PHYSCON 2017, Florence, Italy, 17–19 July, 2017

THE STABILITY OF DISCONTINUOUS SOLUTIONS OF BILINEAR SYSTEMS WITH DELAY

Alexander Sesekin
Department of Optimal Control
Institute of Mathematics and Mechanics of UB RAS
Department of Applied Mathematics
Ural federal University
Russia
sesekin@list.ru

Natalia Zhelonkina
Department of Optimal Control
Institute of Mathematics and Mechanics of UB RAS
Department of Applied Mathematics
Ural federal University
Russia
312115@mail.ru

Abstract
The paper is devoted to the research of the properties of stability and asymptotic stability solutions for bilinear system of the differential equations with impulse actions in the right part of system with delay. The sufficient conditions are received for stability and asymptotic stability solutions for system of such differential equations.

Key words
Differential equations, stability, asymptotic stability, delay, impulse actions.

1 Introduction

The article is dedicated to investigation of nonlinear systems of differential equations with delay and impulse effects. Such systems having discontinuous trajectories are found in various technical systems, in biology and economics [Zavalishchin, Sesekin, 1997; Miller, Rubinovich, 2013]. Properties of asymptotic stability for such systems were considered in [Sesekin, Zhelonkina, 2016]. In this paper the bilinear system was examined. In this systems right part there is a component which contains delay and impulse action which plays a role of disturbance of the right part. The property of asymptotic stability was provided due to asymptotic stability of linear part of the system without an impulse component and without a term containing delay. In this work we assume that the uniform system with delay without impulse is unstable and property of stability and asymptotic stability are reached due to impulse effects. Unlike [Sesekin, Zhelonkina, 2016] in the given paper the generalized effects do not contain a regular component and are a generalized derivative of the step function. In the right part of the system there are incorrect operations of multiplication of discontinuous functions on the generalized function. The decision is understood also as in [Zavalishchin, Sesekin, 1997; Sesekin, Fetisova, 2010], pointwise limit of a sequence of smooth solutions which is generated by a smooth approximation of the generalized impact if this limit does not depend on the sequence approximation selection. Related problems in the formalization of impulse systems, proposed by A.M. Samoilenko, for systems without delay were considered in [Samoilenko, Perestyuk, 1995] and for systems with delay were studied in [Cheng, Deng, Wang, 2013]. We note that the formalization of the solution, which is used in this article, does not allow the dependence of the jump of the trajectory from the delay [Sesekin, Fetisova, 2010]. In this case, the dependence on the delay leads to an ambiguous definition of the jump of the trajectory.

2 The stability of bilinear system with impulse actions in a system matrix

Consider the system of the differential equations

$$\dot{x}(t) = (A + \sum_{i=1}^{m} D_i \dot{v}_i(t))x(t) + A_\tau x(t-\tau), \quad (1)$$

where $A,\ A_\tau\ D_i\ (i \in \overline{1,m})$ — $n \times n$- constant matrix, $v_i(t)$ — vector function components of a piecewise constant continuous at the left $v(t) = (v_1(t), v_2(t)..., v_m(t))^T$, $\{t_k\},\ k = 1, 2, ..., n, ...$ unlimited sequence of function $v(t)$ points of discontinuity.

In this case the equation (1) will take a form

$$\dot{x}(t) = (A + \sum_{k=1}^{\infty}\sum_{i=1}^{m} D_i \Delta v_i(t_k)\delta(t-t_k))x(t)$$

$$+A_\tau x(t-\tau). \quad (2)$$

Let the matrices D_i $(i \in \overline{1, m})$ be mutually commutative. Then there exists an approximable solution $x(t)$ of the equation (2), which will satisfy the integral equation

$$x(t) = \varphi(t_0) + \int_{t_0}^{t} Ax(\xi)d\xi + \int_{t_0}^{t} A_\tau x(\xi - \tau)d\xi$$

