $\{T^2, \beta\}$ has been investigated for $m = 0.2$. For PLL with PIF we compare here the genuine stability domain and its estimate supplied by frequency-algebraic criteria.

In order to obtain the genuine stability domain we use the qualitative analysis of (92) [Leonov et al., 1996].

The system has stable equilibriums $\{\sigma_{eq1} = \arcsin\beta + 2\pi k \quad (k \in \mathbb{Z}); z_{eq1} = 0\}$ and saddle points $\{\sigma_{eq2} = \pi - \arcsin\beta + 2\pi k \quad (k \in \mathbb{Z}); z_{eq2} = 0\}$. Three types of asymptotic behavior are possible here:

i) The system is gradient-like (the phase portrait on the plane $\{\sigma, z\}$ is analogous to one on Fig. 1).

ii) The system has one stable *limit cycle of the second kind*:

$$\exists\, \theta > 0, I \in \mathbb{N} : z(t+\theta) = z(t), \sigma(\theta) = \sigma(t) + 2\pi I,$$

in the half plane $z > 0$.

iii) The system has two limit cycles of the second kind in the half plane $z > 0$. The lower cycle is unstable and the upper one is stable.

The stability of the system can be demonstrated by the analysis of two separatrices of a saddle $(\sigma_{eq2}, 0)$ which adjoin to it with $z(t) > 0$ in its vicinity. The separatrix $z = F(\sigma)$ which "enters" $(\sigma_{eq2}, 0)$ oughts to tend to $+\infty$ as $\sigma \to -\infty$ and the separatrix $z = \tilde{F}(\sigma)$ which "goes out" from $(\sigma_{eq2}, 0)$ oughts to vanish for $\sigma < \sigma_{eq2} + 2\pi$. The numerical analysis of the two separatrices gave the opportunity to obtain the stability domain of (92) on the plain $\{T^2, \beta\}$. Its frontier is shown in Fig. 3 by the red (upper) line. The stability domain is situated under the frontier.

The estimate of the stability domain was obtained here by Theorem 1 and Theorem 2 with $\alpha_1 = \mu_1 = -1$, $\alpha_2 = \mu_2 = 1$. Frequency-domain condition 1) of Theorem 1 takes the form

$$\tau T^2 y^2 + y(T^3 m + \tau T^2 \lambda + \tau P^2 - (\varepsilon + \tau)T^4 m^2 - \\ -\delta T) + (\tau\lambda^2 P^2 + TRP - (\varepsilon + \tau)T^2 R^2 - \\ -\delta P^2) \geq 0, \quad \forall y \geq 0 \tag{95}$$

with

$$P = 1 - T\lambda, R = 1 - Tm\lambda. \tag{96}$$

The varying parameters

$$\tau \geq 0, \varepsilon > 0, \delta > 0, \lambda \in (0; \frac{1}{2T}),$$

satisfy the algebraic condition

$$4\lambda\varepsilon\delta - (1 - \varkappa)^2 \nu^2(\tau_1)\lambda - a_{cr}^2(\beta)\varkappa\delta > 0 \tag{97}$$

where

$$|\nu(x)| = \frac{\left| \int\limits_0^{2\pi} (\sin\sigma - \beta)\, d\sigma \right|}{\int\limits_0^{2\pi} \left| \sin\gamma - \beta \right| \sqrt{(1 + \frac{x}{\varepsilon}\sin^2\sigma)}\, d\sigma}$$

with $\varkappa \in [0, 1], \tau_1 \in [0, \tau]$ and

$$\tau_1 \leq \varepsilon(\nu^{-2}(0) - 1). \tag{98}$$

For any β the value of a_{cr} was established by numerical analysis of separatrices of system (8) (analogous to the analysis done for the stable case of system (92)). Thus we obtained the curve $a_{cr} = a_{cr}(\beta)$.

Then for a couple $(T^2; \beta)$ inequalities (95), (97) were verified for various permissible values of varying parameters.

We scan the interval $(0, \frac{1}{2T})$ for λ to analyse the conditions of Theorem 1.

It is clear that in case when

$$\lambda P > TR \tag{99}$$

and either

$$P^2 + T^2\lambda - T^4 m^2 > 0 \tag{100}$$

or

$$|T^2\lambda + P^2 - T^4 m^2| < 2T\sqrt{\lambda^2 P^2 - T^2 R^2} \tag{101}$$

one can provide that all the conditions of Theorem 1 are fulfilled by choosing the value of τ big enough.

Otherwise the coefficients of quadratic trinomial in (95) and its discriminant determine the upper limits for permissible $\tau, \varepsilon, \delta$. Scanning with small steps the intervals for $\tau, \varepsilon, \delta, \varkappa$ we verify the conditions (95) and (97).

As soon as for a set of varying parameters the inequalities (95), (97) proved to be valid the Lagrange stability of (92) was fixed. Obviously, if $\lambda = 0$ the condition (95) is valid for $\varepsilon, \tau, \delta$ small enough. Then it follows from Theorem 2 that any Lagrange stable system (92) is a gradient-like one. The frontier for the estimating stability domain is drawn in Fig. 3 by the blue (lower) line. The domain is situated under the frontier.

Thus Theorem 1 and Theorem 2 give for mathematical modell (92) rather good estimate of pull-in range for any value of T^2. It should be also mentioned that the estimate of the pull-in range obtained by Theorem 1 and Theorem 2 coordinates with the results of numerical simulation of a concrete PLL given in [Leonov et al., 2015] in Fig 9. The *hidden oscillations*, found in [Leonov et al., 2015] appear beyond the estimate of pull-in range.

6 Conclusion

In the paper we go on with investigation of asymptotic behavior of infinite-dimensional synchronization systems described by integro-differential Volterra equations. Combining basic techniques applied for stability analysis of synchronization systems we demonstrate frequency-domain criteria for gradient-like behavior. New criteria give the opportunity to improve estimates of stability regions in the space of system parameters. They can also be used to improve the tools for estimating of cycle-slipping developed in [Perkin et al., 2013].

Acknowledgements

The work is supported by the Ministry of Science and Higher Education of the Russian Federation (Project No. 075-15-2021-573).

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