$$+ \sum_{t_i < t} S(t_i, x(t_i), \triangle v(t_i + 0)),$$

and discontinuity functions are defined by the following equations

$$S(t_k, x(t_k), \triangle v(t_k + 0)) = z(1) - z(0), \qquad (3)$$

$$\dot{z}(\xi) = \sum_{i=1}^{m} D_i z(\xi) \triangle v_j(t_k), z(0) = x. \qquad (4)$$

where

$$\Delta v_i(t_k) = v_i(t_k + 0) - v_i(t_k).$$

We assume that functions of bounded variation $x(t)$ and $v(t)$ are left continuous functions. We note that the function $x(t)$ on the set $(t_k, t_{k+1}]$ is continuous and satisfies the differential equation

$$\dot{x}(t) = Ax(t) + A_\tau x(t - \tau). \qquad (5)$$

Theorem. We will assume that there exists a positive definite dimensional $n \times n$ the matrix P, the matrix

$$\begin{pmatrix} PA + A^T P - \alpha P & PA_\tau \\ A_\tau^T P & -\beta P \end{pmatrix}, \qquad (6)$$

where α and $\beta > 0$ are some positive constants, is negative definite. In addition, we assume that the derivative, in view of the system (4) of the Lyapunov function

$$V(x(t)) = x^T(t)Px(t), \qquad (7)$$

satisfies the inequality

$$\dot{V}(x(\xi)) \leq -\gamma V(x(\xi), \ \ \gamma > 0 \qquad (8)$$

$t_{k+1} - t_k > \tau \ \forall k = 1, 2, \ldots.$ Let us introduce the following notation

$$\theta_k = k \ln(e^{-\gamma} + \beta\tau) + (\alpha + \beta)(t_k - t_0). \qquad (9)$$

If

$$\theta_k < Q, \qquad (10)$$

where Q is same constant, then the trivial solution of the system (2) will be stabile and if

$$\lim_{k \longrightarrow \infty} \theta_k = -\infty, \qquad (11)$$

then the trivial solution of the system (2) will be asymptotic stabile.

Proof. The derivative Lyapunov function (7) along to system (5) on $[t_0, t_1]$ has an appearance form

$$\dot{V}(x(t)) = (Ax(t) + A_\tau x(t - \tau))^T Px(t)$$

$$+x^T(t)P(Ax(t) + A_\tau x(t - \tau)) = \begin{pmatrix} x(t) \\ x(t - \tau) \end{pmatrix}^T$$

$$\times \begin{pmatrix} PA + A^T P & PA_\tau \\ A_\tau^T P & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ x(t - \tau) \end{pmatrix} \qquad (12)$$

Adding and subtracting in the right part (12) $\alpha x^T(t)Px(t)$ and $x^T(t - \tau)Qx(t - \tau)$ we receive

$$\dot{V}(x(t)) = \begin{pmatrix} x(t) \\ x(t - \tau) \end{pmatrix}^T \begin{pmatrix} PA + A^T P - \alpha P & PA_\tau \\ A_\tau^T P & -\beta P \end{pmatrix}$$

$$\times \begin{pmatrix} x(t) \\ x(t - \tau) \end{pmatrix} + \alpha x^T(t)Px(t)$$

$$+\beta x^T(t - \tau)Px(t - \tau) \qquad (13)$$

Considering that the matrix (6) is negative, from (13) we have:

$$\dot{V}(x(t)) \leq \alpha V(x(t)) + \beta V(x(t - \tau)). \qquad (14)$$

Increasing the right part in (14), we obtain

$$\dot{V}(x(t)) \leq \alpha V(x(t)) + \beta \sup_{\xi \in [t - \tau, t]} V(x(\xi)) \qquad (15)$$

Applying an assessment of differential inequality from [Alwan, Liu, 2013] to (15) we receive the inequality

$$V(x(t)) \leq \sup_{\zeta \in [t_0 - \tau, t_0]} V(x(\zeta))e^{(\alpha + \beta)(t - t_0)}. \qquad (16)$$

Then (16) leads to the following inequality

$$V(x(t_1)) \leq Me^{(\alpha+\beta)(t_1-t_0)}, \quad (17)$$

where

$$M = \sup_{\zeta\in[t_0-\tau,t_0]} V(x(\zeta)) = \sup_{\zeta\in[t_0-\tau,t_0]} \varphi^T(\zeta)P\varphi(\zeta).$$

Let the value of $V(t_k)$ be known. Then after the action of the impulse according to (8) we will to have

$$V(x(t_k+0)) \leq V(x(t_k))e^{-\gamma}. \quad (18)$$

Now we will estimate $V(t)$ on the interval $(t_k, t_k+\tau]$. On this interval the differential inequality (14) is also satisfied.

After the integration procedure we receive

$$V(x(t)) \leq V(x(t_k+0))$$

$$+\alpha\int_{t_k}^{t} V(x(s))\,ds + \beta\int_{t_k}^{t} V(x(s-\tau))\,ds. \quad (19)$$

The value $V(x(s-\tau))$ does not exceed $V(t_k)$ with $s\in[t_k, t_k+\tau]$ (see Figure 1). Then from the inequality

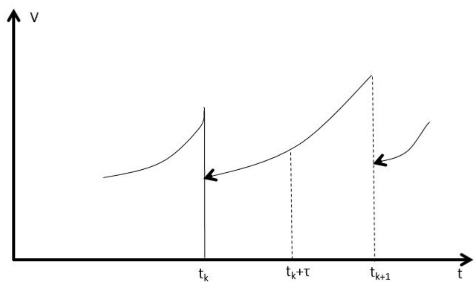

Figure 1.

(19) we obtain

$$V(x(t)) \leq V(x(t_k+0)) + \beta\tau V(x(t_k))$$

$$+\alpha\int_{t_k}^{t} V(x(s))\,ds.$$

Considering (18), from the last inequality we have

$$V(x(t)) \leq V(x(t_k))(e^{-\gamma}+\beta\tau) + \alpha\int_{t_k}^{t} V(x(s))\,ds.$$

Applying Gronwall's inequality [Bellman, 1963] to this inequality we receive

$$V(x(t)) \leq V(x(t_k))(e^{-\gamma}+\beta\tau)e^{\alpha(t-t_k)}.$$

Hence

$$V(x(t_k+\tau)) \leq V(x(t_k))(e^{-\gamma}+\beta\tau)e^{\alpha\tau}. \quad (20)$$

Now we will estimate $V(x(t))$ on the interval $[t_k+\tau, t_{k+1}]$. For this purpose we will use inequality (15) again. Applying Lemma 1 from [Alwan, Liu, 2013] to (15) on the interval $[t_k+\tau, t_{k+1}]$ we receive

$$V(x(t)) \leq \sup_{\zeta\in[t_k,t_k+\tau]} V(x(\zeta))e^{(\alpha+\beta)(t-t_k-\tau)}.$$

Considering that $V(x(t))$ increases on the interval $[t_k, t_k+\tau]$, the previous inequality can be written as

$$V(x(t)) \leq V(x(t_k+\tau))e^{(\alpha+\beta)(t-t_k-\tau)}. \quad (21)$$

Considering (20) from (21) we get

$$V(x(t_{k+1})) \leq V(x(t_k))(e^{-\gamma}+\beta\tau)$$

$$\times e^{(\alpha+\beta)(t_{k+1}-t_k)-\beta\tau}. \quad (22)$$

In accordance with (17) and (22) the following inequality holds

$$V(x(t_k)) \leq M(e^{-\gamma}+\beta\tau)^{k-1}e^{(\alpha+\beta)(t_{k+1}-t_k)-\beta\tau(k-1)}.$$

The last inequality can be written as follows

$$V(x(t_k)) \leq Me^{(k-1)(\ln(e^{-\gamma}+\beta\tau)-\beta\tau)+(\alpha+\beta)(t_k-t_0)}.$$

From this inequality follow statements of the theorem (10) and (11) .

Example.

Consider the differential equation

$$\dot{x}(t) = x(t) + bx(t-\tau)$$

$$+\sum_{k=1}^{\infty} c_k\Delta v_k x(t)\delta(t-k).\ \ t_k = k. \quad (23)$$

Let $V(x(t)) = px^2(t)$. Then for the system

$$\dot{x}(t) = x(t) + bx(t-\tau)$$

$$\dot{V}(x(t)) = \begin{pmatrix} x(t) \\ x(t-\tau) \end{pmatrix}^T \begin{pmatrix} 2p-\alpha p & bp \\ bp & -\beta p \end{pmatrix}$$

$$\times \begin{pmatrix} x(t) \\ x(t-\tau) \end{pmatrix} + px^2(t) + \beta px^2(t-\tau) \qquad (24)$$

The matrix

$$\begin{pmatrix} 2-\alpha & b \\ b & -\beta \end{pmatrix}$$

will be negative definite according to the Hurwitz criterion if $\triangle_1 = 2-\alpha < 0,\ \triangle_2 = \alpha\beta - 2\beta - b^2 > 0$. Let $\alpha = 2.1,\ \beta = \tau = 0.1,\ b = 0.05$. Then the Hurwitz criterion will be satisfied. The derivative in view of the system (4) of the Lyapunov function $V = px^2$ will be

$$\dot{V}(x(s)) = 2c_k \Delta v_k V(x(s)),\ \ -\gamma = 2c_k\Delta v_k. \quad (25)$$

The expression (9) for this example has the form

$$\theta_k = k\ln(e^{-\gamma} + 0,01) + 2,2k.$$

So when $-\gamma \leq \ln 0,1$ the trivial solution of the system (23) will be asymptotic stabile. The last inequality can always be ensured by the choice of Δv_k (see (25)).

Acknowledgements

The research was supported by Russian Science Foundation (RSF)(project No.16-11-10146).

References

Zavalishchin, S. T., Sesekin, A.N. (1997). *Dynamic Impulse Systems: Theory and Applications*. Kluwer Academic Publishers. Dordrecht.

Miller, B. M., Rubinovich, E. Ya. (2013). *Discontinuous solutions in the optimal control problems and their representation by singular space-time transformations*. Automation and Remote Control. v. 74. pp. 1969–2006.

Sesekin, A. N., Zhelonkina, N. I. (2011). *On the stability of linear systems with generalized action and delay*. In *IFAC-PapersOnLine, Proceedings of the 18th IFAC World Congress Milano.* Milano, Italy. pp. 13404–13407.

Sesekin, A. N., Fetisova, Yu. V. (2010)*Functional Differential Equations in the Space of Functions of Bounded Variation*. Proceeding of the Steklov Institute of Mathematics. suppl. 2. pp. 258–265.

Sesekin, A. N., Zhelonkina, N. I. (2016). *Stability of nonlinear dynamical systems containing the product of discontinuous functions and distributions*. In *AIP. Conference Proceeding. 1789.* pp. 1–8.

Samoilenko, A. M., Perestyuk, N. A. *Impulsive Differential Equations*. World Scientific, Singapore, 1995.

Cheng P., Deng F., Wang L. (2013) *The Method of Lyapunov Function and Exponential Stability of Impulsive Delay Systens with Delayed Impulses*. Mathematical Problems in Engineering. V. 2013. Article ID 458047 pp. 1-7.

Alwan M. S, Liu X. (2013). *On stability of liner and weakly nonliner switched systems with time delay*. Mathematical and Computer Modelling. v. 48. pp. 1150–1157.

Bellman, R., Cooke K. (1963). *Differential-difference Equations*. Academic Press. New York. London.